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Rigidity for Differentiable Classification of One-dimensional Dynamical Systems

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Abstract. We prove, using normal forms and embedding flows, some rigidity results for differentiable classification of nonhyperbolic one-dimensional dynamical systems, from which equivalence of lower order of differentiability does imply higher one in some extent.

1 Introduction

Classification of dynamical systems (vector fields, diffeomorphisms etc.) under topological or differentiable equivalence is a central, yet a difficult, problem in the dynamical systems theory. When systems have good 'hyperbolicity', there are several important results. For example, it is known that all the eigenvalues at all periodic points of a toral Anosov diffeomorphism constitute the complete invariants of differentiable conjugacy of toral Anosov diffeomorphisms. See [2, 3]. Another class of dynamical systems for which the classification has been well studied is that of one-dimensional systems. One notable result on differentiable conjugacy of one-dimensional Morse-Smale diffeomorphisms is that the complete invariants consist of the eigenvalues at periodic points and the so-called "time difference functions" which serve as the connecting invariants between periodic points. See [10]. Some interesting global classification results on one-dimensional 'expanding' maps, including the famous Ulam-von Neumann transformation, could be found in [5, 6].

In this paper we are concerned with another interesting phenomenon in the (local) classification results of dynamical systems, i.e., the rigidity results for differentiable equivalence of dynamical systems.

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Let us explain what does mean the rigidity using the systems of diffeomorphisms. Recall that two diffeomorphisms f and g are said to be C^r *conjugate* $(r \geq 0)$ if there is a C^r diffeomorphism H for which the relation

$$H \circ f = g \circ H$$

holds. In case $r = 0$, C^r conjugacy is the usual *topological conjugacy.* It is obvious that if f and g are C^r conjugate then they are also necessarily C^s conjugate for any $0 \leq s \leq r$. Of course, the converse does not hold in general. For example, a nonhyperbolic system may topologically conjugate to a hyperbolic one, while any differentiable conjugacy preserves hyperbolicity. However, some rigidity phenomenon, which means that a C^s conjugacy is actually a C^r conjugacy where $s < r$, does occur for differentiable conjugacy. A basic example is the following one on differentiable (local) conjugacies of one-dimensional systems around hyperbolic fixed points. For a proof of it, one can see, for example, [10].

Theorem A *Let f and g be two C^r $(r \geq 2)$ local diffeomorphisms of $\mathbb{R}$ with hyperbolic fixed point 0, i.e., $|f'(0)| \neq 1$ and $|g'(0)| \neq 1$. Then the C^1 conjugacy between f and g implies the C^r conjugacy between them. In fact, both the C^1 conjugacy and the C^r conjugacy are simply equivalent to $f'(0) = g'(0)$.*

Such a rigidity result also holds for vector fields with hyperbolic singular points in whiuch case the C^r conjugacy should be replaced by the C^r equivalence (see the next section for the definition).

Another important rigidity example is the so-called rigidity phenomenon for strong conjugacy of *families* of diffeomorphisms. This means that even certain *topological* conjugacy for the families does imply the *differentiable* conjugacy on the center manifold of the systems. See [8, 7]. Such a rigidity result has many important applications in bifurcation of families of diffeomorphisms.

In this paper, we aim at generalizing Theorem A to the nonhyperbolic case, including local diffeomorphisms and vector fields of $\mathbb{R}$. The obtained rigidity results depend upon the degree of degeneracy of nonhyperbolic fixed points (singular points). In order to have a feeling on the results, let us consider local C^∞ diffeomorphisms $f(x) = x + ax^n + o(x^n)$ and $g(x) = x + bx^n + o(x^n)$ near $x = 0$, where a and b are nonzero reals and n is an integer ≥ 2. Then our rigidity result says that the C^1 conjugacy implies the C^{n-1} conjugacy, while the C^n conjugacy implies the C^∞ conjugacy. For nonhyperbolic singular points of vector fields, we will prove a similar result.

The proof for vector fields is based on the C^∞ normal forms of nonhyperbolic singular points [4]. The result for local diffeomorphisms is then a corollary of the result for vector fields using the technique of embedding flows [9, 10].

