Front. Math. China 2009, 4(3): 585–598
DOI 10.1007/s11464-009-0034-4

RESEARCH ARTICLE

From ODE to DDE*

Meirong ZHANG

Department of Mathematical Sciences and Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing 100084, China

Abstract In this paper, by considering ordinary differential equation (ODE) as a special case and a starting point of delay differential equation (DDE), we will show that some typical topological methods such as continuation theorems can be applied to detect some dynamics of DDE like periodic solutions. Several problems will be presented.

Keywords Ordinary differential equation (ODE), delay differential equation (DDE), periodic solution, continuation theorem, Sobolev constant, non-degeneracy
MSC 34K13, 34L05, 58J20

1 Introduction

Ordinary differential equations (ODE)

$$x'(t) = f_0(t, x(t)), \quad x(t) \in \mathbb{R}^n, \tag{1.1}$$

are basic models in applied sciences when no delay effects are considered [2]. In this case, equation (1.1) has the finitely dimensional Euclidean space $\mathbb{R}^n$ as its phase space.

When some historical effects or delays are considered, one will be led to delay differential equations (DDE) [7]. For simplicity, we consider the following simplest DDE:

$$x'(t) = f(t, x(t), x(t-r)), \tag{1.2}$$

where $f = f(t, x, y)\colon\ \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $r > 0$ is the delay. In order to study the dynamics of equation (1.2), a general choice of its phase space is

$$C([-r, 0], \mathbb{R}^n) := \{x\colon\ [-r, 0] \to \mathbb{R}^n \text{ is continuous}\},$$

* Received September 18, 2008; accepted November 10, 2008

E-mail: mzhang@math.tsinghua.edu.cn

which is infinitely dimensional. See Ref. [7]. Due to this, DDE presents many new features, and of course is relatively difficult in the study of its dynamics.

Now let us concentrate on a basic dynamics problem for DDE, i.e., the periodic solutions of (1.2). More precisely, let us assume that

$$f(t+1,x,y) \equiv f(t,x,y).$$

The problem is to seek periodic solutions of (1.2) (of period 1). The main idea of this paper is to consider r as a parameter in DDE (1.2). Especially, when $r=0$, equation (1.2) is reduced to ODE (1.1) with $f_0(t,x) := f(t,x,x)$.

As for ODE (1.1), there are many methods to detect periodic solutions. Now we are able to consider (1.2) as a bifurcation problem, where r is understood as a bifurcation parameter, with the starting choice of (1.1). Once equation (1.1) has some periodic solution which is non-degenerate in some sense, we can obtain periodic solutions of (1.2), at least when the delay r is small. It is worth pointing out a crucial difference between this idea and the usual bifurcation treatment. That is, as a bifurcation system, (1.2) is only continuous in the parameter r, but not differentiable in r, because the parameter r appears in the delay way.

However, some topological ideas are applicable to continuous, but not differentiable, problems. A typical tool is the so-called continuation theorems which are deduced from general topology [19]. A nice example of their applications is the global continuation of periodic solutions of the Sitnikov problem [12]. In this paper we will show how we can use continuation theorems to detect a continuum of periodic solutions of (1.2), which is, in most cases, a local continuum, and, while in some cases, a global continuum. This will be illustrated by some typical examples. As we proceed, several problems concerning periodic solutions of DDE will be mentioned.

2 Continuation theorems

The continuation theorems [19] are a basic tool in detecting solutions of (infinitely dimensional) systems. There are many variants of these theorems.

Let $(X, \|\cdot\|)$ be a Banach space and

$$\mathscr{T} = \mathscr{T}(x,\tau) \colon\ X \times [0,1] \to X$$

be a completely continuous operator. The fixed point set of $\mathscr{T}$ is defined as

$$\Sigma := \{(x,\tau) \in X \times [0,1] \colon\ \mathscr{T}(x,\tau) = x\} \subset X \times [0,1].$$

For any $\tau \in [0,1]$, the τ-section of Σ is defined as

$$\Sigma_\tau := \{x \in X \colon\ (x,\tau) \in \Sigma\} \subset X.$$

The local version of continuation theorems is an easy consequence of homotopy invariance of topological index [14].

Proposition 2.1 *Suppose that* Σ_0 *has an isolated point* x_0 *such that the (fixed point) topological index* $\mathrm{index}(\mathscr{T}(\cdot,0),x_0)$ *is non-zero. Then one has a continuum* $\mathscr{C}$ *of* Σ *such that* $x_0 \in \mathscr{C}_0$ *and* $\mathscr{C}_\tau \neq \emptyset$ *for all* $\tau \in (0,\tau_0)$ *where* $\tau_0 > 0$ *is a certain number. Here* $\mathscr{C}_\tau := \mathscr{C} \cap \Sigma_\tau$.

One formulation of (global) continuation theorems is as follows. See, for example, Corollary 1.2 in Ref. [14].

