

Nonuniform Nonresonance of Semilinear Differential Equations¹

Meirong Zhang

*Department of Applied Mathematics, Tsinghua University, Beijing 100084,
People's Republic of China*

E-mail: mzhang@math.tsinghua.edu.cn

Received April 21, 1998; revised February 15, 1999

Consider the Dirichlet problem of nonlinear differential equations with the principal part the p -Laplacian. When the nonlinearity satisfies some semilinearity conditions, the usual nonuniform nonresonance conditions are obtained by comparing nonlinear equations with the classical eigenvalues. In this article, we will introduce some weighted eigenvalues. The nonuniform nonresonance conditions, proved in this article using weighted eigenvalues, will improve the usual ones significantly. © 2000 Academic Press

1. INTRODUCTION

Consider the following Dirichlet problem of nonlinear equation,

$$(\phi_p(x'))' + f(t, x) = 0, \quad \text{a.e. } t \in [0, T], \quad (1.1)$$

$$x(0) = 0 = x(T), \quad (D)$$

where $T > 0$ is fixed, $\phi_p(u) = |u|^{p-2}u$, $1 < p < \infty$, and $f = f(t, u) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function and is semilinear in the following sense: There exist $a(\cdot), b(\cdot) \in L^1(0, T)$ such that

$$a(t) \leq \liminf_{|u| \rightarrow \infty} \frac{f(t, u)}{\phi_p(u)} \leq \limsup_{|u| \rightarrow \infty} \frac{f(t, u)}{\phi_p(u)} \leq b(t) \quad (1.2)$$

uniformly in a.e. $t \in [0, T]$. These equations are the generalization of the second order differential equations where $p = 2$. Most of the existence results of (1.1) + (D) are based on degree theory and variational method, see [2, 4, 5, 8, 10, 11, 14, and the references in 10].

¹ Project supported by the National Natural Science Foundation of China and the Tsinghua University Education Foundation.

A central idea to the existence of problem (1.1) + (D) is to compare Eq. (1.1) with the following eigenvalue problem

$$(\phi_p(x'))' + \lambda \phi_p(x) = 0 \quad (1.3)$$

subject to boundary condition (D). Note that problem (1.3) + (D) has eigenvalues $\lambda_k = (k\pi_p/T)^p$, $k = 1, 2, \dots$, where

$$\pi_p = 2 \int_0^{(p-1)^{1/p}} \frac{ds}{(1-s^p/(p-1))^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)},$$

see [5]. Now the existence results to (1.1) + (D) can be obtained by preventing $f(t, u)$ from the resonant cases $f(t, u) = \lambda_k \phi_p(u) + h(t)$, $k \in \mathbb{N}$.

Let us introduce the following notations. Let $L^\alpha(0, T)$ be the usual Lebesgue space with the corresponding norm $\|\cdot\|_\alpha$, where $1 \leq \alpha \leq \infty$. Let $W_\alpha = W_0^{1,\alpha}(0, T)$ be the usual Sobolev space. For $x \in L^1(0, T)$, the mean value of $x(\cdot)$ is $\bar{x} = (1/T) \int_0^T x(t) dt$. For $w_1, w_2 \in L^1(0, T)$, we write $w_1 < w_2$ if $w_1(t) \leq w_2(t)$ for a.e. $t \in [0, T]$ and $\bar{w}_1 < \bar{w}_2$.

After a delicate analysis, del Pino, Elgueta and Manásevich proved in [5] the existence to problem (1.1) + (D) under the following conditions: There exists $k \in \mathbb{N}$ such that

$$\lambda_{k-1} < a \leq b < \lambda_k, \quad (U)_k$$

where λ_0 means $-\infty$. We call condition $(U)_k$ the k th nonuniform non-resonance condition.

However, these nonuniform nonresonance conditions have some disadvantages. The first one is that conditions $(U)_k$ have no persistence, which means that when $a(t)$ and $b(t)$ have small perturbations, $(U)_k$ are no longer satisfied. Actually, $a(t) - \varepsilon$ and $b(t) + \varepsilon$ may not satisfy $(U)_k$ for $\varepsilon > 0$. Due to degree theory and Property P introduced by Fonda and Habets [6] and Habets and Metzen [7] and to a very general perturbation result concerning with positively homogeneous operators in Banach spaces which is given by the present author in [15], the nonresonance problem has some persistence, which means that if $a(\cdot)$ and $b(\cdot)$ satisfy $(U)_k$, then the existence to (1.1) + (D) can also be guaranteed when $f(t, u)$ only satisfies

$$a(t) - \varepsilon_0 \leq \liminf_{|u| \rightarrow \infty} \frac{f(t, u)}{\phi_p(u)} \leq \limsup_{|u| \rightarrow \infty} \frac{f(t, u)}{\phi_p(u)} \leq b(t) + \varepsilon_0$$

uniformly in a.e. $t \in [0, T]$, where $\varepsilon_0 > 0$ is small.

The second one is that conditions $(U)_k$ naturally imply that $a(\cdot)$ and $b(\cdot)$ are in $L^\infty(0, T)$. Such an implication is not consistent with degree theory, which can deal with L^1 -Carathéodory functions.

The third one is that condition $(U)_k$ with $k > 1$ is not even applicable to the following simple equation

$$x'' + \mu(1 + \cos t)x = h(t, x), \quad t \in [0, 2\pi], \quad (1.4)$$

where h satisfies $\lim_{|x| \rightarrow \infty} h(t, x)/x = 0$. From nonuniform nonresonance condition $(U)_1$, one can obtain the existence of solutions of (1.4) satisfying the boundary condition

$$x(0) = 0 = x(2\pi) \quad (1.5)$$

when $\mu \leq 1/8$. However, as $a(t) = b(t) = \mu(1 + \cos t)$ has zeros in t , we know that $(U)_k$ with $k > 1$ give no any existence result to problem (1.4) + (1.5).

These observations motivate this article. We think that the above disadvantages result from an unsuitable choice of referring equations for Eq. (1.1). Here we recommend one may compare Eq. (1.1) with the following weighted eigenvalue problem

$$(\phi_p(x'))' + \lambda w(t) \phi_p(x) = 0 \quad (1.6)$$

subject to (D). Such an idea is not complicated in theory, but is not developed adequately in the literature.

