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# The Rotation Number Approach to the Periodic Fučik Spectrum[1]

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In this paper, we study the Fučik spectrum of the problem: (*) $\ddot{x} + (\lambda_+ + q_+(t))x_+ + (\lambda_- + q_-(t))x_- = 0$ with the $2\pi$-periodic boundary condition, where $q_\pm(t)$ are $2\pi$-periodic. After introducing a rotation number function $\rho(\lambda_+, \lambda_-)$ for (*), we prove using the Hamiltonian structure and the positive homogeneity of (*) that for any positive integer $n$, the two boundary curves of the domain $\rho^{-1}(n/2)$ in the $(\lambda_+, \lambda_-)$-plane are Fučik curves of (*). The result obtained in this paper shows that such a spectrum problem is much like that of the higher dimensional Fučik spectrum with the Dirichlet condition. In particular, it remains open if the Fučik spectrum of (*) is composed of only these curves.

## 1. INTRODUCTION

Fučik spectrum, a generalization of eigenvalues to asymmetric nonlinearity (jumping nonlinearity), was introduced in 1970s by Fučik [11] and Dancer [5]. By definition, the Fučik spectrum of the Laplacian with the Dirichlet boundary condition means those $(\lambda_+, \lambda_-) \in \mathbb{R}^2$ such that the following problem

$$\begin{cases} \Delta u + \lambda_+ u_+ + \lambda_- u_- = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has nonzero solutions, where $\Omega$ is a domain in $\mathbb{R}^N$, and $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$. As the equation in (1.1) has only the positive homogeneity, not the linearity, the structure of the Fučik spectrum of (1.1) with a general domain $\Omega$ is not known completely even when the dimension $N$ is 2,

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although the explicit formulas of Fučik spectrum can be given for certain domains [18].

When the dimension $N$ is 1, the Fučik spectrum of

$$\ddot{x} + \lambda_+ x_+ + \lambda_- x_- = 0 \tag{1.2}$$

with the Dirichlet boundary condition $x(0) = x(T) = 0$ can be given explicitly because Eq. (1.2) is integrable. Here $\ddot{x} = d^2x/dt^2$. Moreover, the structure of the corresponding Fučik spectrum of the problem

$$\ddot{x} + (\lambda_+ + q_+(t))x_+ + (\lambda_- + q_-(t))x_- = 0 \tag{1.3}$$

with a general two-point boundary condition like

$$x(0)\sin\alpha + \dot{x}(0)\cos\alpha = x(T)\sin\beta + \dot{x}(T)\cos\beta = 0, \qquad (0 \leqslant \alpha,\ \beta < \pi) \tag{$\mathrm{T}_{\alpha\beta}$}$$

has also been studied in some special cases [1, 7, 23], although Eq. (1.3) is now not integrable in general. Using the Prüfer substitution technique and some idea in Section 3 of this paper, one sees that the structure of the Fučik spectrum of $(1.3) + (\mathrm{T}_{\alpha\beta})$ is the same as that of $(1.2) + (\mathrm{T}_{\alpha\beta})$.

In recent years, the study for Fučik spectrum and its generalizations has become a hot topic. Many generalizations to various classes of positively homogeneous equations have been developed, see [8, 10, 24] and the references therein. The variational characterization of Fučik spectra has been proved in some cases [3, 4, 6]. In these works, some properties of Fučik spectra have been revealed and many interesting applications are discussed. For example, in nonresonance problems, Fučik spectrum plays a role similar to eigenvalues, see [2, 4, 6 13, 21, 24]. However, jumping nonlinearities are different from linearities in some dynamics aspects, e.g., the forced motion of asymmetric equations may behave like a typical nonlinear one [16, 20, 22].

In this paper, we consider the Fučik spectrum of (1.3) with the periodic boundary condition

$$(x(2\pi),\ \dot{x}(2\pi)) = (x(0),\ \dot{x}(0)). \tag{P}$$

Here we assume that the period is $T = 2\pi$ and $q_\pm(t)$ are $2\pi$-periodic functions in $L^1(0, 2\pi)$. Denote by $\mathscr{F} = \mathscr{F}(q_\pm)$ the Fučik spectrum of $(1.3) + (\mathrm{P})$. If $q_\pm = 0$ in (1.3), the Fučik spectrum of $(1.3) + (\mathrm{P})$ can be given explicitly because (1.2) is integrable. In fact, the Fučik spectrum of $(1.2) + (\mathrm{P})$ consists of the following Fučik curves: The $\lambda_+$-axis, the $\lambda_-$-axis, and the curves

$$C_n: \qquad \pi\lambda_+^{-1/2} + \pi\lambda_-^{-1/2} = 2\pi/n, \qquad n \in \mathbb{N}. \tag{1.4}$$

So far as we know, there is not much work on the periodic Fučik spectrum problem (1.3) + (P) with $q_\pm \neq 0$. In fact, from considerations to be developed in this paper, it seems that it is not an easy problem to give the complete description of the structure $\mathscr{F}$ of (1.3) + (P). The novelty for $\mathscr{F}$ we find in this paper reveals that problem (1.3) + (P) has some similar difficulty as in the higher dimensional Dirichlet Fučik spectrum problem (1.1). Obviously, $\mathscr{F}$ contains two trivial Fučik lines: $\mathscr{L}^+$: $\lambda_+ = \lambda_0^{\mathrm{P}}(q_+)$ and $\mathscr{L}^-$: $\lambda_- = \lambda_0^{\mathrm{P}}(q_-)$, where $\lambda_0^{\mathrm{P}}(q)$ denotes the zeroth periodic eigenvalue of the following linear Schrödinger operator:

$$(Lx)(t) := -\ddot{x}(t) - q(t)x(t) = \lambda x(t). \tag{1.5}$$

The main contribution of this paper is to find two sequences of Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$, $n \in \mathbb{N}$, in $\mathscr{F}$. However, we do not know if these Fučik curves represent all of $\mathscr{F}$. A crucial difference between such a spectrum problem and problem (1.3) + $(\mathrm{T}_{\alpha\beta})$ will be explained in Remark 3.1.

Let us briefly explain our technique in defining these Fučik curves. Let $q(t)$ be $2\pi$-periodic and $q \in L^1(0, 2\pi)$. The periodic eigenvalues of (1.5) + (P) and antiperiodic eigenvalues of (1.5) with the antiperiodic boundary condition

$$(x(2\pi),\ \dot{x}(2\pi)) = -(x(0),\ \dot{x}(0)) \tag{A}$$

can be analyzed using many methods because Eq. (1.5) is linear so that the Floquet theory for linear periodic equations and the classification theory of $2 \times 2$ symplectic matrices are applicable. Note that the latter two theories do not apply to Eq. (1.3). Thus, we will adopt the rotation number approach to the spectrum of (1.5) [15, 19]. This approach is more geometrical and is very useful in many problems [12, 19, 26]. It has been partially generalized to the periodic and antiperiodic eigenvalues of the one-dimensional $p$-Laplacian with periodic potentials [25]. Now we will give, in this paper, another partial generalization of this approach which enables us to find two sequences of Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$ of problem (1.3) + (P). Due to the asymmetry in (1.3), we will not consider in this paper the Fučik spectrum of (1.3) + (A) because it is more complicated than the periodic case.

In Section 2, we follow the idea in [12, 25, 26] to introduce a rotation number function $\rho(\lambda_+, \lambda_-)$ for Eq. (1.3) and the properties of $\rho(\lambda_+, \lambda_-)$ are discussed. In Section 3, we use the Hamiltonian structure and the positive homogeneity of (1.3) to prove that for any $n \in \mathbb{N}$, $\rho^{-1}(n) := \{(\lambda_+, \lambda_-) \in \mathbb{R}^2\colon \rho(\lambda_-, \lambda_-) = n\}$ is a domain in the $(\lambda_+, \lambda_-)$-plane and the boundary $\partial \rho^{-1}(n)$ consists of, in general, two curves $\mathscr{M}_n$ and $\mathscr{N}_n$ which are necessarily Fučik curves in $\mathscr{F}$. See Theorem 3.1. Some geometric properties of these Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$ are also proved. In particular, all of these periodic Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$ have horizontal and vertical asymptotes which are related with some Dirichlet eigenvalues with potentials being some translations of $q_\pm$. See Theorem 3.3. In the last section, we discuss briefly the abstract setting of our proof and give some further generalization of Theorem 3.1.