Proposition 2.2 *Suppose that one has the following non-degeneracy condition*:

(H_0) Σ_0 *is bounded in* X *and the topological index* $\mathrm{index}(\mathscr{T}(\cdot,0),\Sigma_0)$ *is non-zero.*

Then there exists a continuum $\mathscr{C}$ *of* Σ *such that* $\mathscr{C}_0 \cap \Sigma_0 \neq \emptyset$ *and*

(i) *either,* $\mathscr{C}$ *is unbounded in the space* $X \times [0,1]$,

(ii) *or,* $\mathscr{C}$ *connects the sections* Σ_0 *and* Σ_1. *In this case, for any* $\tau \in [0,1]$, *there exists at least one* $x_\tau \in X$ *such that* $(x_\tau,\tau) \in \mathscr{C} \subset \Sigma$.

Consequently, alternative (ii) shows that, under the assumption above, the fixed point equation

$$x = \mathscr{T}(x,1), \quad x \in X,$$

has at least one solution. In order to study the dependence of solutions of DDE (1.2) on the delay r, we prove the following topological fact.

Proposition 2.3 *Suppose that hypothesis* (H_0) *holds. Moreover, suppose that*

(H_1) Σ *is bounded in* $X \times [0,1]$, *and*

(H_2) *for each* $\tau \in [0,1]$, *the section* Σ_τ *consists of exactly one point.*

Then the set Σ *can be continuously parameterized using* τ, *i.e., there exists a continuous mapping*

$$[0,1] \to X, \quad \tau \to \varphi_\tau,$$

such that

$$\Sigma = \{(\varphi_\tau,\tau):\ \tau \in [0,1]\}.$$

Proof By Proposition 2.2, conditions (H_0) and (H_1) imply that $\Sigma \subset X \times [0,1]$ is a bounded non-empty set. As $\mathscr{T}$ is completely continuous, we know that

$$\Sigma = \mathscr{T}(\Sigma)$$

is compact in the space $X \times [0,1]$. Note that Proposition 2.2 has claimed that Σ contains a continuum $\mathscr{C}$ which connects Σ_0 and Σ_1. By (H_2), for any $\tau \in [0,1]$, there exists a unique $\varphi_\tau \in X$ such that $(\varphi_\tau,\tau) \in \Sigma$. We need to prove that φ_τ is continuous in τ. Otherwise, suppose that there exists some $\tau_0 \in [0,1]$ such that

$$\lim_{\tau \to \tau_0} \varphi_\tau \neq \varphi_{\tau_0}.$$

Hence, there exists a sequence $\tau_n \in [0,1]$ such that $\tau_n \to \tau_0$ and $\lim_{n\to\infty} \varphi_{\tau_n} \neq \varphi_{\tau_0}$. Since Σ is compact, we know that there exists a sub-sequence of φ_{τ_n}, still denoted by φ_{τ_n}, such that

$$\lim_{n\to\infty} \varphi_{\tau_n} = \psi_0 \neq \varphi_{\tau_0}. \tag{2.1}$$

Hence, one has $(\varphi_{\tau_n}, \tau_n) \to (\psi_0, \tau_0)$. Using the continuity of $\mathscr{T}$, the limiting equality of

$$\mathscr{T}(\varphi_{\tau_n}, \tau_n) = \varphi_{\tau_n}, \quad n \in \mathbb{N},$$

is

$$\mathscr{T}(\psi_0, \tau_0) = \psi_0,$$

i.e., $(\psi_0, \tau_0) \in \Sigma$. By ($H_2$) one has $\psi_0 = \varphi_{\tau_0}$, which leads to a contradiction to (2.1). $\square$

3 Local existence principle for DDE

Now we are going to apply the continuation theorems in the preceding section to obtain periodic solutions of DDE (1.2), where

$$f = f(t,x,y)\colon\ \mathbb{S} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbb{S} := \mathbb{R}/\mathbb{Z},$$

is continuous, or, more generally, is L^1 Carathéodory [5]. Note that the choice of the parameter τ in applications of continuation theorems is flexible. For equation (1.2), we can choose, for example, $\tau = r$. The space X for periodic solutions of (1.2) is taken as $X = C(\mathbb{S}, \mathbb{R}^n)$ with the supremum norm $\|\cdot\|_{C^0}$. It is well known that finding periodic solutions of equation (1.2) (of period 1) can be written as a coincidence equation which can be further reduced to a fixed point problem

$$\mathscr{T}(x,r) = x$$

in the space $C(\mathbb{S}, \mathbb{R}^n)$. For details, see Ref. [5]. For example, one can take $\mathscr{T}$ as

$$\begin{aligned}\mathscr{T}(x,r)(t) := x(0) &+ \int_0^t f(s, x(s), x(s-r))\mathrm{d}s\\ &+ (1-t)\int_0^1 f(s, x(s), x(s-r))\mathrm{d}s, \quad t \in [0,1].\end{aligned}$$