When the weight $w(t)$ satisfies $w > 0$, we will use some equations on the circle to study eigenvalue problem (1.6) + (D) in Section 2. It is proved that problem (1.6) + (D) has a sequence of eigenvalues

$$0 < \lambda_1(w) < \lambda_2(w) < \dots < \lambda_k(w) < \dots$$

Based on some comparison results for weighted eigenvalues $\lambda_k(w)$ on weights w , we will prove in Section 3 the following existence results of (1.1) + (D) when $a(\cdot)$ and $b(\cdot)$ in (1.2) satisfy $b \geq a > 0$ and, for some $k \in \mathbb{N}$,

$$\lambda_{k-1}(a) < 1 \quad \text{and} \quad \lambda_k(b) > 1, \quad (N)_k$$

where $\lambda_0(a)$ means $-\infty$. Using the comparison results, one sees $(N)_k$ improve $(U)_k$. Meanwhile, conditions $(N)_k$ have also overcome the disadvantages described above. For example, one can obtain the existence of

(1.4) + (1.5) for μ in a sequence of intervals. Moreover, the nonresonance condition $(U)_1$ for (1.4) + (1.5), $\mu \leq 1/8$, can also be improved as

$$\mu < \left(\frac{6}{\pi^4 + 3\pi^2 - 96} \right)^{1/2} = 0.4398\dots$$

This example shows that the usual nonuniform nonresonance conditions can be improved significantly using weighted eigenvalues.

In Section 4, we will give some lower bounds for the first weighted eigenvalue $\lambda_1(w)$. To this end, we have worked out some best Sobolev constants concerning the spaces W_p , which are known in [13] when $p = 2$.

2. WEIGHTED EIGENVALUE PROBLEMS

Let $1 < p < \infty$ be fixed. For any given $w \in L^1(0, T)$ with $w > 0$, we consider the following weighted eigenvalue problem,

$$(\phi_p(x'))' + \lambda w(t) \phi_p(x) = 0, \quad t \in [0, T], \quad (2.1)$$

subject to the Dirichlet boundary condition (D). As usual, λ is called an eigenvalue of (2.1) + (D) if problem (2.1) + (D) has nonzero solutions. Note that all eigenvalues λ are positive because $w > 0$.

As in [5] and [9], we introduce some functions which may be called the p -cosine and the p -sine. Consider the equation

$$(\phi_p(x'))' + \phi_p(x) = 0. \quad (2.2)$$

Set $\phi_p(x') = -y$ in (2.2). Namely, $x' = -\phi_q(y)$, where $q = p/(p-1)$. Then (2.2) is equivalent to the following system:

$$x' = -\phi_q(y), \quad y' = \phi_p(x). \quad (2.3)$$

Note that (2.3) is a Hamiltonian system with the Hamiltonian $H(x, y) = p^{-1}|x|^p + q^{-1}|y|^q$. For any (x_0, y_0) , the initial value problem of (2.3) with $(x(0), y(0)) = (x_0, y_0)$ has a unique solution $(x(t), y(t))$ which is well defined on the whole line \mathbb{R} . A phase portrait analysis shows that all solutions of (2.3) are periodic with the same period $2\pi_p$. In particular, let $(x, y) = (C_p(t), S_p(t))$ be the unique solution of (2.3) satisfying $(C_p(0), S_p(0)) = (1, 0)$. The functions $C_p(t)$ and $S_p(t)$ are much similar to cosine and sine. We list some properties of the functions $C_p(t)$ and $S_p(t)$.

LEMMA 2.1. *The functions $C_p(t)$ and $S_p(t)$ have the following properties:*

- (1) *Both $C_p(t)$ and $S_p(t)$ are $2\pi_p$ -periodic;*
- (2) *$C_p(t) = 0$ iff $t = \pi_p/2 + n\pi_p$, $n \in \mathbb{Z}$, and $S_p(t) = 0$ iff $t = n\pi_p$, $n \in \mathbb{Z}$;*
- (3) *$C'_p(t) = -\phi_q(S_p(t))$ and $S'_p(t) = \phi_p(C_p(t))$; and*
- (4) *$p^{-1}|C_p(t)|^p + q^{-1}|S_p(t)|^q \equiv p^{-1}$.*

Now we consider eigenvalue problem (2.1) + (D). As explained before, assume that $\lambda > 0$. Set $\phi_p(x') = -\lambda^{1/q}y$ in (2.1). Then $x' = \phi_q(-\lambda^{1/q}y) = -\lambda^{1/p}\phi_q(y)$. Now (2.1) is equivalent to the following system:

$$x' = -\lambda^{1/p}\phi_q(y), \quad y' = \lambda^{1/p}w(t)\phi_p(x). \quad (2.4)$$

As usual, we introduce the ‘‘polar coordinates’’. Let $x = r^{1/p}C_p(\theta)$ and $y = r^{1/q}S_p(\theta)$. Then (2.4) is equivalent to

$$r' = p\lambda^{1/p}(w(t) - 1)\phi_p(C_p(\theta))\phi_q(S_p(\theta))r =: R(t, \theta, r; \lambda), \quad (2.5)$$

$$\theta' = p\lambda^{1/p}(p^{-1}w(t)|C_p(\theta)|^p + q^{-1}|S_p(\theta)|^q) =: \Theta(t, \theta; \lambda). \quad (2.6)$$

Note that $\Theta(t, \theta; \lambda)$ is independent of r and $R(t, \theta, r; \lambda)$ is homogeneous in r . Moreover, $\Theta(t, \theta; \lambda)$ is $2\pi_p$ -periodic in θ . Namely, Eq. (2.6) is an equation on the circle $\mathbb{S}^1 = \mathbb{R}/2\pi_p\mathbb{Z}$. In order to consider eigenvalues, we need only analyze Eq. (2.6). For any given $\theta_0 \in \mathbb{R}$, Eq. (2.6) has a unique solution $\theta = \theta(t; \theta_0, \lambda)$ satisfying the initial condition: $\theta(0; \theta_0, \lambda) = \theta_0$. As $0 \leq \Theta(t, \theta; \lambda) \leq \lambda^{1/p} \max\{w(t), 1\}$ for all $t \in [0, T]$, $\theta(t; \theta_0, \lambda)$ is well defined for all $t \in [0, T]$.