## 2. ROTATION NUMBERS

We introduce some notation. Let

$$\mathscr{P} = \{p : \mathbb{R} \to \mathbb{R} : p(t) \text{ is } 2\pi\text{-periodic and } p \in L^1(0, 2\pi)\}.$$

For $p \in \mathscr{P}$, the mean value is $\bar{p} = (2\pi)^{-1} \int_0^{2\pi} p(t)\, dt$. For $p,\ q \in \mathscr{P}$, write $p \succ q$ if $p(t) \geqslant q(t)$ for a.e. $t \in [0, 2\pi]$ and $p(t) > q(t)$ for $t$ in a subset of $[0, 2\pi]$ of positive measure. For pairs $(p_+, p_-)$ and $(q_+, q_-)$, write $(p_+, p_-) \succ (q_+, q_-)$ if $p_\pm \geqslant q_\pm$, and both $p_+(t) > q_+(t)$ and $p_-(t) > q_-(t)$ hold for $t$ in a *common* subset of $[0, 2\pi]$ of positive measure.

Let $p_\pm \in \mathscr{P}$. Consider the following equation on the circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$:

$$\dot{\theta} = \Phi(t, \theta; p_\pm) := \begin{cases} p_+(t) \cos^2 \theta + \sin^2 \theta & \text{when} \quad + \cos \theta \geqslant 0, \\ p_-(t) \cos^2 \theta + \sin^2 \theta & \text{when} \quad - \cos \theta \geqslant 0. \end{cases} \tag{2.1}$$

Note that $\Phi(t, \theta; p_\pm)$ is continuously differentiable in $\theta \in \mathbb{R}$:

$$\frac{d\Phi(t, \theta; p_\pm)}{d\theta} = (1 - p_\pm(t)) \sin 2\theta \qquad \text{when} \quad \pm \cos \theta \geqslant 0.$$

In particular, $\Phi(t, \theta; p_\pm)$ is globally Lipschitzian in $\theta \in \mathbb{R}$. Thus, for any $\theta_0 \in \mathbb{R}$, the unique solution $\theta(t; \theta_0, p_\pm)$ of (2.1) satisfying the initial condition $\theta(0) = \theta_0$ is well defined for all $t \in \mathbb{R}$. As $\Phi(t, \theta; p_\pm)$ is $2\pi$-periodic in both $t$ and $\theta$,

$$\theta(t; \theta_0 + 2n\pi, p_\pm) \equiv \theta(t; \theta_0, p_\pm) + 2n\pi, \tag{2.2}$$

$$\theta(t + 2n\pi; \theta_0, p_\pm) \equiv \theta(t; \theta(2n\pi; \theta_0, p_\pm), p_\pm) \tag{2.3}$$

for all $t,\ \theta_0 \in \mathbb{R}$ and all $n \in \mathbb{Z}$.

Let $\Theta : \mathbb{R} \to \mathbb{R}$ be the Poincaré map of (2.1) defined by

$$\Theta(\theta_0) := \theta(2\pi; \theta_0, p_\pm).$$

Then $\Theta$ is a diffeomorphism of $\mathbb{R}$ and (2.2) implies that

$$\Theta(\theta_0 + 2n\pi) = \Theta(\theta_0) + 2n\pi \qquad \forall\, \theta_0 \in \mathbb{R}, \ \forall\, n \in \mathbb{Z}. \tag{2.4}$$

As the vector field $\Phi(t, \theta; p_\pm)$ increases when $p_\pm$ increase, the following monotonicity for the solutions of (2.1) follows from the comparison theorem of solutions.

Lemma 2.1. *Let $p_{i\pm} \in \mathscr{P}$ be such that $(p_{1+}, p_{1-}) \succ (p_{2+}, p_{2-})$. Then*

(i) $\theta(t; \theta_0, p_{1\pm}) \geqslant \theta(t; \theta_0, p_{2\pm})$ *for all* $t \geqslant 0$; *and*
(ii) $\theta(t; \theta_0, p_{1\pm}) > \theta(t; \theta_0, p_{2\pm})$ *for all* $t \geqslant 2\pi$.

Let now $q_\pm \in \mathscr{P}$ and $\lambda_\pm \in \mathbb{R}$. We are going to introduce a rotation number function $\rho(\lambda_+, \lambda_-)$ for (1.3). Set $y = -\dot{x}$ in (1.3). Then Eq. (1.3) is equivalent to the following system:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = (\lambda_+ + q_+(t))x_+ + (\lambda_- + q_-(t))x_-. \end{cases} \tag{2.5}$$

In the polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $r$ and $\theta$ satisfy the following equations:

$$\dot{r} = \begin{cases} (\lambda_+ + q_+(t) - 1)\, r\cos\theta\sin\theta & \text{when } \cos\theta \geqslant 0, \\ (\lambda_- + q_-(t) - 1)\, r\cos\theta\sin\theta & \text{when } \cos\theta < 0. \end{cases} \tag{2.6}$$

$$\dot{\theta} = \Phi(t, \theta; \lambda_+, \lambda_-) := \begin{cases} (\lambda_+ + q_+(t))\cos^2\theta + \sin^2\theta & \text{when } \cos\theta \geqslant 0, \\ (\lambda_- + q_-(t))\cos^2\theta + \sin^2\theta & \text{when } \cos\theta < 0. \end{cases} \tag{2.7}$$

For any $\theta_0 \in \mathbb{R}$ and $\lambda_\pm \in \mathbb{R}$, let $\theta(t; \theta_0, \lambda_+, \lambda_-)$ be the unique solution of (2.7) satisfying the initial condition: $\theta(0; \theta_0, \lambda_+, \lambda_-) = \theta_0$. We will write $\theta(t; \theta_0, \lambda_+, \lambda_-)$ as $\theta(t; \theta_0)$ when $\lambda_\pm$ are clear from the context. As the vector field $\Phi(t, \theta; \lambda_+, \lambda_-)$ is $2\pi$-periodic in both $t$ and $\theta$, it is known from Hale [14] that the rotation number of (2.7)

$$\rho(\lambda_+, \lambda_-) = \rho(\lambda_+, \lambda_-; q_\pm) = \lim_{t \to \infty} t^{-1}(\theta(t; \theta_0, \lambda_+, \lambda_-) - \theta_0)$$

exists and is independent of $\theta_0$.

Some properties for the rotation number function $\rho(\lambda_+, \lambda_-)$ are collected in the following lemma.

Lemma 2.2. *Let $q_\pm \in \mathscr{P}$. Then the following hold:*

(i) $\rho(\lambda_+, \lambda_-)$ *is continuous in* $(\lambda_+, \lambda_-) \in \mathbb{R}^2$;
(ii) $\rho(\lambda_+, \lambda_-)$ *is nondecreasing when either* $\lambda_+$ *or* $\lambda_-$ *increases*;

(iii) $\rho(\lambda_+,\lambda_-)\geqslant 0$ *for all* $(\lambda_+,\lambda_-)\in\mathbb{R}^2$;

(iv) $\rho(\lambda_+,\lambda_-)=0$ *if either* $\lambda_+ \ll -1$ *or* $\lambda_- \ll -1$; *and*

(v) $\rho(\lambda_+,\lambda_-)$ *tends to* $+\infty$ *when both* $\lambda_+$ *and* $\lambda_-$ *tend to* $+\infty$.

*Proof.* (i) It can be proved that the homeomorphism $\theta(2\pi;\cdot,p_\pm)$ continuously depends upon $p_+$, $p_-\in\mathscr{P}$ when $\mathscr{P}$ is endowed with the $L^1$-distance: $d(p_1,p_2)=\int_0^{2\pi}|p_1(t)-p_2(t)|\,dt$. Now the continuity of $\rho(\lambda_+,\lambda_-)$ follows from the continuity of rotation numbers on homeomorphisms, see [14].

(ii) Let $(\lambda_{i+},\lambda_{i-})\in\mathbb{R}^2$ be such that $\lambda_{1+}\geqslant\lambda_{2+}$ and $\lambda_{1-}\geqslant\lambda_{2-}$. Applying Lemma 2.1 to $p_{i\pm}(t)=\lambda_{i\pm}+q_\pm(t)$, we have $\theta(t;\theta_0,\lambda_{1+},\lambda_{1-})\geqslant\theta(t;\theta_0,\lambda_{2+},\lambda_{2-})$ for all $t\geqslant 0$. Thus $\rho(\lambda_{1+},\lambda_{1-})\geqslant\rho(\lambda_{2+},\lambda_{2-})$ by definition of rotation numbers.

(iii) It can be proved that $\theta_0>n\pi+\pi/2$ implies that $\theta(t;\theta_0)>n\pi+\pi/2$ for all $t\geqslant 0$. This fact follows essentially from the observation that $\dot\theta(t;\theta_0)=1$ when $\cos(\theta(t;\theta_0))=0$. Now the conclusion that $\rho(\lambda_+,\lambda_-)\geqslant 0$ follows from the independence of the choice of $\theta_0$ in the definition of rotation numbers.

