Existence of SRB measures for expanding maps with weak regularity

WEIGU LI
School of Mathematical Sciences
Peking University
Beijing 100871
People's Republic of China
E-mail: `weigu@sxx0.math.pku.edu.cn`

MEIRONG ZHANG
Department of Mathematical Sciences
Tsinghua University
Beijing 100084
People's Republic of China
E-mail: `mzhang@math.tsinghua.edu.cn`

Abstract. We show that any $C^{1+\mathrm{Dini}}$ expanding map f on any compact manifold admits a unique absolutely continuous invariant probability measure (a.c.i.p.m.) μ_0. Moreover, the system (f, μ_0) is exact and therefore is ergodic. From a nonexistence example of Góra and Schmitt [*Ergod. Th. Dynam. Syst.* **9** (1989), 101–113], one knows that such a Dini regularity on the Jacobian of f is the weakest condition ensuring the existence of a.c.i.p.m.

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1. Introduction and main results. It is well-known that a $C^{1+\alpha}$, $0 < \alpha \leq 1$, (uniformly) expanding map f on a compact manifold M admits a unique absolutely continuous invariant probability measure (a.c.i.p.m.) μ_0 with respect to the Lebesque measure m. Moreover, the system (f, μ_0) is exact, which means that the correlations

$$c_n(\varphi, \psi) = \int (\varphi \circ f^n)\psi d\mu_0 - \int \varphi d\mu_0 \int \psi d\mu_0$$

tend to zero for sufficiently regular observables φ, ψ. In fact, $c_n(\varphi, \psi)$ tend to zero exponentially when φ, $\psi \in C^\beta(M)$, $0 < \beta \leq 1$, are Hölder continuous. As a result, the central limit law holds for $\varphi \circ f^n$ when $\varphi \in C^\beta(M)$. The measure μ_0 is necessarily an SRB (Sinai-Ruelle-Bowen) measure by the Birkhoff ergodic theorem. For these conclusions and related topics on stochastic aspects of differentiable dynamics, see, e.g., the monograph of Viana [21] and the expository article of Baladi [3].

It is also known that the existence and nonexistence of a.c.i.p.m. for a map f do depend upon the following two aspects. The first is the 'hyperbolicity' of the systems f such as uniform expansivity ([1, 11, 18, 19, 21, 22]), uniform hyperbolicity ([6, 21]), almost hyperbolicity ([14, 15]), nonuniform hyperbolicity ([2, 4, 5, 15]) etc. The second aspect is the regularity of the Jacobian $|\det Df(x)|$. Some well-known regularity conditions are the smoothness of class C^2 of f ([1, 11, 17, 18]), the Hölder continuity of $|\det Df(x)|$ ([21] and the references therein), the bounded variation and bounded generalized variation of $|\det Df(x)|$ ([10, 13, 16, 19, 21]) etc.

This paper deals with the second aspect, i.e., under what weak regularity on $|\det Df(x)|$, a C^1 (uniform) expanding map f does admit an a.c.i.p.m.?

Let M be a compact connected Riemannian manifold without boundary. Let $f : M \to M$ be a C^1 expanding map, i.e., we can choose and henceforth fix such a Riemannian metric $\|\cdot\|$ on M that

$$\|Df(x) \cdot v\| \geq \sigma^{-1}\|v\| \quad \text{for all } x \in M \text{ and all } v \in T_xM, \tag{1.1}$$

where σ is some constant less than 1. The induced metric and the Lebesque measure on M are denoted by d and m respectively. As M is compact, M has a finite diameter $d(M) < \infty$ and a finite total volume $m(M) < \infty$. Such an expanding map f admits an a.c.i.p.m. if f is $C^{1+\alpha}$, $0 < \alpha \leq 1$, i.e., the modulus of continuity of $|\det Df(x)|$:

$$\omega(\delta) := \sup\left\{\Big||\det Df(x)| - |\det Df(y)|\Big| : x,\, y \in M,\ d(x,y) \leq \delta\right\} \tag{1.2}$$

is of order $O(\delta^\alpha)$ when $\delta \to 0$. In 1985 Collet and Eckmann proved in [8] the existence of a.c.i.p.m. for a one-dimensional piecewise expanding map f when $Df(x)$ has a weaker regularity (than the Hölder continuity):

$$\omega(\delta) = O(1/(1 + |\log\delta|)^\gamma), \qquad \text{where } \gamma > 1. \tag{1.3}$$

There is also a famous example of Góra and Schmitt [12] of one-dimensional piecewise expanding map f such that f admits *no* a.c.i.p.m. while the modulus of continuity of $Df(x)$ has the type

$$\omega(\delta) = K/(1 + |\log \delta|) \qquad \text{as } \delta \to 0. \tag{1.4}$$

Generally speaking, the smoothness of class C^1 for f cannot guarantee the existence of a.c.i.p.m. In fact, Quas [20] proved in 1999 that there is a residual G_σ subset of C^1 circle expanding maps such that any map in this subset admits *no* a.c.i.p.m.

Our answer to the regularity ensuring the existence of a.c.i.p.m. is:

THEOREM A. *Let f be a C^1 expanding map on a compact manifold M. If the modulus of continuity of $|\det Df(x)|$ (see (1.2)) satisfies the Dini condition, then f admits a unique a.c.i.p.m. μ_0.*

Here the Dini condition refers to the convergence of the following singular integral:

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty.$$

Note that Collet-Eckmann's condition (1.3) satisfies the Dini condition, while Góra-Schmitt's example (1.4) does not. As a result, Theorem A is best possible in the aspect of regularity conditions.

As for the decay of correlations for the system (f, μ_0), we will prove the following exactness result.