2 Statement of Results

Instead of the C^r conjugacy for germs of diffeomorphisms defined before, we say that two germs of vector fields v, w with a singular point at 0 are C^r *equivalent* if there is a germ of C^r diffeomorphism H with a fixed point 0 for which the relation

$$\frac{\partial H}{\partial x} v = w \circ H$$

holds. A germ of vector field v (resp. a diffeomorphism f) with a singular poitn (resp. a fixed point) at 0 is said to be *hyperbolic* if $v'(0) \neq 0$ (resp. $|f'(0)| \neq 1$).

Consider two germs of C^∞ (C^ω) vector fields

$$\dot{x} = a_1 x^n + O(x^{n+1}), \quad a_1 \neq 0 \tag{2.1}$$

and

$$\dot{y} = a_2 y^m + O(y^{m+1}), \quad a_2 \neq 0. \tag{2.2}$$

Our first main result in this paper is the following result on germs of vector fields.

Theorem 1 (i) *The germs (2.1) and (2.2) are C^1 equivalent if and only if*

$$n = m, \quad [\operatorname{sgn} a_1]^n = [\operatorname{sgn} a_2]^n. \tag{2.3}$$

(ii) *If (2.3) holds, then the germs (2.1) and (2.2) are C^{n-1} equivalent.*

(iii) *The germs (2.1) and (2.2) are C^n equivalent if and only if they are C^∞ (resp. C^ω) equivalent.*

Remark The C^∞ equivalent normal form of system (2.1) is

$$\dot{x} = \delta x^n (1 + a x^{n-1})^{-1},$$

where $n,\ a$ and $\delta = (\pm 1)^n$ are invariants, see [4]. Moreover, different parameters $n,\ a,\ \delta$ give different germs with respect to C^∞ equivalence.

Next we consider the classification for nonhyperbolic germs of diffeomorphisms. Set

$$\mathrm{ND}_n = \{f \in C^\infty(\mathbb{R}, 0) \,|\, f(x) = x + ax^n + o(x^n),\ a \neq 0,\ n \geq 2\},$$

$$\mathrm{ND} = \bigcup_{n=2}^{\infty} \mathrm{ND}_n\,.$$

The number n is called the *multiplier* of $f \in \mathrm{ND}_n$ and is denoted by $n = m(f)$.

The following result is on orientation preserving germs of diffeomorphisms.

Theorem 2 (i) $f,\ g \in \mathrm{ND}$ *are C^1 conjugate if and only if*

$$m(f) = m(g) = n, \quad [\operatorname{sgn} f^{(n)}(0)]^n = [\operatorname{sgn} g^{(n)}(0)]^n. \tag{2.4}$$

(ii) *If (2.4) holds, then f and g are C^{n-1} conjugate.*

(iii) *If (2.4) holds, then f and g are C^n conjugate if and only if they are C^∞ conjugate.*

The last result is on nonhyperbolic germs of orientation reversing diffeomorphisms. Consider the germs of C^∞ diffeomorphism

$$f(x) = -x + O(x^2). \tag{2.5}$$

According to normal form theory (see, for example, [1]), there exists a germ of C^∞ diffeomorphism $h = x + O(x^2)$ such that the Taylor series of the germ $g = h^{-1} \circ f \circ h$ contains the terms of odd order in x only. Denote by RD the set of the germs of C^∞ diffeomorphisms f satisfying

(1) $f(0) = 0,\ f'(0) = -1,\ f^{(2n)}(0) = 0,\ \forall n \geq 1;$

(2) There exists a positive integer m such that $f^{(2m+1)}(0) \neq 0$.

Set

$$\mathrm{RD}_n = \{f \in \mathrm{RD} \mid f(x) = -x + ax^{2n+1} + O(x^{2n+3}),\ a \neq 0\},$$

then

$$\mathrm{RD} = \bigcup_{n=1}^{\infty} \mathrm{RD}_n\,.$$

The number n is called the *multiplier* of $f \in \mathrm{RD}_n$ and is denoted by $n = m(f)$.

Theorem 3 (i) *$f,\ g \in \mathrm{RD}$ are C^1 conjugate if and only if*

$$m(f) = m(g) = n, \quad \operatorname{sgn} f^{(2n+1)}(0) = \operatorname{sgn} g^{(2n+1)}(0). \tag{2.6}$$

(ii) *If (2.6) holds, then f and g are C^{2n} conjugate.*
(iii) *If (2.6) holds, then f and g are C^{2n+1} conjugate if and only if they are C^∞ conjugate.*

Remark For the case of germs of diffeomorphisms, we could not consider the analytical conjugacy. The main reason is that an analytical local diffeomorphism may fail to have an analytical embedding vector field, while our Theorems 2 and 3 are proved using the embedding flow technique. See the next section.