(iv) Consider the eigenvalue problem

$$\ddot x+(\lambda+q_+(t))x=0.$$

It is known from [14] that the zeroth periodic eigenvalue $\lambda_0^{\mathrm{P}}(q_+)$ of the above problem has a nowhere vanishing eigenfunction $x_0(t)$. Assume that $x_0(t)>0$ for all $t$. This means that for $\lambda_+=\lambda_0^{\mathrm{P}}(q_+)$ and any $\lambda_-\in\mathbb{R}$, the function $x=x_0(t)$ is a solution of (1.3). Let $\theta(t)=\arg(x_0(t)-i\dot x_0(t))$. Then $|\theta(t)|<\pi/2$ for all $t$ because $x_0(t)>0$ for all $t$. As $\theta(t)$ is a solution of (2.7), we know that $\rho(\lambda_0^{\mathrm{P}}(q_+),\lambda_-)=0$ for all $\lambda_-$. Analogously, $\rho(\lambda_+,\lambda_0^{\mathrm{P}}(q_-))=0$ for all $\lambda_+$. These facts, together with property (ii), show that

$$\rho(\lambda_+,\lambda_-)=0 \qquad \text{if either} \quad \lambda_+\leqslant\lambda_0^{\mathrm{P}}(q_+) \quad \text{or} \quad \lambda_-\leqslant\lambda_0^{\mathrm{P}}(q_-). \tag{2.8}$$

(v) We postpone the proof after the following simple example. ∎

Example 2.1. Let $q_\pm(t)\equiv 0$. Then the rotation number function $\rho(\lambda_+,\lambda_-)$ is

$$\rho(\lambda_+,\lambda_-)=0 \qquad \text{if } \lambda_+\leqslant 0 \text{ or } \lambda_-\leqslant 0, \tag{2.9}$$

$$\rho(\lambda_+,\lambda_-)=\frac{2}{\lambda_+^{-1/2}+\lambda_-^{-1/2}}, \qquad \text{if } \lambda_+>0 \text{ and } \lambda_->0. \tag{2.10}$$

If $\lambda_+ \leqslant 0$ or $\lambda_- \leqslant 0$, (2.9) follows from (2.8). Let now $\lambda_\pm > 0$. Then all solutions $x$ of (1.2) are periodic and $(x, y) = (x, -\dot{x})$ are on arcs of ellipses:

$$\lambda_\pm x^2 + y^2 = \text{const.}$$

Let $\tilde{y} = y/\sqrt{\lambda_\pm}$ if $\pm x \geqslant 0$. Then the arcs of ellipses are now transformed into arcs of circles:

$$x^2 + \tilde{y}^2 = \text{const.}$$

Eq. (2.7) reads now

$$\dot{\theta} = \lambda_\pm \cos^2 \theta + \sin^2 \theta, \qquad \text{if} \quad \pm \cos \theta \geqslant 0. \tag{2.11}$$

Let us introduce another polar coordinates transformation as $x = r \cos \vartheta$, $\tilde{y} = y/\sqrt{\lambda_\pm} = r \sin \vartheta$. Then, Eq. (2.11) is transformed into

$$\dot{\vartheta} = \lambda_\pm^{1/2}, \qquad \text{if} \quad \pm \cos \vartheta \geqslant 0. \tag{2.12}$$

Let $\vartheta(t; \vartheta_0)$ be solutions of (2.12). If $T_n = n\pi(\lambda_+^{-1/2} + \lambda_-^{-1/2})$, then $\vartheta(T_n; \vartheta_0) = \vartheta_0 + 2n\pi$ for all $\vartheta_0 \in \mathbb{R}$ and all $n \in \mathbb{Z}$. From this one has

$$\theta(T_n; \theta_0) = \theta_0 + 2n\pi, \qquad \forall\, \theta_0 \in \mathbb{R}, \ \forall\, n \in \mathbb{Z}.$$

As a result, we have

$$\rho(\lambda_+, \lambda_-) = \lim_{n \to \infty} \frac{\theta(T_n; \theta_0) - \theta_0}{T_n} = \frac{2}{\lambda_+^{-1/2} + \lambda_-^{-1/2}}. \quad \blacksquare$$

Suggested by this example, we complete the proof of Lemma 2.2 (v). Suppose that $\lambda_\pm > 0$. Define a homeomorphism $H_{\lambda_\pm} : \mathbb{R} \to \mathbb{R}$ as follows. Firstly, $H_{\lambda_\pm}$ fixes all of $\{n\pi,\ n\pi + \pi/2\colon n \in \mathbb{Z}\}$. Secondly, for any given $\vartheta \in \mathbb{R}$, $\theta = H_{\lambda_\pm}(\vartheta) \in \mathbb{R}$ is determined by the following equality:

$$(\cos \theta,\ \sin \theta) = \frac{(\cos \vartheta,\ \sqrt{\lambda_\pm} \sin \vartheta)}{\sqrt{\cos^2 \vartheta + \lambda_\pm \sin^2 \vartheta}} \qquad \text{when} \quad \pm \cos \vartheta \geqslant 0.$$

Then $H_{\lambda_\pm}$ is well defined and satisfies (2.13). In particular,

$$\lim_{|\vartheta| \to \infty} \frac{H_{\lambda_\pm}(\vartheta)}{\vartheta} = 1.$$

Now $\vartheta$ satisfies the following equation:

$$\dot{\vartheta} = \Psi(t, \vartheta) := (\lambda_{\pm}^{1/2} + \lambda_{\pm}^{-1/2} q_{\pm}(t)) \cos^2 \vartheta + \lambda_{\pm}^{1/2} \sin^2 \vartheta \qquad \text{if} \quad \pm \cos \vartheta \geqslant 0. \tag{2.13}$$

For any $\vartheta_0 \in \mathbb{R}$, let $\vartheta(t; \vartheta_0)$ be the unique solution of (2.13) satisfying $\vartheta(0; \vartheta_0) = \vartheta_0$. Then we have the following equality:

$$\theta(t; H_{\lambda_{\pm}}(\vartheta_0)) \equiv H_{\lambda_{\pm}}(\vartheta(t; \vartheta_0)).$$

Hence

$$\begin{aligned} \rho(\lambda_+, \lambda_-) &= \lim_{t \to +\infty} \frac{\theta(t; H_{\lambda_{\pm}}(\vartheta_0))}{t} \\ &= \lim_{t \to +\infty} \frac{H_{\lambda_{\pm}}(\vartheta(t; \vartheta_0))}{t} \\ &= \lim_{t \to +\infty} \frac{H_{\lambda_{\pm}}(\vartheta(t; \vartheta_0))}{\vartheta(t; \vartheta_0)} \frac{\vartheta(t; \vartheta_0)}{t} \\ &= \lim_{t \to +\infty} \frac{\vartheta(t; \vartheta_0)}{t}, \end{aligned} \tag{2.14}$$

if one can verify that

$$\lim_{t \to +\infty} \vartheta(t; \vartheta_0) = +\infty. \tag{2.15}$$

Note that

$$\Psi(t, \vartheta) = \lambda_{\pm}^{1/2} + \lambda_{\pm}^{-1/2} q_{\pm}(t) \cos^2 \vartheta \qquad \text{if} \quad \pm \cos \vartheta \geqslant 0.$$

Thus

$$\begin{aligned} \Psi(t, \vartheta) &\geqslant \min\{\lambda_+^{1/2},\ \lambda_-^{1/2}\} - \max\{\lambda_+^{-1/2}\, |q_+(t)|,\ \lambda_-^{-1/2}\, |q_-(t)|\}, \\ \Psi(t, \vartheta) &\leqslant \max\{\lambda_+^{1/2},\ \lambda_-^{1/2}\} + \max\{\lambda_+^{-1/2}\, |q_+(t)|,\ \lambda_-^{-1/2}\, |q_-(t)|\}. \end{aligned}$$

By (2.13), we have, for all $t \geqslant 0$,

$$\vartheta(t; \vartheta_0) \geqslant \vartheta_0 + \min\{\lambda_+^{1/2},\ \lambda_-^{1/2}\}\, t - \max\{\lambda_+^{-1/2}\, Q_+(t),\ \lambda_-^{-1/2}\, Q_-(t)\}, \tag{2.16}$$

$$\vartheta(t; \vartheta_0) \leqslant \vartheta_0 + \max\{\lambda_+^{1/2},\ \lambda_-^{1/2}\}\, t + \max\{\lambda_+^{-1/2}\, Q_+(t),\ \lambda_-^{-1/2}\, Q_-(t)\}, \tag{2.17}$$

where $Q_{\pm}(t) = \int_0^t |q_{\pm}(s)| ds$. By (2.16), one sees that if $\min\{\lambda_+, \lambda_-\} \gg 1$, then (2.15) holds, because $q_{\pm}(t)$ are periodic. It now follows from (2.14), (2.16) and (2.17) that we have the following estimates on $\rho(\lambda_+, \lambda_-)$:

$$\rho(\lambda_+, \lambda_-) \geqslant \min\{\lambda_+^{1/2}, \lambda_-^{1/2}\} - \max\{\lambda_+^{-1/2}\,\overline{|q_+|}, \lambda_-^{-1/2}\,\overline{|q_-|}\}, \tag{2.18}$$

$$\rho(\lambda_+, \lambda_-) \leqslant \max\{\lambda_+^{1/2}, \lambda_-^{1/2}\} + \max\{\lambda_+^{-1/2}\,\overline{|q_+|}, \lambda_-^{-1/2}\,\overline{|q_-|}\}, \tag{2.19}$$

where $\overline{|q_\pm|}$ are mean values of $|q_\pm(t)|$. In particular, (2.18) shows that $\rho(\lambda_+, \lambda_-) \to +\infty$ when $\min\{\lambda_+, \lambda_-\} \to +\infty$. Thus Lemma 2.2(v) is proved. ∎

## 3. PERIODIC FUČIK SPECTRUM

Let now $q_\pm \in \mathscr{P}$ and consider the differential equation (1.3). Recall that the periodic Fučik spectrum of (1.3), denoted by $\mathscr{F} = \mathscr{F}(q_\pm)$, is the set of all those $(\lambda_+, \lambda_-) \in \mathbb{R}^2$ such that Eq. (1.3) has nonzero $2\pi$-periodic solutions.