THEOREM B. *Let f be as in Theorem A. Then the system (f, μ_0) is exact. More precisely, if $\varphi \in L^1(m)$ and $\psi \in C^0(M)$, then $\lim_{n\to\infty} c_n(\varphi, \psi) = 0$.*

Our proofs for Theorems A and B are based on the convergence of the Perron-Frobenius operators [21]. Denote for short $D(x) = |\det Df(x)|$, $x \in M$, where the determinant is with respect to the specific Riemannian metric $\|\cdot\|$. Now the Perron-Frobenius operator associated with f, acting on some convenient space of functions $\varphi : M \to \mathbb{R}$, is defined by

$$(\mathcal{L}\varphi)(y) = \sum_{f(x)=y} \frac{\varphi(x)}{|\det Df(x)|} = \sum_{f(x)=y} \frac{\varphi(x)}{D(x)}.$$

The Perron-Frobenius operator $\mathcal{L}$ is the duality of the operator $U\psi := \psi \circ f$ in the following sense:

$$\int (\mathcal{L}\varphi)\psi dm = \int \varphi(U\psi) dm. \tag{1.5}$$

It is well-known that the (nonnegative) fixed point of $\mathcal{L}$ is the density of an absolutely continuous measure which is invariant under f, and vice versa. Such a technique is nowadays conventional (see [9, 13, 21]). However, as systems considered in this paper have relatively weak regularity, it seems that it is impossible to construct suitable cones of functions on which the Perron-Frobenius operator is a strict contraction with respect to the corresponding projective metric (as in the case of $C^{1+\alpha}$ systems). To overcome this, we have exploited more aspects of topological dynamical behavior of expanding maps.

The paper is organized as follows. In Section 2, we construct spaces and cones of functions for the Perron-Frobenius operators. Some necessary estimations are also established. Theorem A is proved in Section 3. In Section 4, Theorem B is proved and some discussion on decay of correlations is given.

2. Construction of function spaces. Our basis for construction of all spaces of functions is the space $C^0(M)$ of all continuous functions on M. Note that $C^0(M)$ with the maximum norm $\|\cdot\|_{C^0}$ is a Banach space. The Perron-Frobenius operator $\mathcal{L}$ leaves the convex cone of all positive continuous functions in $C^0(M)$ invariant. To normalize such a cone, we introduce the following convex set in $C^0(M)$:

$$C_+ = \left\{ \varphi \in C^0(M) : \min_M \varphi > 0 \text{ and } \int_M \varphi dm = 1 \right\}.$$

Note that C_+ is not complete with respect to $\|\cdot\|_{C^0}$. However, C_+ is complete with respect to the projective metric θ_+ on C_+, see [9, 21]. Explicitly, for any φ, $\psi \in C_+$, the projective metric $\theta_+(\varphi,\psi)$ is given by

$$\begin{aligned}
\alpha_+(\varphi,\psi) &= \min\left\{ \frac{\psi(x)}{\varphi(x)} : x \in M \right\}, \\
\beta_+(\varphi,\psi) &= \max\left\{ \frac{\psi(x)}{\varphi(x)} : x \in M \right\}, \\
\theta_+(\varphi,\psi) &= \log \frac{\beta_+(\varphi,\psi)}{\alpha_+(\varphi,\psi)} = \log \max \left\{ \frac{\psi(x)\varphi(y)}{\varphi(x)\psi(y)} : x,\, y \in M \right\}
\end{aligned}$$

LEMMA 2.1 (C_+, θ_+) *is a complete metric space.*

Proof. This is essentially Proposition 2.6 in [21]. For later use we sketch here the proof.

Suppose that $\{\varphi_n\} \subset C_+$ is a θ_+-Cauchy sequence. It can be proved that there exists a constant $K_0 \geq 1$ such that

$$K_0^{-1} \leq \min \varphi_n \leq \max \varphi_n \leq K_0 \tag{2.1}$$

for all n. This implies that

$$\|\varphi_m - \varphi_n\|_{C^0} \leq K_0 \left(\exp(\theta_+(\varphi_m, \varphi_n)) - 1 \right) \tag{2.2}$$

for all m, n. So $\{\varphi_n\}$ is a Cauchy sequence in $C^0(M)$ because $\theta_+(\varphi_m,\varphi_n) \to 0$ as m, $n \to \infty$. As a result, $\{\varphi_n\}$ has a limit φ_0 in $C^0(M)$. It is obvious that $\int \varphi_0 dm = 1$. By (2.1),

$\min \varphi_0 \geq K_0^{-1} > 0$. Thus $\varphi_0 \in C_+$. Furthermore, $\{\varphi_n\}$ is actually convergent to φ_0 in the space (C_+, θ_+) because

$$\left|\frac{\varphi_n(x)}{\varphi_0(x)} - 1\right| = \frac{|\varphi_n(x) - \varphi_0(x)|}{\varphi_0(x)} \leq K_0 \|\varphi_n - \varphi_0\|_{C^0}, \qquad x \in M,$$

and

$$\theta_+(\varphi_n, \varphi_0) = \log \max_{x,y\in M} \frac{\varphi_n(x)}{\varphi_0(x)} \frac{\varphi_0(y)}{\varphi_n(y)} \leq \log \frac{1 + K_0\|\varphi_n - \varphi_0\|_{C^0}}{1 - K_0\|\varphi_n - \varphi_0\|_{C^0}} \to 0. \tag{2.3}$$

□

By (1.5), $\mathcal{L}$ leaves C_+ invariant and acts on C_+ affinely. As $\mathcal{L}$ maps C_+ into itself, $\mathcal{L}$ does not expand the projective metric θ_+, i.e.,

$$\theta_+(\mathcal{L}\varphi, \mathcal{L}\psi) \leq \theta_+(\varphi, \psi), \qquad \varphi,\, \psi \in C_+,$$

cf. [21]. Thus $\mathcal{L} : (C_+, \theta_+) \to (C_+, \theta_+)$ is continuous.

As for the Hölder continuous functions of exponents $0 < \alpha \leq 1$, we introduce some Hölder-like continuous functions on M.