Note that the vector field $\Theta(t, \theta; \lambda)$ is increasing when λ increases. We have the following monotonicity on solutions of (2.6).

LEMMA 2.2. *If $\lambda_1 > \lambda_2 > 0$, then $\theta(t; \theta_0, \lambda_1) \geq \theta(t; \theta_0, \lambda_2)$ for all $t \in [0, T]$ and all θ_0 . Moreover, $\theta(T; \theta_0, \lambda_1) > \theta(T; \theta_0, \lambda_2)$ for all θ_0 .*

Proof. Assume that $\lambda_1 > \lambda_2 > 0$. Let $\theta_i(t) = \theta(t; \theta_0, \lambda_i)$ and $\theta(t) = \theta_1(t) - \theta_2(t)$. Then $\theta(0) = 0$. By (2.6) we have, for a.e. t ,

$$\begin{aligned} \frac{d\theta(t)}{dt} &= \Theta(t, \theta_1; \lambda_1) - \Theta(t, \theta_2; \lambda_2) \\ &= (\Theta(t, \theta_1; \lambda_2) - \Theta(t, \theta_2; \lambda_2)) + (\Theta(t, \theta_1; \lambda_1) - \Theta(t, \theta_1; \lambda_2)) \\ &=: a(t)\theta(t) + b(t), \end{aligned}$$

where

$$a(t) = \frac{\partial \Theta}{\partial \theta}(t, \xi; \lambda_2) \quad \text{for some } \xi = \xi(t) \in [\theta_1(t), \theta_2(t)],$$

$$b(t) = p(\lambda_1^{1/p} - \lambda_2^{1/p})(p^{-1}w(t) |C_p(\theta_1(t))|^p + q^{-1} |S_p(\theta_1(t))|^q).$$

Thus

$$\theta(t) = \int_0^t b(s) \exp\left(\int_s^t a(\tau) d\tau\right) ds. \quad (2.7)$$

As $b(s) \geq 0$, $\theta(t) \geq 0$ for all $t \in [0, T]$. Now we want to prove $\theta(T) > 0$. Otherwise, if $\theta(T) = 0$, it follows from (2.7) that

$$\int_0^T b(s) \exp\left(\int_s^T a(\tau) d\tau\right) ds = 0.$$

Thus $b(t) = 0$ for a.e. t , i.e.,

$$p^{-1}w(t) |C_p(\theta_1(t))|^p + q^{-1} |S_p(\theta_1(t))|^q = 0 \quad \text{a.e. } t \in [0, T]. \quad (2.8)$$

As a result, $\Theta(t, \theta_1(t); \lambda_1) = 0$ for a.e. $t \in [0, T]$ and $\theta_1(t) \equiv \theta_0$ for all $t \in [0, T]$. Consequently, we get from (2.8) that

$$p^{-1}w(t) |C_p(\theta_0)|^p + q^{-1} |S_p(\theta_0)|^q = 0 \quad \text{a.e. } t \in [0, T].$$

As $w > 0$, we can get $C_p(\theta_0) = S_p(\theta_0) = 0$, a contradiction to Lemma 2.1(4). ■

Now we prove the existence of eigenvalues of (2.1) + (D).

THEOREM 2.1. *Eigenvalue problem (2.1) + (D) has a sequence of eigenvalues:*

$$0 < \lambda_1(w) < \lambda_2(w) < \dots < \lambda_k(w) < \dots$$

Proof. Note that if $x(t)$ is a solution of (2.1), then so does $mx(t)$ for any $m \in \mathbb{R}$. In order to consider eigenvalue problem (2.1) + (D), we need only consider the solution $(x(t), y(t))$ of (2.4) satisfying $(x(0), y(0)) = (0, 1)$. Now $\lambda (> 0)$ is an eigenvalue of problem (2.1) + (D) iff $x(T) = 0$. In the polar coordinates $x = r^{1/p}C_p(\theta)$ and $y = r^{1/q}S_p(\theta)$, $x(t) \equiv r(t)^{1/p}C_p(\theta(t; \pi_p/2, \lambda))$. Thus $x(T) = 0$ is equivalent to

$$\theta(T; \pi_p/2, \lambda) = \pi_p/2 + k\pi_p \quad \text{for some } k \in \mathbb{Z}. \quad (2.9)$$

Note that

$$\lambda^{1/p} w_-(t) \leq \Theta(t, \theta; \lambda) \leq \lambda^{1/p} w_+(t)$$

for all t , where $w_-(t) = \min\{1, w(t)\}$ and $w_+(t) = \max\{1, w(t)\}$. Note that $w_- \succ 0$ because $w \succ 0$. Now it follows from (2.6) that

$$\theta_0 + \lambda^{1/p} \int_0^T w_-(t) dt \leq \theta(T; \theta_0, \lambda) \leq \theta_0 + \lambda^{1/p} \int_0^T w_+(t) dt \quad (2.10)$$

for all θ_0 and all $\lambda > 0$. If Eq. (2.9) has a solution $\lambda > 0$, it is necessary that $k \in \mathbb{N}$.