In this section, we will give a partial description of $\mathscr{F}$. At first, we know from the proof of Lemma 2.2(iv) that the following two straight lines in the $(\lambda_+, \lambda_-)$-plane are always in $\mathscr{F}$:

$$\mathscr{L}^+\colon\ \lambda_+ = \lambda_0^{\mathrm{P}}(q_+), \qquad \mathscr{L}^-\colon\ \lambda_- = \lambda_0^{\mathrm{P}}(q_-).$$

Before giving further results on $\mathscr{F}$, we prove the following conclusion.

Lemma 3.1. *Let $(\lambda_+, \lambda_-) \in \mathscr{F}$. Then there exists $n \in \mathbb{Z}^+ = \{0\} \cup \mathbb{N}$ such that $\rho(\lambda_+, \lambda_-) = n$.*

*Proof.* Suppose that $x(t)$ is a nonzero $2\pi$-periodic solution of (1.3). Let $\theta_0 = \arg(x(0) - i\dot{x}(0))$. Then $\theta(t; \theta_0)$ satisfies

$$\theta(2\pi; \theta_0) = \theta_0 + 2n\pi$$

for some $n \in \mathbb{Z}$. It follows from (2.2) and (2.3) that

$$\theta(2m\pi; \theta_0) = \theta_0 + 2mn\pi$$

for all $m \in \mathbb{Z}$. Thus $\rho(\lambda_+, \lambda_-) = n$. Lemma 2.2(iii) shows that $n \geqslant 0$. ∎

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a homeomorphism satisfying (2.4). Define the rotation number $\rho(h)$ of $h$ by

$$\rho(h) = \lim_{n\to\infty} \frac{h^n(\theta_0) - \theta_0}{2n\pi}$$

(independent of $\theta_0$). The following result is proved in [12].

Lemma 3.2. *Let $h$ be a homeomorphism of $\mathbb{R}$ satisfying* (2.4) *and $n$ an integer. Then*

(i) $\rho(h) \geqslant n$ *if and only if* $\max_{\theta_0 \in \mathbb{R}} (h(\theta_0) - (\theta_0 + 2n\pi)) \geqslant 0$.

(ii) $\rho(h) \leqslant n$ *if and only if* $\min_{\theta_0 \in \mathbb{R}} (h(\theta_0) - (\theta_0 + 2n\pi)) \leqslant 0$.

Note that if one defines

$$\Theta_{\lambda_\pm}(\theta_0) := \theta(2\pi; \theta_0, \lambda_+, \lambda_-),$$

then $\Theta_{\lambda_\pm}$ satisfies (2.4) and $\rho(\lambda_+, \lambda_-) = \rho(\Theta_{\lambda_\pm})$. Introduce the following two functions:

$$M(\lambda_+, \lambda_-) = \max_{\theta_0 \in \mathbb{R}} (\theta(2\pi; \theta_0, \lambda_+, \lambda_-) - \theta_0),$$
$$N(\lambda_+, \lambda_-) = \min_{\theta_0 \in \mathbb{R}} (\theta(2\pi; \theta_0, \lambda_+, \lambda_-) - \theta_0).$$

By Lemma 2.2, it is not difficult to prove that for any given $n \in \mathbb{N}$, the preimage $\rho^{-1}(n)$ is a connected domain in $\mathbb{R}^2$.

Lemma 3.3. *Let $n \in \mathbb{N}$. Then the boundary of $\rho^{-1}(n)$ is given by*

$$\partial\rho^{-1}(n) = \{(\lambda_+, \lambda_-): \textit{ either } M(\lambda_+, \lambda_-) = 2n\pi \textit{ or } N(\lambda_+, \lambda_-) = 2n\pi\}.$$

*Proof.* By Lemmas 2.2 and 3.2, we have

$$\rho^{-1}(n) = \{(\lambda_+, \lambda_-) \in \mathbb{R}^2: N(\lambda_+, \lambda_-) \leqslant 2n\pi \leqslant M(\lambda_+, \lambda_-)\}.$$

Assume that $(\lambda_+, \lambda_-) \in \mathbb{R}^2$ satisfies $N(\lambda_+, \lambda_-) = 2n\pi$. Let $(\mu_+, \mu_-) \in \mathbb{R}^2$ be such that $\mu_+ > \lambda_+$ and $\mu_- > \lambda_-$. By Lemma 2.1,

$$\theta(2\pi; \theta_0, \mu_+, \mu_-) > \theta(2\pi; \theta_0, \lambda_+, \lambda_-)$$

for all $\theta_0$. As a result,

$$\begin{aligned} N(\mu_+, \mu_-) &= \min_{\theta_0} (\theta(2\pi; \theta_0, \mu_+, \mu_-) - \theta_0) \\ &> \min_{\theta_0} (\theta(2\pi; \theta_0, \lambda_+, \lambda_-) - \theta_0) \\ &= N(\lambda_+, \lambda_-) = 2n\pi. \end{aligned}$$

By Lemma 3.2, $\rho(\mu_+, \mu_-) > n$ and $(\mu_+, \mu_-) \notin \rho^{-1}(n)$. Consequently, $(\lambda_+, \lambda_-) \in \partial\rho^{-1}(n)$. Similarly, all $(\lambda_+, \lambda_-)$ satisfying $M(\lambda_+, \lambda_-) = 2n\pi$ are also in $\partial\rho^{-1}(n)$.

Conversely, assume that $(\lambda_+, \lambda_-)$ satisfies

$$N(\lambda_+, \lambda_-) < 2n\pi < M(\lambda_+, \lambda_-). \tag{3.1}$$

As $\theta(2\pi; \theta_0, \mu_+, \mu_-)$ continuously depends on $(\mu_+, \mu_-)$, $N(\mu_+, \mu_-)$ and $M(\mu_+, \mu_-)$ are also continuous. Thus (3.1) implies that

$$N(\mu_+, \mu_-) < 2n\pi < M(\mu_+, \mu_-)$$

for all $(\mu_+, \mu_-)$ near $(\lambda_+, \lambda_-)$. By Lemma 3.2, $\rho(\mu_+, \mu_-) = n$ for all $(\mu_+, \mu_-)$ near $(\lambda_+, \lambda_-)$. As a result, $(\lambda_+, \lambda_-) \in \operatorname{int} \rho^{-1}(n)$. ∎

Now let us introduce the Poincaré map of system (2.5). Note that the vector field $V(t, x, y) = (-y, (\lambda_+ + q_+(t))x_+ + (\lambda_- + q_-(t))x_-)$ is globally Lipschitzian with respect to $(x, y) \in \mathbb{R}^2$. Thus, for any $(x_0, y_0) \in \mathbb{R}^2$, the unique solution $(x(t; x_0, y_0, \lambda_+, \lambda_-), y(t; x_0, y_0, \lambda_+, \lambda_-))$ of (2.5) satisfying the initial value $(x(0), y(0)) = (x_0, y_0)$ is well defined for all $t \in \mathbb{R}$. The Poincaré map $P_{\lambda_\pm} : \mathbb{R}^2 \to \mathbb{R}^2$ of (2.5) is defined by

$$P_{\lambda_\pm}(x_0, y_0) = (x(2\pi; x_0, y_0, \lambda_+, \lambda_-),\ y(2\pi; x_0, y_0, \lambda_+, \lambda_-)).$$

Note that (2.5) is a Hamiltonian system. An approximation technique shows that $P_{\lambda_\pm}$ is an area-preserving homeomorphism although $V(t, x, y)$ may not be differentiable at points $(0, y)$. Moreover, as $V(t, x, y)$ is positively homogeneous in $(x, y)$, then so does $P_{\lambda_\pm}$, i.e.,

$$P_{\lambda_\pm}(kx_0, ky_0) = kP_{\lambda_\pm}(x_0, y_0)$$

for all $k \geqslant 0$ and all $(x_0, y_0) \in \mathbb{R}^2$.