Definition 2.1 Let $h : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a nondecreasing continuous function such that $h(0) = 0$. We call a function $\varphi : M \to \mathbb{R}$ is *Hölder continuous with respect to* h, or simply, h-*Hölder continuous*, if

$$\sup\{|\varphi(x) - \varphi(y)|/h(d(x, y)) : x,\, y \in M,\, 0 < d(x, y) \leq \delta_0\} < \infty.$$

Remark 2.1 Although the h-Hölder continuity in above definition is locally defined, compactness and connectedness of M show that the h-Hölder continuity is actually global on M. In fact, one can prove as in [21] that there exists a constant $A = A(\delta_0, M, h) > 0$ such that if φ is h-Hölder continuous as in Definition 2.1 then

$$\begin{aligned} &\sup\{|\varphi(x) - \varphi(y)|/h(d(x, y)) : x,\, y \in M,\, x \neq y\} \\ &\leq A \sup\{|\varphi(x) - \varphi(y)|/h(d(x, y)) : x,\, y \in M,\, 0 < d(x, y) \leq \delta_0\} < \infty. \end{aligned} \tag{2.4}$$

Denote by $C^h(M)$ the collection of all h-Hölder continuous functions on M. Then $C^h(M)$ is a linear subspace of $C^0(M)$. It is easy to check that if $\varphi,\, \psi \in C^h(M)$ then $\varphi\psi \in C^h(M)$, and if $\varphi \in C^h(M)$ satisfies $\varphi(x) \neq 0$ for all x then $1/\varphi$ is also in $C^h(M)$. Thus $C^h(M)$ is actually an algebra.

Definition 2.2 We call a continuous nondecreasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(0) = 0$, a *Dini-function* if the following singular integral is convergent:

$$\int_0^1 \frac{h(s)}{s} ds < \infty. \tag{2.5}$$

Such a terminology comes from Fourier analysis [23].

EXAMPLE 2.1 Let $0 < \alpha \leq 1$. The function $\ell_\alpha(t) = t^\alpha$ satisfies (2.5) and is thus a Dini-function. In this case, $C^{\ell_\alpha}(M)$ is just the usual space $C^\alpha(M)$ of α-Hölder continuous functions on M.

EXAMPLE 2.2 For any integer $k \geq 1$ and any number $p > 1$, define a function

$$\ell_{k,p}(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{1}{\left(\log^{[k]}(1/t)\right)^p \prod_{i=1}^{k-1} \log^{[i]}(1/t)}, & \text{if } 0 < t \ll 1, \end{cases}$$

where

$$\log^{[k]} s = \underbrace{\log \cdots \log}_{k} s, \qquad s \gg 1.$$

The function $\ell_{k,p}$ is a Dini-function because in this case the singular integral (2.5) corresponds to the following convergent one:

$$\int_0 \frac{dt}{t\left(\log^{[k]}(1/t)\right)^p \prod_{i=1}^{k-1} \log^{[i]}(1/t)} = \int^\infty \frac{ds}{s\left(\log^{[1]} s\right) \cdots \left(\log^{[k-1]} s\right)\left(\log^{[k]} s\right)^p} < \infty.$$

Next let h be a Dini-function. For any given $0 < \nu < 1$, define a function $h_\nu : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h_\nu(t) = \nu \sum_{i=1}^\infty h(\nu^i t). \tag{2.6}$$

Such a definition is simply based on the following equality:

$$h_\nu(\nu t) + \nu h(\nu t) \equiv h_\nu(t). \tag{2.7}$$

The convergence of $h_\nu(t)$ in (2.6) is only ensured by (2.5) because

$$\begin{aligned} h_\nu(t) &= \nu \sum_{i=1}^\infty h(\nu^i t) \\ &\leq \nu \int_0^\infty h(\nu^s t) ds \\ &= \frac{\nu}{-\log \nu} \int_0^t \frac{h(u)}{u} du < \infty. \end{aligned}$$

A lower bound for $h_\nu(t)$ can also be given similarly. In fact one has

$$\frac{\nu}{-\log \nu} \int_0^{\nu t} \frac{h(u)}{u} du \leq h_\nu(t) \leq \frac{\nu}{-\log \nu} \int_0^t \frac{h(u)}{u} du. \tag{2.8}$$

It is obvious that $h_\nu(t)$ is increasing with respect to t or ν. Note that h_ν is not necessarily a Dini-function.

EXAMPLE 2.3 Let $h = \ell_\alpha$ in Example 2.1. Then $h_\nu(t) = (\nu^{1+\alpha}/(1-\nu^\alpha))t^\alpha$. Thus $C^{h_\nu}(M)$ is independent of ν and is just the space $C^\alpha(M)$. When h is the $\ell_{k,p}$ in Example 2.2, by (2.8) one has

$$\begin{aligned} h_\nu(t) &\le \frac{\nu}{-\log\nu}\int_0^t \frac{\ell_{k,p}(u)}{u}du \\ &= \frac{\nu}{-(p-1)\log\nu}\frac{1}{\left(\log^{[k]}(1/t)\right)^{p-1}} \end{aligned}$$

for $0 < t \ll 1$. Similarly

$$h_\nu(t) \ge \frac{\nu}{-(p-1)\log\nu}\frac{1}{\left(\log^{[k]}(1/\nu t)\right)^{p-1}}.$$

Thus $h_\nu(t)$ is of order

$$\frac{\nu}{-(p-1)\log\nu}\frac{1}{\left(\log^{[k]}(1/t)\right)^{p-1}}$$

as $t \to 0+$. The space $C^{h_\nu}(M)$ is also independent of ν in this case and is given by

$$\begin{aligned} C^{h_\nu}(M) = C^{[k,p]}(M) := \Big\{&\varphi \in C^0(M) : \\ &\sup_{0<d(x,y)\le\delta_0} \left(\log^{[k]}(1/d(x,y))\right)^{p-1}|\varphi(x)-\varphi(y)| < \infty\Big\}, \end{aligned}$$

where $\delta_0 > 0$ is a small constant.