By (2.10),

$$\lim_{\lambda \rightarrow 0^+} \theta(T; \pi_p/2, \lambda) = \pi_p/2,$$

$$\lim_{\lambda \rightarrow +\infty} \theta(T; \pi_p/2, \lambda) = +\infty.$$

By Lemma 2.2, the function $\theta(T; \pi_p/2, \lambda)$ is strictly increasing when λ increases. Therefore, for any given $k \in \mathbb{N}$, Eq. (2.9) has a unique solution $\lambda = \lambda_k > 0$, which gives an eigenvalue of problem (2.1) + (D) and is denoted by $\lambda_k(w)$. ■

Remark 2.1. (1) When $w(t) \equiv 1$, $\Theta(t, \theta; \lambda) \equiv \lambda^{1/p}$. Thus $\theta(T; \theta_0, \lambda) = \theta_0 + \lambda^{1/p} T$. As a result, $\lambda_k(1) = (k\pi_p/T)^p$, $k = 1, 2, \dots$

(2) For general $w(t)$, we get from (2.9) and (2.10) the following estimates for $\lambda_k(w)$:

$$\left(\frac{k\pi_p}{T\bar{w}_+} \right)^p \leq \lambda_k(w) \leq \left(\frac{k\pi_p}{T\bar{w}_-} \right)^p, \quad \forall k \in \mathbb{N}. \quad (2.11)$$

Now we give some comparison results for weighted eigenvalues $\lambda_k(w)$ on weights w . The proof is based on the following monotonicity of solutions $\theta(t; \theta_0, \lambda, w)$ of (2.6) with respect to w . Here we write $\theta(t; \theta_0, \lambda)$ as $\theta(t; \theta_0, \lambda, w)$ to emphasize the dependence of solutions upon weights.

LEMMA 2.3. *Let $\lambda > 0$ and $w_i \in L^1(0, T)$ with $w_i \succ 0$. If $w_1 \succ w_2$, then $\theta(T; \theta_0, \lambda, w_1) > \theta(T; \theta_0, \lambda, w_2)$ for all θ_0 .*

Proof. As in the proof of Lemma 2.2, let $\theta_i(t) = \theta(t; \theta_0, \lambda, w_i)$ and $\theta(t) = \theta_1(t) - \theta_2(t)$. Then $\theta(t) \geq 0$ for all $t \in [0, T]$. If $\theta(T) = 0$, the equality corresponding to (2.8) is

$$(w_1(t) - w_2(t)) |C_p(\theta_1(t))|^p = 0 \quad \text{a.e. } t \in [0, T]. \quad (2.12)$$

In the following, it will be proved that the function $C_p(\theta_1(t))$ has only isolated zeros. Thus (2.12) implies that $w_1(t) = w_2(t)$ for a.e. $t \in [0, T]$, a contradiction to the assumption $w_1 \succ w_2$.

Now we prove that $C_p(\theta_1(t))$ has only isolated zeros. Note that the function $C_p(\vartheta)$ has only isolated zeros. As $\theta_1(t)$ satisfies, for a.e. $t \in [0, T]$,

$$\frac{d\theta_1(t)}{dt} = p\lambda^{1/p}(p^{-1} w_1(t) |C_p(\theta_1(t))|^p + q^{-1} |S_p(\theta_1(t))|^q), \quad (2.13)$$

the function $\theta_1(t)$ is nondecreasing in $t \in [0, T]$. If the function $C_p(\theta_1(t))$ has non-isolated zeros in $[0, T]$, there must be some interval $I_0 = [t_1, t_2] \subset [0, T]$ such that $\theta_1(t)$ is constant on I_0 . Assume that $\theta_1(t) \equiv \vartheta_0$ on I_0 . Then $C_p(\vartheta_0) = 0$ and $S_p(\vartheta_0) \neq 0$. It follows from (2.13) that, for a.e. $t \in I_0$,

$$\frac{d\theta_1(t)}{dt} = pq^{-1} \lambda^{1/p} |S_p(\vartheta_0)|^q > 0,$$

i.e., $\theta_1(t)$ is strictly increasing in $t \in I_0$. This is a contradiction to the fact $\theta_1(t) \equiv \vartheta_0$ on I_0 . The assertion is thus proved. ■

THEOREM 2.2. *Assume that $w_i \in L^1(0, T)$ with $w_i \succ 0$. If $w_1 \succ w_2$, then $\lambda_k(w_1) < \lambda_k(w_2)$ for all $k \in \mathbb{N}$.*

Proof. Assume that $w_1 \succ w_2 \succ 0$. By Lemma 2.3 we know that

$$\theta(T; \pi_p/2, \lambda, w_1) > \theta(T; \pi_p/2, \lambda, w_2), \quad \forall \lambda > 0. \quad (2.14)$$

By Theorem 2.1, for any $k \in \mathbb{N}$, the eigenvalues $\mu_i := \lambda_k(w_i) > 0$ are determined by

$$\theta(T; \pi_p/2, \mu_i, w_i) = \pi_p/2 + k\pi_p, \quad i = 1, 2. \quad (2.15)$$

Now the inequality $\mu_1 < \mu_2$ simply follows from (2.14) and (2.15). ■

3. NONRESONANCE RESULTS

In this section we will give some existence results of problem (1.1) + (D) under the semilinearity condition (1.2).

Let us first consider nonresonance results below the first eigenvalues. In this case we can consider differential equations with damping term

$$(\phi_p(x'))' + g(x) x' + f(t, x) = 0, \quad (3.1)$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

THEOREM 3.1. *Assume that the function $f(t, x)$ in (3.1) satisfies, for some $b(\cdot) \in L^1(0, T)$ with $b \succ 0$,*

$$\limsup_{|u| \rightarrow \infty} \frac{f(t, u)}{\phi_p(u)} \leq b(t) \quad (3.2)$$

uniformly in a.e. $t \in [0, T]$. If

$$\lambda_1(b) > 1, \quad (3.3)$$

then problem (3.1) + (D) has at least one solution.

Proof. The proof is based on coincidence degree, cf. [12]. We will deform Eq. (3.1) to the following equation

$$(\phi_p(x'))' + b(t) \phi_p(x) = 0.$$

This leads to the following homotopic equation

$$(\phi_p(x'))' + F(t, x, x'; \tau) = 0, \quad (\tau \in [0, 1]), \quad (\text{H})_\tau$$

where

$$F(t, x, x'; \tau) = \tau g(x) x' + \tau f(t, x) + (1 - \tau) b(t) \phi_p(x).$$

As in [14], problem $(\text{H})_\tau + (\text{D})$ is equivalent to a fixed point equation in the space $X = \{u: [0, T] \rightarrow \mathbb{R} \text{ is } C^1 \text{ on } [0, T] \text{ and satisfies } u(0) = u(T) = 0\}$ with the C^1 -norm:

$$x = \mathcal{M}_\tau(x), \quad x \in X. \quad (3.4)$$

We will not work out the detailed formula of the operator \mathcal{M}_τ and refer the reader to [14]. By assumption (3.3), when $\tau = 0$, problem $(\text{H})_\tau + (\text{D})$ has only the trivial solution $x = 0$. Namely, Eq. (3.4) with $\tau = 0$ has only the trivial solution. As the operator \mathcal{M}_0 is odd in $x \in X$,

$$\deg(\mathcal{M}_0, X, 0) = \text{odd} \neq 0.$$

As a result, the theorem follows if *a priori* bounds for all solutions of $(\text{H})_\tau + (\text{D})$ can be found.