It is easy to verify that $\mathscr{T}\colon\ C(\mathbb{S}, \mathbb{R}^n) \times [0,1] \to C(\mathbb{S}, \mathbb{R}^n)$ is completely continuous. As mentioned in the introduction, even when $f(t,x,y)$ is smooth, the operator $\mathscr{T}(x,r)$ is, in general, not differentiable in r. In fact, let us define a nonlinear operator

$$\mathscr{M}\colon\ (C(\mathbb{S}, \mathbb{R}^n), \|\cdot\|_{C^0}) \times \mathbb{R} \to (C(\mathbb{S}, \mathbb{R}^n), \|\cdot\|_{C^0})$$

by

$$\mathscr{M}(x,r)(t) := x(t-r).$$

By the uniform continuity of any function in $C(\mathbb{S},\mathbb{R}^n)$, one has

$$(x_m, r_m) \to (x_0, r_0) \quad \text{in } (C(\mathbb{S},\mathbb{R}^n), \|\cdot\|_{C^0}) \times \mathbb{R}$$
$$\Longrightarrow \mathscr{M}(x_m, r_m) \to \mathscr{M}(x_0, r_0) \quad \text{in } (C(\mathbb{S},\mathbb{R}^n), \|\cdot\|_{C^0}).$$

However, $\mathscr{M}(x,r)$ is not differentiable in r because

$$\lim_{r\to r_0} \frac{\mathscr{M}(x,r) - \mathscr{M}(x,r_0)}{r - r_0} = \lim_{r\to r_0} \frac{x(\cdot - r) - x(\cdot - r_0)}{r - r_0}$$

does not exist in general.

Now we have the following existence principle for periodic solutions of DDE (1.2).

Theorem 3.1 (Local Existence Principle) *Suppose that the corresponding ODE* (1.1) *with $f_0(t,x) = f(t,x,x)$ has a periodic solution $\varphi(t)$ which is non-degenerate. Then there exists some $R_0 > 0$ such that, for each $r \in [0, R_0)$, DDE* (1.2) *has at least one periodic solution $x_r \in C(\mathbb{S},\mathbb{R}^n)$ and $x_0 = \varphi$.*

The non-degeneracy of $\varphi(t)$ of (1.1) means that the linearization equation

$$y' = \frac{\partial f_0}{\partial x}(t, \varphi(t)) \cdot y \tag{3.1}$$

has only the trivial 1-periodic solution $y = 0$. In this case, we have [5]

$$\operatorname{index}(\mathscr{T}(\cdot, 0), \varphi) = \pm 1.$$

Hence, Theorem 3.1 follows from Proposition 2.1 immediately.

Remark 3.2 (i) The local result above can be extended to DDE with multi-delays:

$$x'(t) = f(t, x(t), x(t - r_1), \ldots, x(t - r_m)),$$

where r_i are small.

(ii) One can write from Proposition 2.2 some alternative existence principle for (1.2). However, in order to obtain a global existence result, one needs to exclude case (i) in Proposition 2.2, which is still a difficult problem for general type of equations (1.2). In the next section, we will develop some idea to obtain a global existence result.

Now we give some applications of Theorem 3.1.

In Refs. [3,4], based on the superlinear Lagrangian equations, Capietto, Mawhin and Zanolin have partially extended some results on ODE to DDE. As a general DDE has no conservative property, and, moreover, the variational structure can be deduced for very limited DDE [6,11], many problems are open for periodic solutions of DDE. For example, it is a classical result that for each $e(t) \in C(\mathbb{S},\mathbb{R})$, the superlinear Lagrangian equation

$$x'' + 2x^3 = e(t) \tag{3.2}$$

has at least one periodic solution (of period 1), see Ref. [16]. In fact, by the Poincaré-Birkhoff twist fixed point theorem, equation (3.2) may admit many n-periodic solutions, where $n \in \mathbb{N}$. Capietto, Mawhin and Zanolin have asked the question that what will happen when one considers delay perturbations of (3.2). To this question, they have obtained some partial, but interesting, answer. By Theorem 3.1, we can assert that for any $e(t) \in C(\mathbb{S}, \mathbb{R})$, there exists some $R_0 = R_0(e) > 0$ such that for any $r \in [0, R_0)$, the following DDE

$$x''(t) + 2x^3(t - r) = e(t)$$

has at least one periodic solution, because the non-degeneracy of the Morris' solutions can be proved. It is then an interesting problem whether or not the range $R_0(e)$ has a uniform positive lower bound for all $e \in C(\mathbb{S}, \mathbb{R})$.