Besides the solutions $\theta(t; \theta_0, \lambda_+, \lambda_-)$ of (2.7), for any given $\theta_0 \in \mathbb{R}$, let $r = r(t; \theta_0, \lambda_+, \lambda_-)$ be the solution of (2.6) satisfying $r(0) = 1$. Then $r(t; \theta_0, \lambda_+, \lambda_-)$ is $2\pi$-periodic in $\theta_0$. Let

$$R_{\lambda_\pm}(\theta_0) := r(2\pi; \theta_0, \lambda_+, \lambda_-).$$

Now the Poincaré map $P_{\lambda_\pm}$ can be written as

$$P_{\lambda_\pm}(k\cos\theta_0,\ k\sin\theta_0) = kR_{\lambda_\pm}(\theta_0)(\cos\Theta_{\lambda_\pm}(\theta_0),\ \sin\Theta_{\lambda_\pm}(\theta_0)) \tag{3.2}$$

for all $k \geqslant 0$ and all $\theta_0$.

The following observation is fundamental in the proof of our main result. It is a result of the area-preserving property of $P_{\lambda_\pm}$.

LEMMA 3.4.

$$\frac{d\Theta_{\lambda_\pm}(\theta_0)}{d\theta_0} \equiv \frac{1}{R^2_{\lambda_\pm}(\theta_0)}. \tag{3.3}$$

*Proof.* For simplicity, rewrite $\Theta_{\lambda_\pm}(\theta_0)$ and $R_{\lambda_\pm}(\theta_0)$ as $\Theta(\theta_0)$ and $R(\theta_0)$, respectively. Let $\theta_0$ be fixed. For any $\theta_1$ ($> \theta_0$) near $\theta_0$, consider the following sector:

$$S = \{(r\cos\vartheta,\ r\sin\vartheta) \in \mathbb{R}^2\colon\ 0 \leqslant r \leqslant 1,\ \ \theta_0 \leqslant \vartheta \leqslant \theta_1\}.$$

Then $S$ has area $\frac{1}{2}(\theta_1 - \theta_0)$. The image $S' = P_{\lambda_\pm}(S)$ of $S$ under $P_{\lambda_\pm}$ is

$$S' = \{(r'\cos\vartheta',\ r'\sin\vartheta') \in \mathbb{R}^2\colon\ 0 \leqslant r' \leqslant R(\Theta^{-1}(\vartheta')),\ \ \Theta(\theta_0) \leqslant \vartheta' \leqslant \Theta(\theta_1)\},$$

where $\Theta^{-1}$ is the inverse of $\Theta$. Thus $S'$ has area

$$\frac{1}{2}\int_{\Theta(\theta_0)}^{\Theta(\theta_1)} R^2(\Theta^{-1}(\vartheta'))d\vartheta' = \frac{1}{2}\int_{\theta_0}^{\theta_1} R^2(\vartheta)\frac{d\Theta(\vartheta)}{d\vartheta}d\vartheta.$$

As $P_{\lambda_\pm}$ is area-preserving,

$$\frac{1}{2}(\theta_1 - \theta_0) \equiv \frac{1}{2}\int_{\theta_0}^{\theta_1} R^2(\vartheta)\frac{d\Theta(\vartheta)}{d\vartheta}d\vartheta.$$

Differentiating this equality with respect $\theta_1$ and evaluating at $\theta_0$, we get (3.3). ∎

Now we consider the Fučik spectrum $\mathscr{F}$ of (1.3). Besides the two straight lines $\mathscr{L}^\pm$, we introduce, for each $n \geqslant 1$, two subsets of $\mathbb{R}^2$ by

$$\mathscr{M}_n\colon\ M(\lambda_+, \lambda_-) = 2n\pi, \qquad \mathscr{N}_n\colon\ N(\lambda_+, \lambda_-) = 2n\pi. \tag{3.4}$$

We will prove in Lemma 3.5 that $\mathscr{M}_n$ and $\mathscr{N}_n$ are actually curves in the $(\lambda_+, \lambda_-)$-plane. By Lemma 3.3, $\mathscr{M}_n$ and $\mathscr{N}_n$ are the boundary curves of $\rho^{-1}(n)$. In some cases, they may coincide. For instance, when $q_\pm(t) \equiv 0$, then $\mathscr{M}_n$ coincides with $\mathscr{N}_n$ and they are the classical periodic Fučik curves $C_n$ given by (1.4).

The following is the main result of this paper.

THEOREM 3.1. *All curves $\mathscr{M}_n$ and $\mathscr{N}_n$ are in $\mathscr{F}$, i.e.,*

$$\mathscr{L}^+ \cup \mathscr{L}^- \cup \left( \bigcup_{n \in \mathbb{N}} (\mathscr{M}_n \cup \mathscr{N}_n) \right) \subset \mathscr{F}.$$

*Proof.* Let $n \in \mathbb{N}$ and $(\lambda_+, \lambda_-) \in \mathscr{M}_n$, i.e.,

$$M_n(\lambda_+, \lambda_-) = \max_{\theta_0} (\Theta_{\lambda_\pm}(\theta_0) - \theta_0) = 2n\pi.$$

Then there exists $\theta_0 \in \mathbb{R}$ such that

$$\Theta_{\lambda_\pm}(\theta_0) = \theta_0 + 2n\pi \qquad \text{and} \qquad \frac{d\Theta_{\lambda_\pm}(\theta_0)}{d\theta_0} = 1. \tag{3.5}$$

By (3.3), $R_{\lambda_\pm}(\theta_0) = 1$. Using expression (3.2),

$$\begin{aligned} P_{\lambda_\pm}(\cos\theta_0, \sin\theta_0) &= R_{\lambda_\pm}(\theta_0)(\cos\Theta_{\lambda_\pm}(\theta_0), \sin\Theta_{\lambda_\pm}(\theta_0)) \\ &= R_{\lambda_\pm}(\theta_0)(\cos(\theta_0 + 2n\pi), \sin(\theta_0 + 2n\pi)) \\ &= (\cos\theta_0, \sin\theta_0). \end{aligned}$$

Namely, $P_{\lambda_\pm}$ has a nonzero fixed point $(\cos\theta_0, \sin\theta_0)$. As a result, (2.5) has a nonzero $2\pi$-periodic solution and $(\lambda_+, \lambda_-) \in \mathscr{F}$. Similarly, $\mathscr{N}_n \subset \mathscr{F}$. ∎

REMARK 3.1. We do not know if $\mathscr{F}$ is just composed of these Fučik curves $\mathscr{L}^\pm, \mathscr{M}_n, \mathscr{N}_n$, $n \in \mathbb{N}$. Let us mention a recent work by Rynne [23] on the Fučik spectrum of (1.3) with two-point boundary condition $(T_{\alpha\beta})$. In this case, when $(\lambda_+, \lambda_-)$ is a Fučik point with the corresponding eigenfunction $x(t)$, then $x(t)$ satisfies the linear equation

$$\ddot{x} + ((\lambda_+ + q_+(t))\chi_+(t) + (\lambda_- + q_-(t))\chi_-(t))x = 0, \tag{3.6}$$

where $\chi_+(t)$ is the characteristic function of the set $\{t : x(t) > 0\}$ and $\chi_-(t)$ is similarly defined. Using the two-point boundary condition $(T_{\alpha\beta})$, a 'non-degeneracy' condition

$$\dim \ker \left( \frac{d^2}{dt^2} + (\lambda_+ + q_+(t))\chi_+(t) + (\lambda_- + q_-(t))\chi_-(t) \right) = 1 \tag{3.7}$$

holds. See (2.2) of [23]. Such a nondegeneracy condition (3.7) enables one to apply the Implicit Function Theorem to obtain the complete description of $(1.3) + (\mathrm{T}_{\alpha\beta})$, i.e. the corresponding Fučik spectrum is composed of two sequences of Fučik curves. Thus the presence of $q_\pm$ does not change the structure of the spectrum in this case. However, for the periodic problem

(1.3) + (P), the nondegeneracy condition (3.7) does not hold in general. For example, the dimension in the left-hand side of (3.7) is 2 in case that all solutions of (3.6) are $2\pi$-periodic. This may be the main difference between (1.3) + (P) and (1.3) + ($\mathrm{T}_{\alpha\beta}$). Note that for the partial differential equations, the non-degeneracy condition (3.7) does not hold in general. In view of this, the Fučik spectrum problem (1.3) + (P) has some similarity with the higher dimensional problems [21].

By the expression of the Poincaré map (3.2) and the important equality (3.3), the Fučik spectrum $\mathscr{F}(q_\pm)$ of (1.3) + (P) is known in theory.