The following estimates are conventional. Let $\varepsilon_0 > 0$ be sufficiently small. By (3.2), there is $\psi \in L^1(0, T)$ with $\psi \geq 0$ such that

$$xf(t, x) \leq (b(t) + \varepsilon_0) |x|^p + \psi(t)$$

for all $x \in \mathbb{R}$ and a.e. $t \in [0, T]$. Thus

$$x(\tau f(t, x) + (1 - \tau) b(t) \phi_p(x)) \leq (b(t) + \varepsilon_0) |x|^p + \psi(t)$$

for all $x \in \mathbb{R}$, a.e. $t \in [0, T]$, and all $\tau \in [0, 1]$. Let now $x(\cdot)$ be a solution of $(H)_\tau + (D)$ for some $\tau \in [0, 1]$. On the one hand,

$$\int_0^T x(\phi_p(x'))' dt = - \int_0^T \phi_p(x') x' dt = - \|x'\|_p^p, \quad \int_0^T \tau x g(x) x' dt = 0.$$

On the other hand, we observe that $\lambda_1(w)$ has the following characterization:

$$\lambda_1(w) = \inf_{u \in W_p \setminus \{0\}} \frac{\int_0^T |u'|^p dt}{\int_0^T w(t) |u|^p dt}. \quad (3.5)$$

Thus

$$\begin{aligned} \int_0^T x(\tau f(t, x) + (1 - \tau) b(t) \phi_p(x)) dt &\leq \int_0^T ((b(t) + \varepsilon_0) |x|^p + \psi(t)) dt \\ &\leq (1/\lambda_1(b) + \varepsilon_0/\lambda_1(1)) \|x'\|_p^p + \|\psi\|_1. \end{aligned}$$

Now we get from $(H)_\tau$ that

$$\|x'\|_p^p \leq (1/\lambda_1(b) + \varepsilon_0/\lambda_1(1)) \|x'\|_p^p + \|\psi\|_1.$$

Thus, if $\varepsilon_0 < \lambda_1(1)(1 - 1/\lambda_1(b))$, then

$$\|x'\|_p^p \leq \frac{\|\psi\|_1}{1 - 1/\lambda_1(b) - \varepsilon_0/\lambda_1(1)} =: C_0. \quad (3.6)$$

As $x(0) = 0 = x(T)$, there is some $C_1 > 0$ such that

$$\|x\|_\infty \leq C_1. \quad (3.7)$$

Since $f(t, u)$ is an L^1 -Carathéodory function, (3.7) implies there is some $\varphi \in L^1(0, T)$ with $\varphi \geq 0$ such that

$$|\tau f(t, x(t)) + (1 - \tau) b(t) \phi_p(x(t))| \leq \varphi(t) \quad (3.8)$$

for all t, τ and all solutions $x(\cdot)$ of $(H)_\tau + (D)$. As $x(0) = x(T) = 0$, $x'(t_*) = 0$ for some $t_* \in [0, T]$. Now it follows from $(H)_\tau$, (3.6) and (3.8) that, for all $t \in [0, T]$,

$$\begin{aligned} |\phi_p(x'(t))| &= \left| \int_{t_*}^t (\tau g(x(s)) x'(s) + \tau f(s, x(s)) + (1 - \tau) b(s) \phi_p(x(s))) ds \right| \\ &\leq G_0 \|x'\|_1 + \|\varphi\|_1 \leq G_0 T^{1/q} \|x'\|_p + \|\varphi\|_1 \\ &\leq G_0 T^{1/q} C_0^{1/p} + \|\varphi\|_1 =: C_2, \end{aligned}$$

where $G_0 = \max_{|u| \leq C_1} |g(u)|$. Thus $|x'(t)| \leq \phi_q(C_2)$ for all t . Hence all solutions of $(H)_\tau + (D)$ are *a priori* bounded in X and the theorem follows. ■

Remark 3.1. We remark here that weighted eigenvalues are also useful in the nonresonance problem of positive periodic solutions of differential equations with singularities like

$$x'' + g(x) x' + f(t, x) = cx^{-\gamma}, \quad (3.9)$$

where $c > 0$, $\gamma \geq 1$, and $f(t, x)$ satisfies the semilinearity condition: There exist $a(\cdot)$ and $b(\cdot)$ such that

$$a(t) \leq \liminf_{x \rightarrow +\infty} \frac{f(t, x)}{x} \leq \limsup_{x \rightarrow +\infty} \frac{f(t, x)}{x} \leq b(t).$$

In [16], the existence of positive T -periodic solutions of (3.9) is proved when $a(\cdot)$ satisfies $\bar{a} > 0$ and $b(\cdot)$ satisfies the condition (M_D) in [16]. In the present notation, the condition (M_D) is equivalent to

$$\min_{s \in \mathbb{R}} \lambda_1(b(\cdot + s)) > 1.$$

Now we give the nonresonance results of (1.1) which correspond to conditions $(U)_k$ with $k > 1$.

THEOREM 3.3. *Let $b \geq a > 0$ in (1.2). If there exists an integer $k \geq 2$ such that*

$$\lambda_{k-1}(a) < 1 \quad \text{and} \quad \lambda_k(b) > 1, \quad (N)_k$$

then problem (1.1) + (D) has at least one solution.