Let now $f : M \to M$ be an C^1 expanding map. Assume that (1.1) holds for some constant $\sigma < 1$. Then f expands locally the metric d: There exists a constant $\delta_0 > 0$ such that

$$d(f(x), f(y)) \ge \sigma^{-1}d(x,y), \quad \forall\, x,\, y \in M \text{ with } d(x,y) \le \delta_0. \tag{2.9}$$

Here the constant σ may be a little bit larger than that in (1.1). It follows from (1.1) that $D(x) \ge \sigma^{-1}$ for all x. Suppose further that the modulus of continuity of $D(x) := |\det Df(x)|$ is a Dini-function. So there is some Dini-function h such that

$$\sup\{|D(x)-D(y)|/h(d(x,y)) : x,\, y \in M,\ 0 < d(x,y) \le \delta_0\} \le 1. \tag{2.10}$$

We construct a family of convex sets $\{C_{h_\nu}\}_{0<\nu<1}$ as follows. Let

$$C_{h_\nu} = \{\varphi \in C_+ : \log\varphi(x) \text{ satisfies condition (2.11)}\},$$

where condition (2.11) means that $\log\varphi \in C^{h_\nu}(M)$ and

$$\sup\{|\log\varphi(x) - \log\varphi(y)|/h_\nu(d(x,y)) : x,\, y \in M,\ 0 < d(x,y) \le \delta_0\} \le 1. \tag{2.11}$$

It is obvious that C_{h_ν} is a convex subset for each $0 < \nu < 1$. We will prove that the family $\{C_{h_\nu}\}$ of convex sets has the following properties:

(P1) $\overline{C_{h_\nu}} \cap \left(-\overline{C_{h_\nu}}\right) = \{0\}$.

(P2) If $0 < \nu < \bar{\nu} < 1$ then $C_{h_\nu} \subset C_{h_{\bar{\nu}}} \subset C_+$.

(P3) C_{h_ν} is compact in (C_+, θ_+) for each $0 < \nu < 1$.

(P4) $\mathcal{L}$ maps C_{h_ν} to itself whenever $\sigma \le \nu < 1$.

(P5) The convex set C_{h_ν} spans the space $C^{h_\nu}(M)$ for each $0 < \nu < 1$.

It is easy to prove properties (P1) and (P2). Now we prove the compactness property (P3).

Lemma 2.2 *Property* (P3) *is satisfied for all* $0 < \nu < 1$.

Proof. Let $\{\varphi_n\}$ be any sequence in C_{h_ν}. From (2.4) in Remark 2.1,

$$\sup\{|\log\varphi_n(x) - \log\varphi_n(y)|/h_\nu(d(x,y)) : x,\, y \in M,\ x \ne y\} \le A < \infty,$$

where A is a constant independent of n. As a result,

$$\max\varphi_n/\min\varphi_n \le \exp(Ah_\nu(\operatorname{diam} M)) < \infty.$$

Since $\int \varphi_n dm = 1$, one has

$$\min\varphi_n \le 1/m(M) \le \max\varphi_n, \qquad n \ge 1.$$

These show that there exists a constant $K_1 > 0$ such that

$$K_1^{-1} \le \min\varphi_n \le \max\varphi_n \le K_1, \qquad n \ge 1.$$

So the sequence $\{\varphi_n\}$ is bounded in the space $C^0(M)$. By the definition of C_{h_ν}, $\{\varphi_n\}$ is also equi-continuous on M. Now the Ascoli-Arzela theorem shows that $\{\varphi_n\}$ has a subsequence converging to some φ_0 in $C^0(M)$. The limit function φ_0 satisfies $\min\varphi_0 \ge K_1^{-1} > 0$ and is in C_{h_ν} because C_{h_ν} is a closed subset of $C^0(M)$. Furthermore, $\{\varphi_n\}$ is actually convergent to φ_0 in the space (C_+, θ_+), cf. (2.3). This proves that C_{h_ν} is a compact subset of (C_+, θ_+). □

Let now α_ν, β_ν and $\theta_\nu = \log\beta_\nu/\alpha_\nu$ be the corresponding objects in the projective metric for C_{h_ν}. Explicitly, for any $\varphi_1,\ \varphi_2 \in C_{h_\nu}$,

$$\alpha_\nu(\varphi_1,\varphi_2) = \inf\left\{\frac{\varphi_2(x)}{\varphi_1(x)},\ \frac{\exp\left(h_\nu(d(x,y))\right)\varphi_2(x) - \varphi_2(y)}{\exp\left(h_\nu(d(x,y))\right)\varphi_1(x) - \varphi_1(y)} : x,\, y \in M,\ 0 < d(x,y) \le \delta_0\right\},$$

$$\beta_\nu(\varphi_1,\varphi_2) = \sup\left\{\frac{\varphi_2(x)}{\varphi_1(x)},\ \frac{\exp\left(h_\nu(d(x,y))\right)\varphi_2(x) - \varphi_2(y)}{\exp\left(h_\nu(d(x,y))\right)\varphi_1(x) - \varphi_1(y)} : x,\, y \in M,\ 0 < d(x,y) \le \delta_0\right\}.$$

Now we prove the invariance property (P4).

Lemma 2.3 *Property* (P4) *is satisfied for all* $\sigma \le \nu < 1$.

Proof. Let y_1, $y_2 \in M$ be such that $d(y_1, y_2) \leq \delta_0$. Denote $f^{-1}(y_j) = \{x_{j1}, \cdots, x_{jk}\}$, $j = 1, 2$, where $k = \# f^{-1}(y)$ is finite and is independent of $y \in M$. By the expansivity (2.9) one has

$$d(x_{1i}, x_{2i}) \leq \sigma d(y_1, y_2), \qquad 1 \leq i \leq k, \tag{2.12}$$

because $d(y_1, y_2) \leq \delta_0$.