3 Proof of Theorems

Lemma 4 *If (2.1) and (2.2) are C^1 equivalent, then (2.3) holds.*

Proof Let $y = h(x) \in C^1(\mathbb{R}, 0)$ with $h'(0) = c \neq 0$ be the diffeomorphism which transforms (2.1) into (2.2). Substituting $y = h(x) = cx + o(x)$ into (2.2) we obtain

$$(c + o(1))(a_1 x^n + o(x^n)) = a_2 c^m x^m (1 + o(1))$$

or

$$a_1 c x^n (1 + o(1)) = a_2 c^m x^m (1 + o(1)).$$

This implies

$$n = m, \quad a_1 c = a_2 c^m, \quad (a_1 a_2)^n = a_2^{2n} c^{(n-1)n} > 0.$$

□

Lemma 5 *If (2.3) holds, then (2.1) and (2.2) are C^{n-1} equivalent.*

Proof Assume that (2.3) holds. By normal form theory (see [4]), systems (2.1) and (2.2) are C^∞ equivalent to their normal forms respectively as follows

$$\dot{x} = \sigma x^n (1 + b_1 x^{n-1})^{-1} \tag{3.1}$$

and

$$\dot{y} = \sigma y^n (1 + b_2 y^{n-1})^{-1}, \tag{3.2}$$

where $\sigma = [\operatorname{sgn} a_1]^n = [\operatorname{sgn} a_2]^n$. We put the system (3.1) and (3.2) together as a planar differential system:

$$\frac{dy}{dx} = \frac{y^n (1 + b_2 y^{n-1})^{-1}}{x^n (1 + b_1 x^{n-1})^{-1}}. \tag{3.3}$$

We claim that all integral curves of (3.3) lying in the first or the third quadrant are tangent to the line $y = x$ with order $n - 1$ at the origin. Indeed, by integrating

(3.3) we get

$$\begin{aligned} f(x,y) &\overset{\text{def}}{=} y(1+(1-n)b_2y^{n-1}\log|y|)^{\frac{1}{1-n}} \\ &\quad - x(1+(1-n)b_1x^{n-1}\log|x| + cx^{n-1})^{\frac{1}{1-n}} \\ &= 0, \end{aligned} \tag{3.4}$$

where c is the integral constant. Define $f(0,0)=0$. Then f is of the class C^{n-1}. Applying the Implicit Function Theorem to f at the origin $(0,0)$ we obtain from (3.4) a function of class C^{n-1}

$$y = h(x,c)$$

satisfying

$$h(0,c)=0,\ h'(0,c)=1,\ h^{(i)}(0,c)=0,\ 2\le i\le n-1.$$

The C^{n-1} diffeomorphism h transforms (3.1) into (3.2). □

Lemma 6 *If (3.1) and (3.2) are C^n equivalent, then $b_1 = b_2$.*

Proof Let $y=h(x)$ be a C^n diffeomorphism which transforms (3.1) into (3.2). Then $f(x,h(x))=0$, where f is the function defined by (3.4). Therefore, $h(0)=0$, $h'(0)=1$, $h^{(i)}(0)=0$ for $2\le i\le n-1$. Substituting $y=h(x)=x+kx^n+o(x^n)$ into (3.3) we obtain

$$1+knx^{n-1}+o(x^{n-1}) = (1+kx^{n-1}+o(x^{n-1}))^n(1+b_1x^{n-1})(1+b_2x^{n-1}+o(x^{n-1}))^{-1},$$

which implies $b_1=b_2$. □

Proof of Theorem 1 This is a corollary of Lemmas 4, 5 and 6. □

Definition 7 A smooth flow $\{f^t\}$ $(t\in\mathbb{R})$ is said to be an embedding flow of a diffeomorphism f if $f^1=f$. The corresponding vector field, $v(x)=\frac{\partial}{\partial t}f^t(x)|_{t=0}$, is called an embedding vector field of f.