Next let us consider some singular equations. Let $\nu \geqslant 1$ be a constant and $\alpha(t) \in C(\mathbb{S}, \mathbb{R})$ be such that $\alpha(t) \geqslant 0$. Consider the following singular ODE:

$$x'' + \alpha(t)x = 1/x^\nu, \quad x > 0. \tag{3.3}$$

In case $\nu = 3$, (3.3) is called the Ermakov-Pinney equation:

$$x'' + \alpha(t)x = 1/x^3, \quad x > 0, \tag{3.4}$$

which has a variety of important applications in applied sciences. See, for example, the commentary article [9]. In the study of nonuniform non-resonance of singular ODE [20,22] and twist character of elliptic periodic solutions of nonlinear Lagrangian equations [10], the present author has established a basic result for the Ermakov-Pinney equation (3.4), which can be stated as follows. Suppose that the corresponding Hill's equation

$$x'' + \alpha(t)x = 0 \tag{3.5}$$

is elliptic. Then Ermakov-Pinney equation (3.4) has the unique positive periodic solution $\varphi(t)$ which is also non-degenerate in the sense of (3.1), i.e., the corresponding linearization equation

$$x'' + (\alpha(t) + 3\varphi^{-4}(t))x = 0$$

has only the trivial 1-periodic solution [22]. Based on this, it is also proved in Ref. [22] that, under the ellipticity of (3.5), equation (3.3) of the Ermakov-Pinney type has at least one positive periodic solution. Note that when $\nu \neq 3$, equation (3.3) may admit more than one periodic solutions. By Theorem 3.1, we have the following result on the DDE version of (3.3).

Theorem 3.3 *Let $\nu \geqslant 1$ and $\alpha(t) \in C(\mathbb{S}, \mathbb{R})$, $\alpha(t) \geqslant 0$, be such that equation (3.5) is elliptic. Then there exists $R_0 > 0$ such that for any r_i with $|r_i| < R_0$, $i = 1, 2$, the following singular DDE*

$$x''(t) + \alpha(t)x(t - r_1) = 1/(x(t - r_2))^\nu \tag{3.6}$$

has at least one positive periodic solution.

Remark 3.4 It is then an interesting problem if singular DDE (3.6) has at least one positive periodic solution for any $r_i \in \mathbb{R}$.

4 Global results for DDE of Landesman-Lazer type

By Proposition 2.2, one can obtain a global continuation by excluding unbounded continua. Let us recall a well-studied problem for ODE, called the Landesman-Lazer problem [8,18]. For simplicity, let us take the following ODE:

$$x'' + g(x) = h(t), \tag{4.1}$$

where the external force $h(t) \in L^1(\mathbb{S}, \mathbb{R})$. Equation (4.1) is called Landesman-Lazer type, if $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, monotone and one-sided bounded. The latter means that either

$$\inf_x g(x) > -\infty$$

or

$$\sup_x g(x) < +\infty.$$

In case (4.1) has a periodic solution, it is necessary that the mean value of $h(t)$ satisfies

$$\overline{h} := \int_0^1 h(t)\mathrm{d}t \in g(\mathbb{R}). \tag{4.2}$$

The converse is almost true for the Landesman-Lazer equations.

Theorem 4.1 [18] (i) *Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing and $\inf_x g(x) > -\infty$. Then equation* (4.1) *has at least one periodic solution if and only if $h(t)$ satisfies* (4.2).

(ii) *Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is non-decreasing and $\inf_x g(x) > -\infty$. Then equation* (4.1) *has at least one periodic solution if $h(t)$ satisfies*

$$\overline{h} \in \mathrm{int}(g(\mathbb{R})). \tag{4.3}$$

Here, $\mathrm{int}(J)$ *denotes the interior of J and condition* (4.3) *is a little bit stronger than the necessary condition* (4.2).

4.1 Global continuation result

Now let us take (4.1) as a starting point and consider the following type of DDE:

$$x''(t) + g_1(x(t)) + g_2(x(t-r)) = h(t), \tag{4.4}$$

where $g_i\colon \mathbb{R} \to \mathbb{R}$ are continuous, $h \in L^1(\mathbb{S}, \mathbb{R})$ and $r \in \mathbb{R}$. For simplicity, we say that (4.4) is Landesman-Lazer type, if

- both g_1 and g_2 are non-decreasing, and

- both g_1 and g_2 have lower bounds

$$\inf_x g_i(x) > -\infty, \quad i = 1, 2.$$

For DDE (4.4), one can introduce $g\colon\ \mathbb{R} \to \mathbb{R}$ by

$$g(x) := g_1(x) + g_2(x). \tag{4.5}$$

Then $g(x)$ is also non-decreasing. Moreover, it is easy to see that

$$\begin{aligned} \inf g &= \inf g_1 + \inf g_2 \in (-\infty, \infty), \\ \sup g &= \sup g_1 + \sup g_2 \in (-\infty, \infty]. \end{aligned} \tag{4.6}$$

Hence the ODE

$$x''(t) + g_1(x(t)) + g_2(x(t)) = h(t) \tag{4.7}$$

is the Landesman-Lazer type. By considering the delay r as a homotopy parameter, DDE (4.4) is deformed from ODE (4.7).