Proof. Let us explain what $(N)_k$ does mean. For any $c \in L^1(0, T)$ satisfying

$$a(t) \leq c(t) \leq b(t) \quad \text{a.e. } t \in [0, T], \quad (3.10)$$

we assert that the following equation

$$(\phi_p(x'))' + c(t)\phi_p(x) = 0 \quad (3.11)$$

has only the trivial solution satisfying (D). In fact, $c \geq a$ implies that $\lambda_{k-1}(c) \leq \lambda_{k-1}(a) < 1$ by Theorem 2.2. Similarly, $c \leq b$ implies $\lambda_k(c) \geq \lambda_k(b) > 1$. Thus $\lambda_n(c) \neq 1$ for all n . Consequently, (3.11) + (D) has only the trivial solution.

When $p = 2$, the above assertion means that the pair $\{a, b\}$ satisfies Property P in [6], [7] and [15]. As a result, problem (1.1) + (D) has at least one solution, cf. Theorem 5.1 in [15]. For general case $p \neq 2$, one can also introduce Property P . Here we omit the details and refer the reader to [5] and [15]. ■

Remark 3.2. (1) We remark here that conditions $(U)_k$ imply $(N)_k$ correspondingly. In fact, by Theorem 2.2, $a > A_{k-1} := \lambda_{k-1} = ((k-1)\pi_p/T)^p$ implies that $\lambda_n(a) < \lambda_n(1)/A_{k-1} = (n/(k-1))^p$. In particular, $\lambda_{k-1}(a) < 1$. Similarly, $b < \lambda_k$ implies that $\lambda_k(b) > 1$.

(2) Conditions $(N)_k$ have overcome the disadvantages described in Section 1. For example, $(N)_k$ are persistent under small perturbations of $a(\cdot)$ and $b(\cdot)$. The reason is as follows. From Eq. (2.6), it can be proved that the solutions $\theta(t; \theta_0, \lambda) = \theta(t; \theta_0, \lambda, w)$ are continuously dependent on $w(\cdot)$ (with the L^1 -norm). Thus $\lambda_k(w)$ are also continuous in $w(\cdot)$ by (2.9).

4. ESTIMATES AND EXAMPLES

We will give some lower bounds for the first eigenvalue $\lambda_1(w)$ of problem (2.1) + (D). Let $1 < p < \infty$ be fixed. For any $1 \leq \alpha \leq \infty$, let $K(\alpha, p)$ be the best Sobolev constant in the following inequality:

$$C \|x\|_\alpha^p \leq \|x'\|_p^p, \quad \forall x \in W_p,$$

i.e.,

$$K(\alpha, p) = \inf_{x \in W_p \setminus \{0\}} \frac{\|x'\|_p^p}{\|x\|_\alpha^p}.$$

Note that $K(p, p)$ is the first eigenvalue $\lambda_1(1) = (\pi_p/T)^p$ of (1.3) + (D). When $p = 2$, the constants $K(\alpha, 2)$ are known, see Talenti [13]. Now we will give the explicit formula of $K(\alpha, p)$.

THEOREM 4.1.

$K(\alpha, p) =$

$$\begin{cases} \frac{2^p(p-1)(p\alpha+p-\alpha)^{p/\alpha-1} \left(\frac{\Gamma(1/\alpha) \Gamma(1-1/p)}{\Gamma(1+1/\alpha-1/p)} \right)^p, & \text{if } 1 \leq \alpha < \infty \\ \frac{2^p}{T^{p-1}}, & \text{if } \alpha = \infty. \end{cases} \quad (4.1)$$

Proof. Let us consider the case $1 < \alpha < \infty$. As usual, consider the following functional

$$J[x] = \|x'\|_p^p - \mu \|x\|_\alpha^p = \int_0^T |x'(t)|^p dt - \mu \left(\int_0^T |x(t)|^\alpha dt \right)^{p/\alpha}, \quad x \in W_p,$$

where $\mu \in \mathbb{R}$ is a constant. For any $x, y \in W_p$ and $\varepsilon \in \mathbb{R}$,

$$\begin{aligned} \int |x' + \varepsilon y'|^p &= \int |x'|^p + \varepsilon p \int \phi_p(x') y' + o(\varepsilon) \\ &= \int |x'|^p - \varepsilon p \int (\phi_p(x'))' y + o(\varepsilon), \\ \left(\int |x + \varepsilon y|^\alpha \right)^{p/\alpha} &= \left(\int |x|^\alpha \right)^{p/\alpha} + \varepsilon p \left(\int |x|^\alpha \right)^{p/\alpha-1} \left(\int \phi_\alpha(x) y \right) + o(\varepsilon). \end{aligned}$$

Thus $K(\alpha, p)$ is the first eigenvalue μ_1 of

$$(\phi_p(x'))' + \mu \left(\int_0^T |x|^\alpha dt \right)^{p/\alpha-1} \phi_\alpha(x) = 0 \quad (4.2)$$

subject to (D).

Let $x \in W_p \setminus \{0\}$ be an eigenfunction corresponding to the first eigenvalue μ_1 of (4.2) + (D). Then we can assume that $x'(0) = 1$. Moreover, $x(t) > 0$ on $(0, T)$, $x'(T/2) = 0$, and $x(T-t) \equiv x(t)$ on $[0, T]$. Set

$$M = \left(\int_0^T x(t)^\alpha dt \right)^{p/\alpha-1}. \quad (4.3)$$

Then $x(t)$ satisfies

$$(\phi_p(x'))' + \mu_1 M \phi_\alpha(x) = 0. \quad (4.4)$$

As in Section 2, Eq. (4.4) is equivalent to a Hamiltonian system. As a result,

$$\frac{(x')^p}{q} + \mu_1 M \frac{x^\alpha}{\alpha} \equiv \frac{1}{q}.$$

Thus

$$x(T/2) = \left(\frac{\alpha}{\mu_1 q M} \right)^{1/\alpha} =: A \quad (4.5)$$

and

$$\frac{dx}{dt} = x' = \left(1 - \frac{\mu_1 q M}{\alpha} x^\alpha \right)^{1/p} = (1 - (x/A)^\alpha)^{1/p}, \quad t \in [0, T/2].$$

Consequently,

$$\frac{T}{2} = \int_0^{T/2} dt = \int_0^A \frac{dx}{(1 - (x/A)^\alpha)^{1/p}} = A \int_0^1 \frac{dx}{(1 - x^\alpha)^{1/p}} = \frac{A}{\alpha} \frac{\Gamma(1/\alpha) \Gamma(1 - 1/p)}{\Gamma(1 + 1/\alpha - 1/p)}. \quad (4.6)$$