Lemma 8[9] *The germ of C^∞ diffeomorphism*

$$f(x) = x + ax^n + o(x^n) \quad (a\neq 0)$$

can be embedded into a flow of the germ of a C^∞ vector field

$$\dot{x} = ax^n + o(x^n).$$

Lemma 9[10] *Two germs of smooth diffeomorphisms in real line are C^k, $k\ge 1$, conjugate if and only if their embedding vector fields are C^k equivalent.*

Proof of Theorem 2 This is a corollary of Theorem 1, Lemmas 8 and 9. □

Next we give the **the proof of Theorem 3**. Let $f=-x+ax^{2n+1}+O(x^{2n+3})\in \mathrm{RD}_n$ and let $h=cx$, $c=|a|^{1/2n}$. Then

$$h^{-1}\circ f\circ h = -x+\sigma x^{2n+1}+O(x^{2n+3}),\ \sigma=\operatorname{sgn} a.$$

Assume that $f\in\mathrm{RD}_n$ has the following form

$$f = -x+\sigma x^{2n+1}+cx^{2m+1}+O(x^{2m+3}),\ n<m<2n.$$

Let $h=x+bx^{2m-2n+1}$, $b=\frac{\sigma c}{4n-2m}$, then

$$h^{-1}\circ f\circ h = -x+\sigma x^{2n+1}+O(x^{2m+3}).$$

Therefore, there exists a polynomial transformation $h = x + O(x^2)$ such that

$$h^{-1} \circ f \circ h = -x + \sigma x^{2n+1} + \lambda x^{4n+1} + O(x^{4n+3}).$$

Set

$$B^{\sigma}_{n,\lambda} = \{f \in \mathrm{RD}^{\infty} \mid f = -x + \sigma x^{2n+1} + \lambda x^{4n+1} + O(x^{4n+3})\}.$$

Lemma 10 *Any two germs in $B^{\sigma}_{n,\lambda}$ are C^{∞} conjugate.*

Proof First we note that all germs in $B^{\sigma}_{n,\lambda}$ are formally conjugate to each other. Therefore in order to prove the lemma it is sufficient to prove that the germs

$$g(x) = -x + \sigma x^{2n+1} + \lambda x^{4n+1}$$

and $f = g + \text{flatgerm}$ are C^{∞} conjugate. Since $f \circ f = g \circ g + \text{flatgerm}$, by Takens Theorem[9], there exists a germ of C^{∞} diffeomorphism $h(x) = x + \text{flatgerm}$ such that $h \circ g \circ g = f \circ f \circ h$. Let

$$H(x) = \begin{cases} h(x), & \text{for } x \geq 0 \\ f \circ h \circ g^{-1}(x), & \text{for } x < 0. \end{cases} \tag{3.5}$$

Then H is a C^{∞} diffeomorphism with $H(0) = 0$, $H'(0) = 1$ and satisfies $H \circ g = f \circ H$. □

Now we prove conclusion (iii) of Theorem 3. It is sufficient to prove that if (2.6) holds and f and g are C^{2n+1} conjugate, then they have the same normal form. Let

$$f = -x + \sigma x^{2n+1} + \lambda_1 x^{4n+1}, \ g = -x + \sigma x^{2n+1} + \lambda_2 x^{4n+1}. \tag{3.6}$$

Then

$$f \circ f = x - 2\sigma x^{2n+1} + (2n + 1 - 2\lambda_1)x^{4n+1} + O(x^{4n+3})$$
$$g \circ g = x - 2\sigma x^{2n+1} + (2n + 1 - 2\lambda_2)x^{4n+1} + O(x^{4n+3}).$$

Since f and g are C^{2n+1} conjugate, then so are $f \circ f$ and $g \circ g$. By Theorem 1, they have the same normal form, which implies $\lambda_1 = \lambda_2$.

The necessity of conclusion (i) of Theorem 3 is a corollary of conclusion (i) of Theorem 1.

The sufficiency of conclusion (i) is a corollary of conclusion (ii), for which we give the proof.

Assume now that (2.6) holds and f and g are given by (3.6). By Lemma 5 there exists a C^{2n} diffeomorphism $h(x) = x + O(x^2)$ with $h^{(i)}(0) = 0$, $2 \leq i \leq 2n$, such that $h \circ g \circ g = f \circ f \circ h$. Let H be the diffeomorphism given by (3.5). Then H is C^{2n} and satisfies $H \circ g = f \circ H$.

All conclusions of Theorem 3 are thus proved.

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