Suppose that (4.4) has a periodic solution $x(t)$. By integrating (4.4) over one period, we know that

$$\begin{aligned} \overline{h} &= \int_0^1 g_1(x(t))\mathrm{d}t + \int_0^1 g_2(x(t-r))\mathrm{d}t \\ &= \int_0^1 g_1(x(t))\mathrm{d}t + \int_0^1 g_2(x(t))\mathrm{d}t \quad \text{(as } x \text{ is 1-periodic)} \\ &= \int_0^1 (g_1(x(t)) + g_2(x(t)))\mathrm{d}t \\ &= g_1(x(t_0)) + g_2(x(t_0)) \\ &= g(x(t_0)) \end{aligned} \tag{4.8}$$

for some $t_0 = t_0(x) \in [0, 1]$. It yields a necessary condition on h:

$$\overline{h} \in g(\mathbb{R}) \tag{4.9}$$

with g as in (4.5). This is also the necessary condition for the starting ODE (4.7). We will show that condition (4.9) is almost sufficient, not only for ODE (4.7), but also for DDE (4.4).

In the following, for any $h \in L^\gamma(\mathbb{S}, \mathbb{R}),\ 1 \leqslant \gamma \leqslant \infty$, we use $\|h\|_\gamma$ to denote the L^γ norm $\|h\|_{L^\gamma(0,1)}$. We can obtain a global continuation of periodic solutions of DDE (4.4).

Theorem 4.2 (Global Continuation Result) *Let g_i be as above. Suppose that $h \in L^1(\mathbb{S}, \mathbb{R})$ satisfies*

$$\overline{h} \in \mathrm{int}(g(\mathbb{R})), \tag{4.10}$$

with g as in (4.5)*. Then, for each $r \in \mathbb{R}$, DDE* (4.4) *has at least one periodic solution.*

Proof Following the proof for ODE in Ref. [18], we will prove that all possible periodic solutions of (4.4) are bounded in the space $C^1(\mathbb{S},\mathbb{R})$ with the norm

$$\|x\|_{C^1} := \max(\|x\|_{C^0}, \|x'\|_{C^0}).$$

Suppose that $x(t)$ is a 1-periodic solution of (4.4) for some r. The (uniform) bound of $\|x'\|_{C^0}$ can be obtained without assumption (4.10). As $x(t)$ is periodic, we may assume that $x'(0) = 0$. By equation (4.4), we have, for all $t \in [0,1]$,

$$\begin{aligned} x'(t) =& \int_0^t (h(s) - g_1(x(s)) - g_2(x(s-r)))\mathrm{d}s \\ \leqslant& \int_0^t (h(s) - \inf g_1 - \inf g_2)\mathrm{d}s \\ =& \int_0^t h(s)\mathrm{d}s - t \cdot \inf g \quad \text{(by (4.6))} \\ \leqslant& \|h\|_1 + \max(0, -\inf g) \\ =:& \, C_1 < \infty. \end{aligned}$$

As $x'(1) = x'(0) = 0$, by equation (4.4) again, we have, for all $t \in [0,1]$,

$$\begin{aligned} x'(t) =& -\int_t^1 x''(s)\mathrm{d}s \\ =& \int_t^1 (g_1(x(s)) + g_2(x(s-r)) - h(s))\mathrm{d}s \\ \geqslant& \int_t^1 (\inf g_1 + \inf g_2 - h(s))\mathrm{d}s \\ \geqslant& -C_1 \\ >& -\infty. \end{aligned}$$

Hence, one has always

$$\|x'\|_{C^0} \leqslant C_1. \tag{4.11}$$

Next we show that $\|x\|_{C^0}$ has also a (uniform) bound under assumption (4.10). By (4.8), $g(x(t_0)) = \overline{h}$ for some $t_0 = t_0(x) \in [0,1]$. That is,

$$x(t_0) \in g^{-1}(\{\overline{h}\}) =: J_{\overline{h}}.$$

By assumption (4.10) on $\overline{h}$, one sees that

$$J_{\overline{h}} = g^{-1}(\{\overline{h}\}) = [a_{\overline{h}}, b_{\overline{h}}]$$

is necessarily a bounded interval of $\mathbb{R}$ because g is non-decreasing. We conclude that any periodic solution $x(t)$ of (4.4) satisfies

$$|x(t_0)| \leqslant c_{\overline{h}} := \max(|a_{\overline{h}}|, |b_{\overline{h}}|) \tag{4.12}$$

for some $t_0 = t_0(x) \in [0,1]$. From (4.11) and (4.12), we have

$$\|x\|_{C^0} \leqslant C_0 := c_{\overline{h}} + C_1 \tag{4.13}$$

for all possible periodic solutions $x(t)$ of (4.4).