THEOREM 3.2. *Let* $q_\pm$ *and* $\Theta_{\lambda_\pm}$ *be as before. Then*

$$\mathscr{F}(q_\pm)\backslash(\mathscr{L}^+ \cup \mathscr{L}^-) = \{(\lambda_+, \lambda_-) \in \mathbb{R}^2 : \exists\, n \in \mathbb{N} \text{ and } \theta_0 \in \mathbb{R} \text{ such that (3.5) is satisfied}\}, \tag{3.8}$$

*with the following inclusion in a quadrant*:

$$\mathscr{F}(q_\pm)\backslash(\mathscr{L}^+ \cup \mathscr{L}^-) \subset (\lambda_0^{\mathrm{P}}(q_+), \infty) \times (\lambda_0^{\mathrm{P}}(q_-), \infty). \tag{3.9}$$

Property (3.8) can be proved similarly as in Theorem 3.1, while (3.9) follows from (2.8) and Lemma 3.1.

REMARK 3.2. When the period $T \neq 2\pi$ is considered, one needs only to modify the sets $\rho^{-1}(n)$ by $\rho^{-1}(2n\pi/T)$, and the corresponding Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$ can be defined similarly.

Now we give some geometric properties for the periodic Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$.

LEMMA 3.5. *For each* $n \geqslant 1$, $\mathscr{M}_n$ *and* $\mathscr{N}_n$ *defined by* (3.4) *are curves in the* $(\lambda_+, \lambda_-)$*-plane.*

*Proof.* By a curve, we mean that it intersects every horizontal line $\lambda_- =$ const. (respectively, every vertical line $\lambda_+ =$ const.) at most at one point.

Let us prove that $\mathscr{N}_n$ is a curve in the above sense. Assume that $(\lambda_+, \lambda_-) \in \mathscr{N}_n$. Let $\mu \neq \lambda_+$. For definiteness, assume that $\lambda_+ > \mu$. We need only to prove that

$$\theta(2\pi; \theta_0, \mu, \lambda_-) < \theta(2\pi; \theta_0, \lambda_+, \lambda_-), \qquad \forall\, \theta_0 \in \mathbb{R}, \tag{3.10}$$

because this implies that $N(\mu, \lambda_-) < N(\lambda_+, \lambda_-) = 2n\pi$. As a result, $(\mu, \lambda_-) \notin \mathcal{N}_n$.

By Lemma 2.1, we have $\theta_1(t) := \theta(t; \theta_0, \mu, \lambda_-) \leqslant \theta(t; \theta_0, \lambda_+, \lambda_-) =: \theta_2(t)$ for all $t \in [0, 2\pi]$. If (3.10) does not hold, we would have $\theta_1(t) = \theta_2(t)$, $t \in [0, 2\pi]$. It follows from the equations for $\theta_1(t)$ and $\theta_2(t)$, cf. (2.7), that $(\lambda_+ - \mu)\cos^2\theta_2(t) = 0$ for all $t \in [0, 2\pi]$ satisfying $\cos\theta_2(t) \geqslant 0$. As $\lambda_+ \neq \mu$, we see that $\cos\theta_2(t) \geqslant 0$ implies that $\cos\theta_2(t) = 0$, $t \in [0, 2\pi]$. Since the function $\cos\theta_2(t)$ has only isolated zeros, we know that $\cos\theta_2(t) \leqslant 0$ for all $t \in [0, 2\pi]$. As $\theta_2(t)$ is continuous, there must exist some $n_0 \in \mathbb{Z}$ such that

$$(2n_0 + 1)\pi - \pi/2 \leqslant \theta_2(t) = \theta(t; \theta_0, \lambda_+, \lambda_-) \leqslant (2n_0 + 1)\pi + \pi/2$$

for all $t \in [0, 2\pi]$. Taking $t = 0$ in the above inequality, we have

$$-(2n_0 + 1)\pi - \pi/2 \leqslant -\theta_0 \leqslant -(2n_0 + 1)\pi + \pi/2.$$

These imply that

$$-\pi \leqslant \theta(2\pi; \theta_0, \lambda_+, \lambda_-) - \theta_0 \leqslant \pi.$$

Thus $N(\lambda_+, \lambda_-) \leqslant \pi < 2n\pi$. This contradiction proves the lemma. ∎

By Lemma 3.5 and Theorem 3.1, $\mathcal{M}_n$ and $\mathcal{N}_n$ $(n \in \mathbb{N})$ are called the *nth Fučik curves* of (1.3), while $\mathcal{L}^{\pm}$ are called the *zeroth Fučik curves* of (1.3). It follows from Lemma 3.5 and the monotonicity of the rotation number function in Lemma 2.2 that, for any $n \geqslant 1$, $\mathcal{M}_n$ (respectively, $\mathcal{N}_n$) can be written as

$$\lambda_+ = M_n^+(\lambda_-) \qquad \text{or} \qquad \lambda_- = M_n^-(\lambda_+)$$

(respectively,

$$\lambda_+ = N_n^+(\lambda_-) \qquad \text{or} \qquad \lambda_- = N_n^-(\lambda_+)\,),$$

where $M_n^{\pm}$ and $N_n^{\pm}$ are strictly decreasing.

Remark 3.3. The curve $\mathcal{M}_n$ is 'on the left' of $\mathcal{N}_n$, which means that

$$M_n^+(\lambda_-) \leqslant N_n^+(\lambda_-)$$

when $\lambda_-$ is in the common domain of the functions $M_n^+$ and $N_n^+$. If $\mathcal{M}_n$ and $\mathcal{N}_n$ intersect at some point $(\lambda_+, \lambda_-)$, then all solutions of (1.3) would be $2\pi$-periodic. Moreover, one can give some estimates on the location of these Fučik curves using estimates (2.9) and (2.10) for the rotation number function.

In the following we discuss the asymptotes of the periodic Fučik curves $\mathcal{M}_n$ and $\mathcal{N}_n$. By Theorem 3.2, it is obvious that all curves $\mathcal{M}_n$ and $\mathcal{N}_n$ both have the horizontal and vertical asymptotes, i.e. for each $n \in \mathbb{N}$, all of the following four limits exist:

$$\lim_{\lambda_- \to \infty} M_n^+(\lambda_-) = \xi_n^+, \tag{3.11}$$

$$\lim_{\lambda_+ \to \infty} M_n^-(\lambda_+) = \xi_n^-, \tag{3.12}$$

$$\lim_{\lambda_- \to \infty} N_n^-(\lambda_-) = \eta_n^+, \tag{3.13}$$

$$\lim_{\lambda_+ \to \infty} N_n^-(\lambda_+) = \eta_n^-. \tag{3.14}$$

Thus the functions $M_n^\pm$ map $(\xi_n^\pm, \infty)$ onto $(\xi_n^\mp, \infty)$ and $N_n^\pm$ map $(\eta_n^\pm, \infty)$ onto $(\eta_n^\mp, \infty)$ in a decreasing way.

When $q_\pm \neq 0$ and the boundary condition $(\mathrm{T}_{\alpha\beta})$ are considered, Rynne [23] has recently obtained the asymptotes of the corresponding Fučik curves. It is found that the asymptotes depend upon the boundary conditions $(\mathrm{T}_{\alpha\beta})$ in a delicate way. See Theorem 3.1 and Corollary 3.3 of [23].

For the periodic case, if $q_\pm = 0$, then all of those asymptotes in (3.11)–(3.14) coincide and they are given by $\xi_n^\pm = \eta_n^\pm = (n/2)^2$, cf. (1.4). One sees that it is the $n$th Dirichlet eigenvalue of (1.5) with potential $q = 0$. For $q_\pm \neq 0$, it will be proved in the next theorem that the asymptotes in (3.11)–(3.14) are related with the Dirichlet eigenvalues with suitable choice of potentials. Since most parts of the proof of Theorem 3.1 in [23] also work in this case, we give only the sketch of the proof of this result.

Although the next theorem holds for a more wider class of $q_\pm$, we consider only the case that $q_\pm(t)$ are $2\pi$-periodic and continuous.

Before giving the result, we introduce some notation as in [23]. Let $L^\gamma(a,b)$, $1 \leqslant \gamma \leqslant \infty$, be the Lebesgue spaces with the corresponding norms denoted by $|\cdot|_{\gamma,(a,b)}$. Let $H^k(a,b)$ be the usual Sobolev spaces and $H^k_{2\pi}$ be the Sobolev spaces of $2\pi$-periodic functions. Let $C^0_{2\pi}$ be the space of all continuous $2\pi$-periodic functions with the supremum norm $|\cdot|_\infty$.

Let $n \in \mathbb{N}$ and $(\lambda_+, \lambda_-) \in \mathcal{M}_n \cup \mathcal{N}_n$. We call in the sequel a nonzero $2\pi$-periodic solution $x(t) = x(t; \lambda_+, \lambda_-)$ of (1.3) an *eigenfunction* (with the spectrum $(\lambda_+, \lambda_-)$). As $x(t)$ satisfies piecewise linear equations, $x(t)$ has only simple zeros. As usual, the zeros of $x(t; \lambda_+, \lambda_-)$ are called the *nodes* of $x(t; \lambda_+, \lambda_-)$. An interval $I = (t_1, t_2)$ is called a *positive nodal interval* (respectively, a *negative nodal interval*) of $x(t)$ if $x(t_1) = x(t_2) = 0$ and $x(t) > 0$ on $(t_1, t_2)$ (respectively, if $x(t_1) = x(t_2) = 0$ and $x(t) < 0$ on $(t_1, t_2)$).