Let $\varphi \in C_{h_\nu}$. Then $\mathcal{L}\varphi \in C^0(M)$ and $\min \mathcal{L}\varphi > 0$ because f is onto M. By (1.5), $\mathcal{L}\varphi$ satisfies also $\int \mathcal{L}\varphi dm = 1$. Suppose now that $\nu \in [\sigma, 1)$. Then

$$\begin{aligned}
(\mathcal{L}\varphi)(y_1) &= \sum_{i=1}^{k} \frac{\varphi(x_{1i})}{D(x_{1i})} \\
&= \sum_{i=1}^{k} \varphi(x_{2i}) \exp\left[\log \varphi(x_{1i}) - \log \varphi(x_{2i})\right] \frac{\exp\left[\log D(x_{2i}) - \log D(x_{1i})\right]}{D(x_{2i})} \\
&\leq \sum_{i=1}^{k} \frac{\varphi(x_{2i})}{D(x_{2i})} \exp\left[\log \varphi(x_{1i}) - \log \varphi(x_{2i})\right] \exp\left[\nu |D(x_{2i}) - D(x_{1i})|\right] \\
&\leq \sum_{i=1}^{k} \frac{\varphi(x_{2i})}{D(x_{2i})} \exp\left[h_\nu(d(x_{1i}, x_{2i})) + \nu h(d(x_{1i}, x_{2i}))\right] \\
&\leq \sum_{i=1}^{k} \frac{\varphi(x_{2i})}{D(x_{2i})} \exp\left[h_\nu(\nu d(y_1, y_2)) + \nu h(\nu d(y_1, y_2))\right] \\
&= (\mathcal{L}\varphi)(y_2) \exp\left[h_\nu(d(y_1, y_2))\right],
\end{aligned}$$

where the inequalities $1/D(x) \leq \sigma \leq \nu$, (2.12) and $d(x_{1i}, x_{2i}) \leq \sigma d(y_1, y_2) \leq \nu d(y_1, y_2)$, the monotonicity of $h_\nu(t)$ and the equality (2.7) are used. This proves that $\psi(y) := (\mathcal{L}\varphi)(y)$ satisfies

$$|\log \psi(y_1) - \log \psi(y_2)| \leq h_\nu(d(y_1, y_2)) \qquad \text{when } 0 < d(y_1, y_2) \leq \delta_0,$$

i.e., $\psi = \mathcal{L}\varphi \in C_{h_\nu}$. □

LEMMA 2.4 *Property* (P5) *is satisfied for all* $0 < \nu < 1$.

Proof. Let $\varphi \in C^{h_\nu}(M)$. Denote

$$B := \sup\{|\varphi(x) - \varphi(y)|/h_\nu(d(x, y)) : 0 < d(x, y) \leq \delta_0\} < \infty.$$

Let $\varphi_\pm = \varepsilon(|\varphi| \pm \varphi) + 1$, where $\varepsilon > 0$ is sufficiently small. Let $\gamma_\pm > 0$ be constants such that

$$\int \gamma_\pm \varphi_\pm dm = 1.$$

Then $\gamma_\pm \varphi_\pm \in C_+$. Moreover, if $0 < d(x, y) \leq \delta_0$, then

$$\left|\log(\gamma_\pm \varphi_\pm(x)) - \log(\gamma_\pm \varphi_\pm(y))\right| / h_\nu(d(x, y))$$

$$
\begin{aligned}
&= \left|\log(\varphi_{\pm}(x)) - \log(\varphi_{\pm}(y))\right| / h_\nu(d(x,y)) \\
&= \frac{1}{\eta_\pm} \left|\varphi_\pm(x) - \varphi_\pm(y)\right| / h_\nu(d(x,y)) \\
&\le \frac{2\varepsilon}{\eta_\pm} \left|\varphi(x) - \varphi(y)\right| / h_\nu(d(x,y)) \\
&\le 2\varepsilon B,
\end{aligned}
$$

where $\eta_\pm = \eta_\pm(x,y)$ is between $\varphi_\pm(x)$ and $\varphi_\pm(y)$ and therefore $\eta_\pm \ge 1$. If one takes $\varepsilon \le 1/(2B)$, then $\psi_\pm := \gamma_\pm \varphi_\pm \in C_{h_\nu}$. Consequently,

$$\varphi = \frac{1}{2\varepsilon}\left(\psi_+/\gamma_+ - \psi_-/\gamma_-\right) \in \operatorname{span}(C_{h_\nu}).$$

This proves (P5). □

3. Existence of a.c.i.p.m. After properties (P1)–(P5) for convex sets $\{h_\nu\}$ and the Perron-Frobenius operator $\mathcal{L}$ were established in Section 2, we can now give the existence of a.c.i.p.m. Theorem A is contained in the following result.

Theorem 3.1 *Let $f : M \to M$ be a C^1 expanding map. Assume that $D(x) = |\det Df(x)|$ satisfies for some Dini-function h that*

$$A_0 := \sup\{|D(x) - D(y)|/h(d(x,y)) : x,\, y \in M,\ 0 < d(x,y) \le \delta_0\} < \infty. \tag{3.1}$$

Then

(i) *f admits a unique a.c.i.p.m. μ_0. Moreover, the density $\varphi_0 = d\mu_0/dm$ is continuous, strictly positive and φ_0 has the regularity that $\varphi_0 \in C^{h_\sigma}(M)$, i.e.,*

$$\sup\{|\varphi(x) - \varphi(y)|/h_\sigma(d(x,y)) : x,\, y \in M,\ 0 < d(x,y) \le \delta_0\} < \infty. \tag{3.2}$$

(ii) *The system (f, μ_0) is ergodic.*

Proof. We give the proof in several steps.