By (4.3),

$$\begin{aligned} M &= \left(\int_0^T x(t)^\alpha dt \right)^{p/\alpha - 1} = \left(2 \int_0^{T/2} x(t)^\alpha dt \right)^{p/\alpha - 1} \\ &= \left(2 \int_0^A \frac{x^\alpha dx}{(1 - (x/A)^\alpha)^{1/p}} \right)^{p/\alpha - 1} = \left(\frac{2A^{\alpha+1}}{\alpha} \frac{\Gamma(1 + 1/\alpha) \Gamma(1 - 1/p)}{\Gamma(2 + 1/\alpha - 1/p)} \right)^{p/\alpha - 1}. \end{aligned} \quad (4.7)$$

By (4.5), (4.6), and (4.7), we can get (4.1) for the case $1 < \alpha < \infty$. For the cases $\alpha = 1$ and $\alpha = \infty$, they can be proved by a limiting procedure. ■

THEOREM 4.2. *Suppose that $w \in L^\alpha(0, T)$ for some $1 \leq \alpha \leq \infty$ and $w > 0$. Then*

$$\lambda_1(w) \geq \frac{K(p\alpha^*, p)}{\|w\|_\alpha} \quad (\alpha^* = \alpha/(\alpha - 1)) \quad (4.8)$$

Proof. For any $u \in W_p$,

$$\int_0^T w |u|^p dt \leq \|w\|_\alpha \| |u|^p \|_{\alpha^*} = \|w\|_\alpha \|u\|_{p\alpha^*}^p \leq (\|w\|_\alpha / K(p\alpha^*, p)) \|u'\|_p^p.$$

Now (4.8) follows from the characterization (3.5) of $\lambda_1(w)$. ■

EXAMPLE 4.1. Let $1 < p < \infty$. Consider the following nonlinear equation

$$(\phi_p(x'))' + \mu(1 + \cos t) \phi_p(x) = h(t, x), \quad t \in [0, 2\pi], \quad (4.9)$$

with the Dirichlet condition (1.5), where $\mu \in \mathbb{R}$ is a constant and $h(t, x) : [0, 2\pi] \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 Carathéodory function satisfying $\lim_{|x| \rightarrow \infty} h(t, x)/\phi_p(x) = 0$ uniformly in a.e. $t \in [0, 2\pi]$. Eq. (4.9) satisfies (1.2) with $a(t) = b(t) = \mu w(t)$, where $w(t) = 1 + \cos t$.

As $w < 2$, condition $(U)_1$ shows that problem (4.9) + (1.5) has at least one solution when

$$\mu \leq \frac{1}{2} \lambda_1(1) = \frac{1}{2} \left(\frac{\pi_p}{2\pi} \right)^p = \frac{p-1}{2(p \sin \pi/p)^p} =: H_1(p). \quad (4.10)$$

As explained in Section 1, the conditions $(U)_k$ ($k \geq 2$) are not applicable to this example.

However, for this example, conditions $(N)_k$ yield the existence of (4.9) + (1.5) when $\mu \neq \lambda_k(w)$, $k \in \mathbb{N}$. Namely, we have a sequence of non-resonance intervals $\mu \in (\lambda_{k-1}(w), \lambda_k(w))$, $k \in \mathbb{N}$. Even for the first non-resonance interval, one can improve (4.10) using the estimates (2.11) and (4.8). By Theorem 3.1 and (2.11), problem (4.9) + (1.5) has solutions when μ satisfies

$$\mu < \left(\pi_p / \int_0^{2\pi} w_+(t) dt \right)^p = \left(\frac{\pi}{\pi+1} \right)^p \frac{p-1}{(p \sin \pi/p)^p} =: H_2(p). \quad (4.11)$$

One sees that condition (4.11) improves (4.10) when $1 < p < p_0 := (\log 2)/\log(1 + \pi^{-1})$.

Now we apply Theorem 4.2 to obtain lower bounds for $\lambda_1(w)$. Let us think $w(t) = 1 + \cos t$ is in $L^\alpha(0, 2\pi)$, where $1 \leq \alpha \leq \infty$. Note that

$$\|w\|_\alpha = \begin{cases} 2^{1+1/\alpha} \pi^{1/2\alpha} \left(\frac{\Gamma(\frac{1}{2} + \alpha)}{\alpha \Gamma(\alpha)} \right)^{1/\alpha} & \text{if } 1 \leq \alpha < \infty, \\ 2 & \text{if } \alpha = \infty. \end{cases} \quad (4.12)$$

We know from (4.1), (4.8), and (4.12) that

$$\lambda_1(w) \geq H(\alpha, p) := K(p\alpha^*, p) / \|w\|_\alpha, \quad \forall 1 \leq \alpha \leq \infty.$$

Thus we can obtain the existence of (4.9) + (1.5) when

$$\mu < \max_{1 \leq \alpha \leq \infty} H(\alpha, p) =: H_3(p). \quad (4.13)$$

As $H(\infty, p) = K(p, p)/\|w\|_\infty = H_1(p)$, $H_3(p) > H_1(p)$ for all $1 < p < \infty$. Thus (4.13) is better than (4.10) for all p . A numerical computation shows that (4.13) improves (4.10) significantly. ■

When $p = 2$, we have another method for estimating the lower bounds of $\lambda_1(w)$. This is based on the Opial's inequality [1], see also Theorem A of [3].