In order to apply Proposition 2.2, we need to prove that hypothesis (H_0) is satisfied by ODE (4.7). As (4.7) is Landesman-Lazer type and condition (4.10) is the same as (4.3), the non-degeneracy condition on (4.7) is fulfilled. In fact the Landesman-Lazer result for ODE is just obtained from this. Now the theorem follows from Proposition 2.2. □

Compared with the necessary condition (4.9), the global continuation condition (4.10) for DDE (4.4) is optimal in some sense. From the proof of Theorem 4.2, the solution set Σ of (4.4) is contained in

$$\{(x,r) \in C^1(\mathbb{S},\mathbb{R}) \times \mathbb{R} \colon \ \|x\|_{C^1} \leqslant C_0\}.$$

See (4.11) and (4.13).

4.2 Global and unique continuation result

Denote $x_+ = \max(x,0)$ for $x \in \mathbb{R}$ and $x_+(t) = \max(x(t),0)$ for a function $x(t)$. We consider the following example of DDE:

$$x''(t) + a x_+^p(t) + b x_+^q(t-r) = \overline{h} + \widetilde{h}(t), \tag{4.14}$$

where the constants are

$$a \geqslant 0, \quad b \geqslant 0, \quad a+b>0, \quad p \in (1,\infty), \quad q \in (1,\infty), \tag{4.15}$$

and the external force is decomposed into $\overline{h} + \widetilde{h}$, where $\overline{h} \in \mathbb{R}$ and

$$\widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R}) := \{e \in L^1(\mathbb{S},\mathbb{R}) \colon \ \bar{e} = 0\}.$$

Equation (4.14) is a Landesman-Lazer DDE in the sense above. From Theorem 4.2, we have the following existence result.

Lemma 4.3 *Suppose that $\overline{h} \in (0,\infty)$. Then, for each $\widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R})$, equation (4.14) has at least one periodic solution.*

For the ODE case of (4.14), i.e., $a>0$ and $b=0$, by a scaling of x, we need only to consider

$$x'' + x_+^p = \overline{h} + \widetilde{h}(t), \quad \widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R}). \tag{4.16}$$

In a recent paper by Ortega and Zhang [17], it is proved that when the mean value $\overline{h}$ satisfies

$$0 < \overline{h} < \sigma_p := (4p^{-1}S(2p))^{p^*}, \quad p^* := \frac{p}{p-1}, \tag{4.17}$$

equation (4.16) has exactly one periodic solution for each $\widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R})$. Moreover, the bound σ_p of (4.17) is optimal to guarantee the uniqueness of periodic solutions for all $\widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R})$. Here, for $\gamma \in [1,\infty]$, $S(\gamma)$ is the following Sobolev constant [1]:

$$S(\gamma) := \inf_{u\in H_0^1(0,1), u\neq 0} \frac{\|u'\|_2^2}{\|u\|_\gamma^2}. \tag{4.18}$$

The explicit formula of $S(\gamma)$ can be found, for example, in Ref. [21]. In doing so, we are led to some explicit non-degeneracy conditions for the linearization equations of (4.16) at all possible periodic solutions. These have been constructed in Ref. [21] for ODE and its p-Laplacian variant.

We can undertake a similar analysis on DDE (4.14). Suppose that $\varphi(t)$ is a periodic solution of (4.14). The linearization of (4.14) is the following linear DDE:

$$x''(t) + \alpha(t)x(t) + \beta(t)x(t-r) = 0, \tag{4.19}$$

where

$$\alpha(t) = ap\dot{(}\varphi_+(t))^{p-1}, \quad \beta(t) = bq\dot{(}\varphi_+(t-r))^{q-1}$$

are non-negative periodic coefficients. As in Refs. [15,21], we are also able to establish some explicit non-degeneracy conditions for (4.19) using $\|\alpha\|_\gamma$ and $\|\beta\|_{\gamma'}$ only. Here γ and γ' are some suitable exponents deduced from p and q. Now we can apply the idea in Ref. [17] to obtain the following result.

Theorem 4.4 (Global and Unique Continuation Result) *Given a, b, p, q as in* (4.15). *Then there exists a constant $\sigma_{a,b,p,q} > 0$ such that if*

$$0 < \overline{h} < \sigma_{a,b,p,q}, \tag{4.20}$$

then, for each $\widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R})$ and each $r \in \mathbb{R}$, DDE (4.14) *has the unique* 1*-periodic solution $x = \varphi_r(t)$. Moreover, as a functional of r,*

$$\mathbb{R} \to (C^1(\mathbb{S},\mathbb{R}), \|\cdot\|_{C^1}), \quad r \to \varphi_r \tag{4.21}$$

is continuous.