Existence. We prove the existence of a.c.i.p.m. $\mu_0 = \varphi_0 m$. Without loss of generality, assume that the constant A_0 in (3.1) is 1. Otherwise one may replace the Dini-function h by $A_0 h$. So condition (2.10) is satisfied for h. Recall that $\mathcal{L} : (C_+, \theta_+) \to (C_+, \theta_+)$ is a continuous operator. From (P3), the set C_{h_σ} is a compact convex subset of (C_+, θ_+). By (P4), $\mathcal{L}$ maps C_{h_σ} into itself. Now the Schauder fixed point theorem shows that $\mathcal{L}$ has at least one fixed point φ_0 in C_{h_σ} which gives an a.c.i.p.m. $\mu_0 = \varphi_0 m$ of f. In fact, let $\varphi \in C_{h_\sigma}$ be any given function. Then the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}^j \varphi \tag{3.3}$$

has a subsequence converging in C_+ to some $\varphi_0 \in C_{h_\sigma}$. Now such a φ_0 is the desired fixed point of $\mathcal{L}$. Note that the regularity (3.2) holds because $\varphi_0 \in C_{h_\sigma} \subset C^{h_\sigma}(M)$.

Uniqueness. We prove at the present stage that $\mathcal{L}$ has only one fixed point in C_{h_σ}. Let $\varphi_0,\ \psi_0 \in C_{h_\sigma}$ be fixed points of $\mathcal{L}$. Then $\mathcal{L}^n\varphi_0 = \varphi_0$ and $\mathcal{L}^n\psi_0 = \psi_0$. As a result,

$$\alpha_+(\mathcal{L}^n\varphi_0, \mathcal{L}^n\psi_0) = \alpha_+(\varphi_0, \psi_0), \qquad \beta_+(\mathcal{L}^n\varphi_0, \mathcal{L}^n\psi_0) = \beta_+(\varphi_0, \psi_0), \qquad n \geq 0.$$

On the other hand, let $x_0 \in M$ be such that

$$\beta_+(\varphi_0, \psi_0) = \max_{x \in M} \frac{\varphi_0(x)}{\psi_0(x)} = \frac{\varphi_0(x_0)}{\psi_0(x_0)}.$$

Then

$$\begin{aligned}
\beta_+(\varphi_0, \psi_0) &= \beta_+(\mathcal{L}^n\varphi_0, \mathcal{L}^n\psi_0) \\
&= \frac{\mathcal{L}^n\varphi_0(x_0)}{\mathcal{L}^n\psi_0(x_0)} \\
&= \frac{\sum_{f^n(y)=x_0} \varphi_0(y)/|\det Df^n(y)|}{\sum_{f^n(y)=x_0} \psi_0(y)/|\det Df^n(y)|} \\
&\leq \frac{\sum_{f^n(y)=x_0} \beta_+(\varphi_0, \psi_0)\psi_0(y)/|\det Df^n(y)|}{\sum_{f^n(y)=x_0} \psi_0(y)/|\det Df^n(y)|} \\
&= \beta_+(\varphi_0, \psi_0).
\end{aligned}$$

This implies that

$$\varphi_0(y) = \beta_+(\varphi_0, \psi_0)\psi_0(y) \tag{3.4}$$

for all y in the *inverse orbit*:

$$\mathcal{O}^-(x_0) := \{y \in M : \text{there exists some integer } n \geq 0 \text{ such that } f^n(y) = x_0\}.$$

As f is expanding, $\mathcal{O}^-(x_0)$ is dense in the whole manifold M. Now (3.4) implies that

$$\varphi_0(y) = \beta_+(\varphi_0, \psi_0)\psi_0(y) \qquad \text{for all } y \in M$$

because φ_0 and ψ_0 are continuous. Consequently $\beta_+(\varphi_0, \psi_0) = 1$ because $\int \varphi_0 dm = \int \psi_0 dm = 1$. Therefore $\varphi_0 = \psi_0$.

Ergodicity. We prove ergodicity of the system (f, μ_0). Let $A \subset M$ be any invariant set under f and χ_A be its characteristic function. Then $\chi_A \circ f^j = \chi_A$ for all $j \geq 0$. Let φ be any given function in C_{h_σ}. As in the proof of the existence, for any sequence $\{n_k\}$ of integers with $n_k \to \infty$, the sequence

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathcal{L}^j\varphi \tag{3.5}$$

has a subsequence converging in the space (C_+, θ_+) to some fixed point of $\mathcal{L}$ in C_{h_σ}. As f has a unique fixed point φ_0 in C_{h_σ}, the sequence (3.5) has a subsequence converging in (C_+, θ_+) to φ_0. As a result, the sequence (3.3) itself converges in (C_+, θ_+) to φ_0, i.e.,

$$\theta_+\Big(\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j\varphi,\, \varphi_0\Big) \to 0 \qquad \text{as } n \to \infty.$$

One then has

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j\varphi - \varphi_0\Big\|_{C^0} = 0, \tag{3.6}$$

see (2.2) in the proof of Lemma 2.1. This shows that

$$\int\Big(\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j\varphi - \varphi_0\Big)\chi_A dm \to 0. \tag{3.7}$$

Note that $\int \varphi dm = 1$. Then

$$\begin{aligned}
&\int\Big(\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j\varphi - \varphi_0\Big)\chi_A dm \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\int(\mathcal{L}^j\varphi)\chi_A dm - \Big(\int \chi_A d\mu_0\Big)\Big(\int \varphi dm\Big) \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\int \varphi(\chi_A \circ f^j) dm - \int\Big(\int \chi_A d\mu_0\Big)\varphi dm \\
&= \int \varphi\chi_A dm - \int\Big(\int \chi_A d\mu_0\Big)\varphi dm \\
&= \int\Big[\chi_A - \int \chi_A d\mu_0\Big]\varphi dm.
\end{aligned}$$

Now (3.7) shows that

$$\int\Big[\chi_A - \int \chi_A d\mu_0\Big]\varphi dm = 0 \tag{3.8}$$

for all $\varphi \in C_{h_\sigma}$. By the linearity, (3.8) holds for all $\varphi \in \mathrm{span}\,(C_{h_\sigma}) = C^{h_\sigma}(M)$, cf. Lemma 2.4. Thus we get from (3.8) that

$$\chi_A(x) = \int \chi_A d\mu_0 = \mu_0(A) \qquad m\text{-a.e. } x.$$

Consequently, $\mu_0(A)$ is either 0 or 1, and μ_0 is an ergodic measure.