The rotation number $\rho(\lambda_+, \lambda_-)$ can be described using the number of nodes of the eigenfunctions:

$$\rho(\lambda_+, \lambda_-) = \tfrac{1}{2}\,\#\{t \in [0, 2\pi) : x(t; \lambda_+, \lambda_-) = 0\}.$$

Thus any eigenfunction $x(t)$ with $(\lambda_+, \lambda_-) \in \mathscr{M}_n \cup \mathscr{N}_n$ has exactly $2n$ nodes in $[0, 2\pi)$ and has exactly $n$ positive nodal intervals and $n$ negative nodal intervals within one period.

Let $q \in \mathscr{P}$. For $t_0 \in \mathbb{R}$, $q_{t_0}(t)$ denotes the translation of $q(t)$: $q_{t_0}(t) = q(t + t_0)$. We use $\lambda_n^{\mathrm{D}}(q)$ to denote the $n$th eigenvalue of (1.5) with the Dirichlet boundary condition

$$x(0) = x(2\pi) = 0. \tag{D}$$

Theorem 3.3. *For any $n \in \mathbb{N}$, there exist $t_n^\pm$, $s_n^\pm \in \mathbb{R}$ such that*

$$\xi_n^\pm = \lambda_n^{\mathrm{D}}(q_{+, t_n^\pm}), \tag{3.15}$$

$$\eta_n^\pm = \lambda_n^{\mathrm{D}}(q_{-, s_n^\pm}). \tag{3.16}$$

*Proof.* Let us prove (3.15) for $\xi_n^+$ because the others are similar. So $\lambda_- \to +\infty$. As in Section 3 of [23], let us take a sequence $(\lambda_+^k, \lambda_-^k) \in \mathscr{M}_n$ such that $\lambda_-^k \nearrow +\infty$ as $k \to \infty$. Thus $\lambda_+^k \searrow \xi_n^+$. Take a sequence of eigenfunctions $x^k(t)$ with the spectra $(\lambda_+^k, \lambda_-^k)$, $k = 1, 2, \ldots$ . Without loss of generality, we assume that each $x^k$ is normalized for the $L^2$ norm, i.e., $|x^k|_2 := |x^k|_{2,(0,2\pi)} = 1$ for all $k$. Each $x^k(t)$ has exactly $n$ positive nodal intervals and $n$ negative nodal intervals within one period. As $\lambda_-^k \to \infty$, the length of negative nodal intervals of $x^k(t)$, which is of order $O((\lambda_-^k)^{-1/2})$, goes to 0 as $k \to \infty$. Meanwhile, the length of positive nodal intervals of $x^k(t)$ is bounded away from 0, [23, see Lemma 3.4]. It can be proved that the sequence $x^k(t)$ is bounded in $H_{2\pi}^1$. Thus $x^k(t)$, if necessary going to a subsequence, converges weakly to some $x(t)$ in $H_{2\pi}^1$ and converges strongly to $x(t)$ in $C_{2\pi}^0$.

Since the length of negative nodal intervals of $x^k(t)$ goes to 0, we may assume, if necessary going to a subsequence, that as $k \to \infty$ these negative nodal intervals shrink to some points $\{t_i\}$, from which only exactly $n$ points are inside $[0, 2\pi)$. So there are exactly $n$ intervals from $\mathbb{R}\backslash\{t_i\}$ within one period. On each interval $P$ of $\mathbb{R}\backslash\{t_i\}$, a standard regularity argument for ordinary differential equations shows that the limiting function $x(t)$ satisfies the limiting equation

$$\ddot{x} + (\xi_n^+ + q_+(t))x = 0 \tag{3.17}$$

on $P$. In fact, the restriction $x|\bar{P}$ of $x(t)$ to $\bar{P}$ is in $H^2(P) \cap H_0^1(P) \cap C^1(\bar{P})$. It can be proved that $x \neq 0$ on each $P$. See p. 99 of [23]. In fact, $x(t)$ is strictly positive on $P$. Note that $x(t)$ is not a classical solution of (3.17) on $\mathbb{R}$ because $x(t)$ is not even $C^1$ on $\mathbb{R}$.

An important fact on $x(t)$ is that if $P = (\gamma, \delta)$ is an interval from $\mathbb{R} \backslash \{t_i\}$, then besides proving the existence of derivatives $\dot{x}(\gamma+)$ and $\dot{x}(\delta-)$, one can prove that they are always nonzero. This can be seen from Lemma 3.7 of [23] and has been neglected in the original proof of Theorem 3.1 of [23].

Finally, let $0 \leqslant t_0 < t_1 < \cdots < t_{n-1} < t_n = t_0 + 2\pi$ be the points from $\{t_i\}$ lying in one period. Then $x(t_i) = 0$ and $x(t)$ satisfies (3.17) for a.e. $t \in (t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$. Since $\lim_{t \to t_i \pm} \dot{x}(t)$ exist and are nonzero, one can then choose nonzero constants $\gamma_i$ so that the following function

$$\tilde{x}(t) = \gamma_i x(t), \qquad t \in [t_{i-1}, t_i], \quad i = 1, 2, \ldots, n,$$

is $C^1$ on $[t_0, t_n]$. So $\tilde{x}(t)$ is a classical nonzero solution of (3.17) on $[t_0, t_n]$. Generally speaking, $\tilde{x}(t)$ has different derivatives at $t_0$ and $t_n = t_0 + 2\pi$. Let $y(t)$ be the function

$$y(t) \equiv \tilde{x}(t + t_0), \qquad t \in [0, 2\pi].$$

Then $y(t)$ is a nonzero solution of the following equation:

$$\ddot{y} + (\xi_n^+ + q_{+,t_0}(t))y = 0, \qquad t \in [0, 2\pi] \tag{3.18}$$

with the Dirichlet boundary condition (D) satisfied. Moreover, $y(t)$ has exactly $n$ zeros within $[0, 2\pi)$. It follows from (3.18) + (D) that $\xi_n^+$ is just the $n$th Dirichlet eigenvalue of (3.18). Let $t_n^+ = t_0$ in this case. The theorem is thus proved. ∎

Remark 3.4. Comparing Theorem 3.3 with Theorem 3.1 and Corollary 3.3 in [23], the asymptotes in Theorem 3.2 are relatively simpler than those for two-point boundary value problems because $x(t)$ satisfies the Dirichlet boundary conditions on all nodal intervals $I$: $x(t) = 0$ for $t \in \partial I$. So part of the proof in [23] can be simplified accordingly.

In the following, we discuss briefly the Fučik spectrum $\mathscr{F}^{\alpha\beta} = \mathscr{F}^{\alpha\beta}(q_\pm)$ of (1.3) with two-point boundary condition $(\mathrm{T}_{\alpha\beta})$, and a relationship between $\mathscr{F}$ and $\mathscr{F}^{\alpha\beta}$ is established. Note that $\mathscr{F}^{\alpha\beta}$ has been studied in [23] in detail.

Observe that $(\lambda_+, \lambda_-)$ is in $\mathscr{F}^{\alpha\beta}$ if and only if $(\lambda_+, \lambda_-)$ satisfies, for some $n \in \mathbb{Z}$, either

$$\theta(2\pi; \alpha, \lambda_+, \lambda_-) = \beta + n\pi \tag{3.19}$$

or

$$\theta(2\pi; \alpha+\pi, \lambda_+, \lambda_-) = (\beta+\pi) + n\pi. \tag{3.20}$$

By Lemma 2.2, all $n$ in (3.19) and in (3.20) are nonnegative. Thus the structure of $\mathscr{F}^{\alpha\beta}$ is clear from the monotonicity of $\theta(2\pi; \theta_0, \lambda_+, \lambda_-)$ with respect to $\lambda_+$, $\lambda_-$, cf. Lemma 2.1.

When $\beta = \alpha$ in $(\mathrm{T}_{\alpha\beta})$, we write, for simplicity, $(\mathrm{T}_{\alpha\alpha})$ and $\mathscr{F}^{\alpha\alpha}$ as $(\mathrm{T}_\alpha)$ and $\mathscr{F}^\alpha$, respectively. Now $(\lambda_+, \lambda_-) \in \mathscr{F}^\alpha$ is determined by either

$$\mathscr{M}_n^\alpha\colon\ \theta(2\pi; \alpha, \lambda_+, \lambda_-) = \alpha + n\pi \tag{3.21}$$

or

$$\mathscr{N}_n^\alpha\colon\ \theta(2\pi; \alpha+\pi, \lambda_+, \lambda_-) = (\alpha+\pi) + n\pi, \tag{3.22}$$

where $n \in \mathbb{Z}^+$.