Note that the constant $\sigma_{a,b,p,q}$ can be constructed explicitly using some Sobolev constants corresponding to DDE. However, the constant $\sigma_{a,b,p,q}$ constructed as in Refs. [15,17,21] is not optimal. The continuity of the solution mapping (4.21) follows immediately from Proposition 2.3 because condition (4.20) has guaranteed the uniqueness of solutions of (4.14). Theorem 4.4 has provided a nice example for nonlinear DDE. By considering DDE (4.14) as a continuation of ODE (4.14) (with $r = 0$), there exists a unique continuous curve $r \to \varphi_r$, parameterized by the delay r, in the space $(C^1(\mathbb{S},\mathbb{R}), \|\cdot\|_{C^1})$ such that φ_r is the unique 1-periodic solution of DDE (4.14) for each delay r.

For the ODE case (4.16) of (4.14), the bound σ_p found from the Sobolev constants of (4.18) is optimal, as mentioned before. However, even for the simplest DDE case of (4.14), i.e., $a = 0$ and $b > 0$, though we have some careful analysis on the non-degeneracy of linear DDE

$$x''(t) + \beta(t)x(t-r) = 0, \tag{4.22}$$

the results for (4.22) in Ref. [15] are moderate, especially compared with the optimal results in Ref. [21] for the linear ODE (3.5). Due to the displacement caused by the delay r, it is found in Ref. [15] that some variants of the standard Sobolev inequalities of ODE and PDE are important in the study of (4.22). For example, the following Sobolev constants

$$S(\gamma, r) := \inf_{u\in \widetilde{H}^1(\mathbb{S},\mathbb{R}),u\neq 0} \frac{\|u'\|_2^2}{\|(u(\cdot)u(\cdot - r))_+\|_\gamma} \tag{4.23}$$

are also useful. Here,

$$\widetilde{H}^1(\mathbb{S},\mathbb{R}) = \{u \in H^1(\mathbb{S},\mathbb{R}):\ \bar{u} = 0\}$$

is the Hilbert space of periodic functions of mean value 0. When $r \notin \mathbb{Z}$, problem (4.23) loses the variational structure. The explicit formulas of $S(\gamma, r)$ are only known to the author for some very special cases. See Proposition 4.1 in Ref. [15].

The results of Theorem 4.4 can be rephrased in another way. Give $\widetilde{h} \in \widetilde{L}^1(\mathbb{S},\mathbb{R})$. Then there exists a bifurcation value $\mathscr{H}(\widetilde{h}, r) > 0$ such that

$$0 < \overline{h} < \mathscr{H}(\widetilde{h}, r) \Longrightarrow \text{ (4.14) has a unique 1-periodic solution.}$$

For ODE (4.16), one can define the corresponding bifurcation value $\mathscr{H}(\widetilde{h})$ in a similar way. Now the optimality of (4.17) is the same as

$$\inf_{\widetilde{h}\in \widetilde{L}^1(\mathbb{S},\mathbb{R})} \mathscr{H}(\widetilde{h}) = \sigma_p,$$

while Theorem 4.4 asserts that

$$\inf_{\widetilde{h}\in \widetilde{L}^1(\mathbb{S},\mathbb{R}),r\in\mathbb{R}} \mathscr{H}(\widetilde{h}, r) \geqslant \sigma_{a,b,p,q}.$$

Thus we have the following interesting problems for DDE like (4.14).

Remark 4.5 (i) Find the corresponding Sobolev inequalities and their constants for the non-degeneracy of DDE (4.19).

(ii) Given a, b, p, q as in (4.15), find the optimal bifurcation values

$$\inf\nolimits_{\widetilde{h}\in \widetilde{L}^1(\mathbb{S},\mathbb{R})} \mathscr{H}(\widetilde{h}, r), \quad r \in \mathbb{R},$$
$$\inf\nolimits_{\widetilde{h}\in \widetilde{L}^1(\mathbb{S},\mathbb{R}),r\in\mathbb{R}} \mathscr{H}(\widetilde{h}, r).$$

To attack these problems, it seems that the following basic problem on the evolution of linear equations (4.19) plays an important role. Suppose that $x(t)$ is a non-zero solution of (4.19). The problem is whether all zeros of $x(t)$ are non-degenerate. Note that for the ODE case ($r = 0$), this is true due to the uniqueness of solutions of initial value problems. Once this is true for DDE, the idea in Ref. [21] can be applied to obtain better constant $\sigma_{a,b,p,q}$.