Uniqueness revisited. As we have proved that μ_0 is equivalent to m and (f, μ_0) is ergodic, it can be proved as in [21] that f has a unique a.c.i.p.m. In fact, let μ be an invariant probability measure such that $\mu \ll m$. Then $\mu \ll \mu_0$. Since μ_0 is ergodic, one has $\mu = \mu_0$. □

In Theorem 4.1 of the next section, we will prove that the system (f, μ_0) is actually exact.

4. Decay of correlations. In this section, we give the proof of Theorem B. Namely we will prove that correlations

$$c_n(\varphi,\psi)=\int(\varphi\circ f^n)\psi d\mu_0-\Big(\int\varphi d\mu_0\Big)\Big(\int\psi d\mu_0\Big) \tag{4.1}$$

always decay to zero as $n\to\infty$ for all continuous observables $\varphi,\ \psi\in C^0(M)$. As a result, the system (f,μ_0) is exact. It is well-known that if $c_n(\varphi,\psi)$ has further decay rates such as exponential rate, then the central limit law holds for $\{\varphi\circ f^n\}$ as a sequence of random variables. It thus is an important problem to examine the decay rates for correlations. We refer to [3, 7, 14, 21] for this topic.

As $\mu_0=\varphi_0 m$ in Theorem B is absolutely continuous, the following equalities

$$\begin{aligned}c_n(\varphi,\psi)&=\int(\varphi\circ f^n)\psi d\mu_0-\int\varphi d\mu_0\int\psi d\mu_0\\&=\int(\varphi\circ f^n)(\psi\varphi_0)dm-\int\varphi\varphi_0 dm\int\psi\varphi_0 dm\\&=\int\varphi\Big[\mathcal{L}^n(\psi\varphi_0)-\Big[\int\psi\varphi_0 dm\Big]\varphi_0\Big]dm \qquad (4.2)\end{aligned}$$

show that the decay of correlations can be estimated by the convergence of the iterates of Perron-Frobenius operator $\mathcal{L}$:

$$|c_n(\varphi,\psi)|\le\Big\|\mathcal{L}^n(\psi\varphi_0)-\Big[\int\psi\varphi_0 dm\Big]\varphi_0\Big\|_{C^0}\|\varphi\|_{L^1(m)}\,. \tag{4.3}$$

Therefore the most important matter is to examine the decay of the C^0-norms of

$$D_n(\psi):=\mathcal{L}^n\psi-\Big[\int\psi dm\Big]\varphi_0. \tag{4.4}$$

When ψ are in convex sets C_{h_ν}, the decay of $D_n(\psi)$ can be controlled using the quantities

$$\Theta_n(\psi):=\theta_+(\mathcal{L}^n\psi,\varphi_0), \tag{4.5}$$

cf. (2.2) in the proof of Lemma 2.1.

Theorem B is contained in the following result.

THEOREM 4.1 *Let f be as in Theorem A and $D(x)=|\det Df(x)|$ satisfies* (3.1) *for some Dini-function h. Then for any $\varphi\in L^1(m)$ and any $\psi\in C^0(M)$, one has*

$$\lim_{n\to\infty}c_n(\varphi,\psi)=0. \tag{4.6}$$

Proof. By (4.2)–(4.4), we see that (4.6) can be achieved by proving that

$$\lim_{n\to\infty}\Big\|\mathcal{L}^n\psi-\varphi_0\Big\|_{C^0}=0 \tag{4.7}$$

for all $\psi\in C^0(M)$.

The proof is based on the following important property for the operator $\mathcal{L}$: If $\varphi,\ \psi \in C^0(M)$ with $\min\varphi > 0$, $\min\psi > 0$, then

$$\max_{x\in M}\frac{\mathcal{L}\varphi}{\mathcal{L}\psi}(x) \le \max_{x\in M}\frac{\varphi}{\psi}(x), \tag{4.8}$$

$$\min_{x\in M}\frac{\mathcal{L}\varphi}{\mathcal{L}\psi}(x) \ge \min_{x\in M}\frac{\varphi}{\psi}(x). \tag{4.9}$$

Let us first prove (4.7) for $\psi \in C_{h_\sigma}$. Let $\psi_n = \mathcal{L}^n\psi \in C_{h_\sigma}$. Set

$$M_n = \max_{x\in M}\frac{\psi_n(x)}{\varphi_0(x)} \qquad \text{and} \qquad m_n = \min_{x\in M}\frac{\psi_n(x)}{\varphi_0(x)}.$$

It follows from (4.8) and (4.9) that M_n decreases and m_n increases when n increases. Let

$$M_\infty = \lim_{n\to\infty} M_n, \qquad m_\infty = \lim_{n\to\infty} m_n.$$

Since C_{h_σ} is compact in (C_+,θ_+), for any sequence $n_k \to \infty$, the sequence $\mathcal{L}^{n_k}\psi$ has a subsequence $\mathcal{L}^{n_{k_j}}\psi$ converging in (C_+,θ_+) to some $\psi_0 \in C_{h_\sigma}$:

$$\lim_{j\to\infty}\psi_{n_{k_j}} = \psi_0.$$

For any $m \in \mathbb{N}$, one has

$$\begin{aligned}\max\frac{\mathcal{L}^m\psi_0}{\varphi_0} &= \max\frac{\lim_{j\to\infty}\psi_{m+n_{k_j}}}{\varphi_0}\\ &= \lim_{j\to\infty}\max\frac{\psi_{m+n_{k_j}}}{\varphi_0} = \lim_{j\to\infty} M_{m+n_{k_j}}\\ &= M_\infty = \max\frac{\psi_0}{\varphi_0}.\end{aligned} \tag{4.10}$$