THEOREM 4.3 [1]. *Let $\psi(\cdot) \in L^2(a, b)$. Let $x(\cdot)$ be absolutely continuous on $[a, b]$ with $x(a) = 0$. Then*

$$\int_a^b \psi(t) |x(t)| |x'(t)| dt \leq \kappa \int_a^b |x'(t)|^2 dt,$$

where

$$\kappa = \left(\frac{1}{2} \int_a^b (t-a) \psi(t)^2 dt \right)^{1/2}.$$

If the boundary condition $x(a) = 0$ is replaced by $x(b) = 0$, we need only to replace κ in the above inequality by

$$\kappa' = \left(\frac{1}{2} \int_a^b (b-t) \psi(t)^2 dt \right)^{1/2}.$$

In the following we assume that $p = 2$ in the eigenvalue problem. We try to use Theorem 4.3 to derive a lower bound for $\lambda_1(w)$. Let $\Phi(t)$ be a primitive of $w(t)$. For any $v \in \mathbb{R}$ and any $x \in W_2$, we get from Theorem 4.3 that

$$\begin{aligned} \int_0^T w(t) x^2 dt &= \int_0^T x^2 d(\Phi(t) - v) = -2 \int_0^T (\Phi(t) - v) x x' dt \\ &\leq 2 \int_0^T |\Phi(t) - v| |x| |x'| dt \\ &= 2 \left[\int_0^{T/2} |\Phi(t) - v| |x| |x'| dt + \int_{T/2}^T |\Phi(t) - v| |x| |x'| dt \right] \\ &\leq \kappa_1(v) \int_0^{T/2} |x'|^2 dt + \kappa_2(v) \int_{T/2}^T |x'|^2 dt \\ &\leq \kappa(v) \int_0^T |x'|^2 dt, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned}\kappa_1(v) &= \left(2 \int_0^{T/2} t(\Phi(t) - v)^2 dt \right)^{1/2}, \\ \kappa_2(v) &= \left(2 \int_{T/2}^T (T-t)(\Phi(t) - v)^2 dt \right)^{1/2}, \\ \kappa(v) &= \max\{\kappa_1(v), \kappa_2(v)\}.\end{aligned}$$

From (3.5) and (4.14) we know that $\lambda_1(w) \geq 1/\kappa(v)$ for all $v \in \mathbb{R}$. Thus we have

THEOREM 4.4. *Let $p = 2$ and $w \in L^1(0, T)$ with $w > 0$. Then*

$$\lambda_1(w) \geq \frac{1}{\min_{v \in \mathbb{R}} \kappa(v)}. \quad (4.15)$$

EXAMPLE 4.2. Let us consider Example 4.1 with $p = 2$, i.e., the problem (1.4) + (1.5). By (U)₁, one can obtain the existence of (1.4) + (1.5) when $\mu \leq 1/8 = 0.1250$

Now we apply Theorem 4.4. Let $w(t) = 1 + \cos t$ and $\Phi(t) = t + \sin t$. We have

$$\begin{aligned}\kappa_1(v) &= \left(\frac{6\pi^2 v^2 - (8\pi^3 + 24\pi)v + (3\pi^4 + 27\pi^2 - 96)}{6} \right)^{1/2}, \\ \kappa_2(v) &= \left(\frac{6\pi^2 v^2 - (16\pi^3 - 24\pi)v + (11\pi^4 - 21\pi^2 - 96)}{6} \right)^{1/2}.\end{aligned}$$

Thus, it follows from (4.15) that

$$\begin{aligned}\lambda_1(w) &\geq \frac{1}{\min_{v \in \mathbb{R}} \max\{\kappa_1(v), \kappa_2(v)\}} = \frac{1}{\kappa_1(\pi)} \\ &= \left(\frac{6}{\pi^4 + 3\pi^2 - 96} \right)^{1/2} = 0.4398\dots\end{aligned} \quad (4.16)$$

Now Theorem 3.1 gives the existence of (1.4) + (1.5) when $\mu \leq 0.4398$. This is much better than the usual existence result. ■

We remark that the lower bound for $\lambda_1(w)$ in (4.16) is very sharp because a numerical computation shows that $\lambda_1(w) = 0.4449\dots$.

ACKNOWLEDGMENT

The author thanks the referee for many helpful suggestions.

REFERENCES

1. P. R. Beesack and K. M. Das, Extensions of Opial's inequality, *Pacific J. Math.* **26** (1968), 215–232.
2. L. Boccardo, P. Dràbek, D. Giachetti, and M. Kuček, Generalization of Fredholm alternative for nonlinear differential operators, *Nonlinear Anal.* **10** (1986), 1083–1103.
3. R. C. Brown and D. B. Hinton, Opial's inequality and oscillation of 2nd order equations, *Proc. Amer. Math. Soc.* **125** (1997), 1123–1129.
4. M. Cuesta and J.-P. Gossez, A variational approach to nonresonance with respect to the Fučík spectrum, *Nonlinear Anal.* **19** (1992), 487–500.
5. M. del Pino, M. Elgueta, and R. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0$, $u(0) = u(T) = 0$, $p > 1$, *J. Differential Equations* **80** (1989), 1–13.
6. A. Fonda and P. Habets, Periodic solutions of asymptotically positively differential equations, *J. Differential Equations* **81** (1989), 68–97.
7. P. Habets and G. Metzen, Existence of periodic solutions of Duffing equations, *J. Differential Equations* **78** (1989), 1–32.
8. M. García-Huidobro, R. Manásevich, and F. Zanolin, A Fredholm-like result for strongly nonlinear second order ODE's, *J. Differential Equations* **114** (1994), 132–167.
9. B. Liu, The stability of the equilibrium of a conservative system, *J. Math. Anal. Appl.* **202** (1996), 133–149.
10. R. Manásevich and J. Mawhin, Periodic solutions for nonlinear systems with p -Laplacian-like operators, *J. Differential Equations* **145** (1998), 367–393.
11. R. Manásevich and F. Zanolin, Time-mappings and multiplicity of solutions for the one-dimensional p -Laplacian, *Nonlinear Anal.* **21** (1993), 269–291.
12. J. Mawhin, “Topological Degree Methods in Nonlinear Boundary Value Problems,” CBMS, Vol. 40, Amer. Math. Soc., Providence, RI, 1979.
13. G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **110** (1976), 353–372.
14. M. Zhang, Nonuniform nonresonance at the first eigenvalue of the p -Laplacian, *Nonlinear Anal.* **29** (1997), 41–51.
15. M. Zhang, Nonresonance conditions for asymptotically positively homogeneous differential systems: The Fučík spectrum and its generalization, *J. Differential Equations* **145** (1998), 332–366.
16. M. Zhang, A relationship between the periodic and the Dirichlet BVPs of singular differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 1099–1114.