Suppose that $n \in \mathbb{N}$ is even. Applying properties (2.2) and (2.3), we get from (3.21) or (3.22) that $\rho(\lambda_+, \lambda_-) = n/2 \in \mathbb{N}$. Thus, both the curves $\mathscr{M}_n^\alpha$ and $\mathscr{N}_n^\alpha$ are 'between' the periodic Fučik curves $\mathscr{M}_{n/2}$ and $\mathscr{N}_{n/2}$ in this case. Such a fact is well known in the linear case. Conversely, the periodic Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$ can be recovered from the Fučik curves $\mathscr{M}_{2n}^\alpha(q_{\pm,t_0})$ and $\mathscr{N}_{2n}^\alpha(q_{\pm,t_0})$, $t_0 \in \mathbb{R}$. We state this fact as the following theorem.

Theorem 3.4. *For any $n \in \mathbb{N}$ and any $t_0 \in \mathbb{R}$, the Fučik curves of $\mathscr{M}_{2n}^\alpha(q_{\pm,t_0})$ and $\mathscr{N}_{2n}^\alpha(q_{\pm,t_0})$ in $\mathscr{F}^\alpha(q_{\pm,t_0})$ are between the periodic Fučik curves $\mathscr{M}_n(q_\pm)$ and $\mathscr{N}_n(q_\pm)$. Conversely, for any $(\lambda_+, \lambda_-) \in \mathscr{M}_n(q_\pm) \cup \mathscr{N}_n(q_\pm)$, there exists some $t_0$ such that $(\lambda_+, \lambda_-) \in \mathscr{M}_{2n}^\alpha(q_{\pm,t_0}) \cup \mathscr{N}_{2n}^\alpha(q_{\pm,t_0})$.*

This theorem can be proved using the same trick as in [25], which is essentially based on the simple fact that for any periodic solution $x(t)$ of (1.3), there exists some $t_0$ such that $x_{t_0}(t) \equiv x(t+t_0)$ satisfies the boundary condition $(T_\alpha)$.

## 4. CONCLUDING REMARKS

In Theorem 3.1, besides the trivial Fučik lines $\mathscr{L}^+$ and $\mathscr{L}^-$, we have constructed two sequences of Fučik curves $\mathscr{M}_n$ and $\mathscr{N}_n$ ($n \in \mathbb{N}$) of $\mathscr{F}(q_\pm)$ for general $q_\pm$. However, we do not know if $\mathscr{F}(q_\pm)$ consists of only these curves. It is thus an interesting problem to verify if the converse part of Theorem 3.1 holds. Note that our proof has only exploited the Hamiltonian structure and the positive homogeneity of systems (2.5). In order to verify the converse part of Theorem 3.1, one may need to find some further structure of (2.5), which should be some generalization of linearity in some sense.

As only the Hamiltonian structure and the positive homogeneity of systems (2.5) are used in our proof, the method in this paper also applies to the periodic Fučik spectrum of

$$(p(t)x')' + (\lambda_+ w_+(t) + q_+(t))x_+ + (\lambda_- w_-(t) + q_-(t))x_- = 0, \qquad (4.1)$$

where $p(t)$, $w_\pm(t)$, $q_\pm(t) \in \mathscr{P}$ and $p > 0$, $(w_+, w_-) \succ (0,0)$. A theorem analogous to Theorem 3.1 can be proved for (4.1) + (P).

An abstract form of our result can be stated as the next theorem. Let $\{P_\lambda \colon \lambda \in \mathbb{R}^m\}$ be a family of homeomorphisms of $\mathbb{R}^2$. Suppose that

- $P_\lambda(x) \neq 0$ if $x \neq 0$,
- $P_\lambda$ is area-preserving for each $\lambda$,
- $P_\lambda$ is positively homogeneous for each $\lambda$, i.e., $P_\lambda(kx) = kP_\lambda(x)$ for all $k \geqslant 0$ and all $x \in \mathbb{R}^2$, and
- the induced family of homeomorphisms $\{\Theta_\lambda\}$ of $\mathbb{S}$ is differentiable and is monotone with respect to $\lambda$, where

$$\Theta_\lambda(x) = \frac{P_\lambda(x)}{\|P_\lambda(x)\|}, \qquad x \in \mathbb{R}^2,\ \|x\| = 1.$$

For such a family, one may lift $\Theta_\lambda$ to $\mathbb{R}$ so that the rotation number function $\rho(\lambda)$ is well defined. Then we have the following result.

Theorem 4.1. *Let $n \in \mathbb{Z}$. If $\lambda \in \partial\rho^{-1}(n)$, then $P_\lambda$ has 0 as a parabolic fixed point in the sense that there exists $x_0 \neq 0$ such that $P_\lambda x_0 = x_0$. If, in addition, $P_\lambda$ is homogeneous in the following sense*:

$$P_\lambda(kx) = kP_\lambda(x) \quad \text{for all } k \in \mathbb{R} \text{ and all } x \in \mathbb{R}^2,$$

*then for any odd integer $n \in \mathbb{N}$ and any $\lambda \in \partial\rho^{-1}(n/2)$, $P_\lambda$ has 0 as a parabolic fixed point in the sense that there exists $x_0 \neq 0$ such that $P_\lambda x_0 = -x_0$.*

In a recent work [25], the author of the present paper has partially generalized the rotation number approach to another famous spectrum problem, i.e. the periodic and the antiperiodic eigenvalues of the $p$-Laplacian with periodic potentials:

$$(|x'|^{p-2}x')' + (\lambda + q(t))|x|^{p-2}x = 0, \qquad (4.2)$$

where $p > 1$. A similar result as in this paper has been obtained for eigenvalue problems (4.2) + (P) and (4.2) + (A), i.e. one can use the rotation number function to find two sequences of eigenvalues of (4.2) + (P) and (4.2) + (A) as in the linear case, although we do not know if these represent all periodic and antiperiodic eigenvalues of (4.2).

Following some ideas in [25], for the Fučik spectrum problem of the $p$-Laplacian:

$$(p(t)|x'|^{p-2}x')' + (\lambda_+ w_+(t) + q_+(t))|x_+|^{p-2}x_+ + (\lambda_- w_-(t) + q_-(t))|x_-|^{p-2}x_- = 0$$

with the periodic boundary condition (P), one can also obtain a result similar to Theorem 3.1. For some recent progress of higher dimensional Dirichlet Fučik spectrum problem, see [3, 21].

Finally, it can be expected that the Fučik curves defined in this paper will play an important role in nonresonance problems of the following *nonautonomous* equations:

$$\ddot{x} + f(t, x) = 0, \tag{4.3}$$

where $f(t, x)$ $(\equiv f(t + 2\pi, x))$ satisfies

$$\phi_\pm(t) \leqslant \liminf_{x \to \pm\infty} \frac{f(t, x)}{x} \leqslant \limsup_{x \to \pm\infty} \frac{f(t, x)}{x} \leqslant \Phi_\pm(t). \tag{4.4}$$

Most of existence results of periodic solutions of (4.3) are obtained by assuming that the functions $\phi_\pm(t)$ and $\Phi_\pm(t)$ in (4.4) are between two points in two consecutive Fučik curves $C_n$ and $C_{n+1}$ (see [4]). These results can be explained using the Fučik curves in this paper from a point of view of *nonautonomous equations*. Due to a general result on positively homogeneous operators in [24] (for the linear case, see [9]), the existence of (4.3) + (P) is essentially reduced to verifying the following family of equations:

$$\ddot{x} + \psi_+(t)x_+ + \psi_-(t)x_- = 0 \tag{4.5}$$

has only the trivial $2\pi$-periodic solution for each pair $\psi_\pm \in \mathscr{P}$ with

$$\phi_\pm(t) \leqslant \psi_\pm(t) \leqslant \Phi_\pm(t) \quad \text{for all } t. \tag{4.6}$$

Such a triviality of (4.5) + (P) is equivalent to $(0,0) \notin \mathscr{F}(\psi_\pm)$ for all $\psi_\pm$ satisfying (4.6). By Lemma 2.1 and the characterization of Fučik curves $\mathscr{M}_n(q_\pm)$ and $\mathscr{N}_n(q_\pm)$, one sees that $\mathscr{M}_n(q_\pm)$ and $\mathscr{N}_n(q_\pm)$ ‘decrease’ when $q_\pm$ ‘increase’. Such a fact corresponds to the comparison result of eigenvalues in the linear case. Using this observation, one can obtain the existence of (4.3) + (P) when (0,0) is between $\mathscr{N}_n(\phi_\pm)$ and $\mathscr{M}_{n+1}(\Phi_\pm)$ for some $n$. This condition generalizes the usual ones from the point of view of nonautonomous equations.

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