In the following we proceed the proof as for that of the uniqueness in Theorem 3.1. Let $x_0 \in M$ be such that

$$M_\infty = \max\frac{\psi_0}{\varphi_0} = \frac{\psi_0(x_0)}{\varphi_0(x_0)}.$$

For any $m \in \mathbb{N}$, it follows from (4.10) that

$$\begin{aligned}M_\infty = \frac{\psi_0(x_0)}{\varphi_0(x_0)} &= \max\frac{\mathcal{L}^m\psi_0(x_0)}{\varphi_0(x_0)}\\ &= \frac{\sum_{f^m(y)=x_0}\psi_0(y)/|\det Df^m(y)|}{\sum_{f^m(y)=x_0}\varphi_0(y)/|\det Df^m(y)|}\\ &\le \frac{\sum_{f^m(y)=x_0}M_\infty\varphi_0(y)/|\det Df^m(y)|}{\sum_{f^m(y)=x_0}\varphi_0(y)/|\det Df^m(y)|}\\ &= M_\infty.\end{aligned}$$

This implies that

$$\psi_0(y) = M_\infty\varphi_0(y), \qquad \text{for all } y \in \mathcal{O}^-(x_0).$$

As a result, one has

$$\psi_0(y) = M_\infty \varphi_0(y), \qquad y \in M$$

because of the continuity of ψ_0, φ_0 and of the density of $\mathcal{O}^-(x_0)$ in M. As $\int \psi_0 dm = \int \varphi_0 dm = 1$, we have $M_\infty = 1$ and $\psi_0 = \varphi_0$.

We have proved that any sequence $\{\mathcal{L}^{n_k}\psi\}$ has a subsequence converging in (C_+, θ_+) to φ_0. Consequently, the sequence $\{\mathcal{L}^n\psi\}$ itself converges in (C_+, θ_+) to φ_0. It now follows from (2.2) that (4.7) holds when $\psi \in C_{h_\sigma}$.

Now we prove (4.7) for general $\psi \in C_+$. For any $\varepsilon > 0$, choose functions $\psi_\pm \in C^{h_\sigma}(M)$ such that

$$\psi_- \le \psi \le \psi_+, \qquad \|\psi_+ - \psi_-\|_{C^0} < \frac{\varepsilon}{2}. \tag{4.11}$$

Applying (4.7) to functions $\psi_\pm / \int \psi_\pm dm \in C_{h_\sigma}$, one has

$$\lim_{n\to\infty} \Big\| \mathcal{L}^n \psi_\pm - \Big[\int \psi_\pm dm\Big]\varphi_0 \Big\|_{C^0} = 0.$$

Thus there exists $N \gg 1$ such that

$$\Big[\int \psi_- dm - \frac{\varepsilon}{2}\Big]\varphi_0 < \mathcal{L}^n\psi_- \le \mathcal{L}^n\psi_+ < \Big[\int \psi_+ dm + \frac{\varepsilon}{2}\Big]\varphi_0, \qquad n \ge N. \tag{4.12}$$

It now follows from (4.11) and (4.12) that

$$[1-\varepsilon]\varphi_0 < \Big[\int \psi_- dm - \frac{\varepsilon}{2}\Big]\varphi_0 < \mathcal{L}^n\psi_- \le \mathcal{L}^n\psi \le \mathcal{L}^n\psi_+ \le \Big[\int \psi_+ dm + \frac{\varepsilon}{2}\Big]\varphi_0 < [1+\varepsilon]\varphi_0.$$

This implies that

$$\|\mathcal{L}^n\psi - \varphi_0\|_{C^0} < \varepsilon \|\varphi_0\|_{C^0}, \qquad n \ge N.$$

Namely, (4.7) holds for all $\psi \in C_+$.

Finally, since (4.7) is linear in ψ, one then knows that (4.7) holds for all $\psi \in \mathrm{span}(C_+) = C^0(M)$. This proves the theorem. □

When an expanding map f is $C^{1+\alpha_0}$ for some $0 < \alpha_0 \le 1$, the condition (3.1) is satisfied for $h(t) = \ell_\alpha(t) = at^\alpha$, where $a > 0$ and $0 < \alpha \le \alpha_0$. In this case, it can be proved that the operator $\mathcal{L}$ maps (C_{h_ν}, θ_ν), $\sigma \le \nu < 1$, into $C_{h_{\lambda\nu}}$, where $0 < \lambda < 1$ is some constant independent of ν. As $C_{h_{\lambda\nu}}$ has finite diameter in (C_{h_ν}, θ_ν), $\mathcal{L}$ maps (C_{h_ν}, θ_ν) $(\sigma \le \nu < 1)$ into itself in a contraction way, cf. [21]. One thus has

$$\theta_+(\mathcal{L}^n\psi, \varphi_0) \le \theta_\nu(\mathcal{L}^n\psi, \varphi_0) \le \Lambda^n \theta_\nu(\psi, \varphi_0) \tag{4.13}$$

for all $n \ge 0$ and all $\psi \in C_{h_\nu}$, where $\Lambda \in (0,1)$ is some constant. Now (4.3), (4.4), (4.5) and (4.13) show that the correlations $c_n(\varphi, \psi)$ in (4.1) have an exponential decay in the spaces $C^\alpha(M)$, $0 < \alpha \le \alpha_0$. We recover in this way the well-known result for $C^{1+\alpha_0}$ expanding maps.

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Note. After this manuscript was finished, the authors found that Aihua Fan and Yunping Jiang have also proved a similar result in their preprint, *Convergence speeds of Ruelle-Perron-Frobenius operators*, Queens College, CUNY, February 2000.

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