Finally, let us briefly mention an important dynamics problem, i.e., the stability of periodic solutions $\varphi(t)$ of (1.2). By linearizing (1.2) along $\varphi(t)$, we arrive at the linear DDE (4.19) with periodic coefficients $\alpha(t)$ and $\beta(t)$. When $r = 0$, equation (4.19) is a Hill's equation, whose stability can be simply studied using the Floquet theory (in finitely dimensional spaces). Therefore, the estimates of eigenvalues like those in Ref. [21] are useful. The extension of Floquet theory (in infinitely dimensional spaces) to linear DDE has been established in Ref. [13]. However, the similar estimates for DDE have not been established in an adequate way. Like the continuation of periodic solutions, can we study the stability of periodic solutions of DDE by considering them as a continuation of ODE?

Acknowledgements The author would like to thank the editors of this special issue for inviting him to write this paper. Help from Gang Meng, Rafael Ortega, and Ping Yan during the preparation of this paper is gratefully acknowledged. Taking this opportunity, the author would like to point out that the name 'Emarkov' in the title of Ref. [22] should be spelled as Ermakov, who was a scientist in the nineteenth century. This work was supported by the National Basic Research Program of China (Grant No. 2006CB805903), the National Natural Science Foundation of China (Grant No. 10531010) and the 111 Project of China (2007).

References

1. Adams R A, Fournier J J F. Sobolev Spaces. 2nd Ed. Pure Appl Math (Amsterdam), Vol 140. Amsterdam: Elsevier/Academic Press, 2003
2. Arnold V I. Mathematical Methods of Classical Mechanics. Graduate Texts Math, Vol 60. New York-Heidelberg: Springer-Verlag, 1978
3. Capietto A, Mawhin J, Zanolin F. The coincidence degree of some functional-differential operators in spaces of periodic functions and related continuation theorems. In: Delay Differential Equations and Dynamical Systems (Claremont, CA, 1990). Lecture Notes Math, Vol 1475. Berlin: Springer, 1991, 76–87
4. Capietto A, Mawhin J, Zanolin F. Periodic solutions of some superlinear functional differential equations. In: Yoshizawa T, Kato J, eds. Proc Intern Symp Functional Differential Equations (Kyoto, Japan, 30 Aug–2 Sept, 1990). Singapore: World Scientific, 1991, 19–31
5. Gaines R E, Mawhin J L. Coincidence Degree and Nonlinear Differential Equations. Lecture Notes Math, Vol 568. Berlin-New York: Springer-Verlag, 1977
6. Guo Z, Yu J. Multiplicity results for periodic solutions of delay differential equations via critical point theory. J Differential Equations, 2005, 218: 15–35
7. Hale J K. Theory of Functional Differential Equations. New York: Springer-Verlag, 1977
8. Jiang M -Y. A Landesman-Lazer type theorem for periodic solutions of the resonant asymmetric p-Laplacian equation. Acta Math Sinica, Engl Ser, 2005, 21: 1219–1228
9. Leach P G L, Andropoulos K. The Ermakov equation, a commentary. Appl Anal Discrete Math, 2008, 2: 146–157
10. Lei J, Li X, Yan P, Zhang M. Twist character of the least amplitude periodic solution of the forced pendulum. SIAM J Math Anal, 2003, 35: 844–867
11. Li J, He X. Proof and generalization of Kaplan-Yorke's conjecture on periodic solutions of differential delay equations. Sci China, Ser A, 1999, 42: 957–964
12. Llibre J, Ortega R. On the families of periodic orbits of the Sitnikov problem. SIAM J Appl Dynam Syst, 2008, 7: 561–576

13. Mallet-Paret J, Sell G R. Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions. J Differential Equations, 1996, 125: 385–440
14. Mawhin J. Continuation theorems and periodic solutions of ordinary differential equations. In: Topological Methods in Differential Equations and Inclusions. Dordrecht: Kluwer, 1995, 291–375
15. Meng G, Yan P, Lin X, Zhang M. Non-degeneracy and periodic solutions of semilinear differential equations with deviation. Adv Nonlinear Stud, 2006, 6: 563–590
16. Morris G R. An infinite class of periodic solutions of $x'' + 2x^3 = p(t)$. Proc Cambridge Phil Soc, 1965, 61: 157–164
17. Ortega R, Zhang M. Some optimal bounds for bifurcation values of a superlinear periodic problem. Proc Royal Soc Edinburgh, Sect A, 2005, 135: 119–132
18. Ward J R. Asymptotic conditions for periodic solutions of ordinary differential equations. Proc Amer Math Soc, 1981, 81: 415–420
19. Whyburn G T. Analytic Topology. Amer Math Soc Colloq Publ, Vol 28. New York: Amer Math Soc, 1942
20. Yan P, Zhang M. Higher order non-resonance for differential equations with singularities. Math Meth Appl Sci, 2003, 26: 1067–1074
21. Zhang M. Certain classes of potentials for p-Laplacian to be non-degenerate. Math Nachr, 2005, 278: 1823–1836
22. Zhang M. Periodic solutions of equations of Emarkov-Pinney type. Adv Nonlinear Stud, 2006, 6: 57–67