LYAPUNOV STABILITY FOR CONSERVATIVE SYSTEMS WITH LOWER DEGREES OF FREEDOM

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Abstract. It is a central theme to study the Lyapunov stability of periodic solutions of nonlinear differential equations or systems. For dissipative systems, the Lyapunov direct method is an important tool to study the stability. However, this method is not applicable to conservative systems such as Lagrangian equations and Hamiltonian systems. In the last decade, a method that is now known as the ‘third order approximation’ has been developed by Ortega, and has been applied to particular types of conservative systems including time periodic scalar Lagrangian equations (Ortega, J. Differential Equations, 128(1996), 491-518). This method is based on Moser’s twist theorem, a prototype of the KAM theory. Latter, the twist coefficients were re-explained by Zhang in 2003 through the unique positive periodic solutions of the Ermakov-Pinney equation that is associated to the first order approximation (Zhang, J. London Math. Soc., 67(2003), 137-148). After that, Zhang and his collaborators have obtained some important twist criteria and applied the results to some interesting examples of time periodic scalar Lagrangian equations and planar Hamiltonian systems. In this survey, we will introduce the fundamental ideas in these works and will review recent progresses in this field, including applications to examples such as swing, the (relativistic) pendulum and singular equations. Some unsolved problems will be imposed for future study.

1. Introduction. Lyapunov stability of periodic solutions is a central topic in differential equations and dynamical systems, and therefore has been attracting many attentions. See, for example, [8, 9, 14, 15, 16, 17, 18, 31]. Roughly speaking, there are two classes of dynamical systems. One is dissipative systems and the another...
is conservative systems. For dissipative systems, by the stability we always refer to stability for time in one direction (either positive or negative), and the method of Lyapunov functions is an important tool to study the stability. Unlike cases in dissipative systems, for a solution $u(t)$ of a conservative system, the stability means that all solutions of the system with initial conditions close to that of $u(t)$ always stay at a neighborhood of $u(t)$ in the phase space for all time $t$, including positive and negative. In this situation, the Lyapunov function method is, in general, not applicable.

As an example, let us consider nonlinear scalar Lagrangian equation

$$x'' + g(t, x) = 0,$$  \hspace{1cm} (1.1)

where $g = g(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $T$-periodic in $t$ and of class $C^{0.5}$ in $(t, x)$. Suppose that $x = u(t)$ is a $T$-periodic solution of (1.1). For the Lyapunov stability of $u(t)$, a classical idea is to study the associated Poincaré map $P$ of (1.1). The $T$-periodic solution $u(t)$ corresponds to a fixed point of $P$, say $(x_0, y_0)$. Thus, one may instead consider the stability of this fixed point of the Poincaré map, which can be studied through the Birkhoff normal form of $P$ near $(x_0, y_0)$. According to the well known Moser’s twist theorem, if the periodic solution $u(t)$ is of twist type, i.e., the first twist coefficient (to be defined later) of $u(t)$ is non-zero, then $u(t)$ is Lyapunov stable [28]. Furthermore, equation (1.1) has many subharmonic solutions and quasi-periodic solutions near $u(t)$, which are typical scenarios for conservative systems [32].

Though this is a classical idea, it is usually not easy to find explicit twist criteria for conservative systems. The reason is that twist coefficients depend on the nonlinear approximation of (1.1) in a complicated way. As far as the authors know, such an idea is not applicable to the periodic solutions that are found in the restricted 3-body problem [7, 19]. Since 1992, a practical method, now known as the third order approximation, has been developed by Ortega based on the Birkhoff normal forms and the Moser’s twist theorem [26, 27, 28, 29]. During the last decade, there have been a considerable progresses on this topic. It is the aim of this paper to give a survey on these results. Applications of these results will be emphasized on the forced pendulum, the swing, and singular differential equations. For more applications, one can refer [4, 13, 23, 24, 25, 33, 41].

The rest part of this survey is organized as follows. In Section 2, we will sketch the fundamental ideas of the method of third order approximation. In Section 3, we will discuss twist criteria for Lagrangian equations and planar Hamiltonian systems. Some interesting examples which have stable periodic solutions will be given in Section 4. In Section 5, we will give several unsolved problems for future research.

2. **Fundamental ideas for the third order approximation.** In this section, we will outline the fundamental ideas for the third order approximation. In particular, two basic facts on linear systems will be given.

2.1. **Reduction from ellipticity to $R$-ellipticity for linear systems.** Consider planar linear systems

$$\begin{align*}
\dot{x} &= b(t)y, \\
\dot{y} &= -a(t)x,
\end{align*}$$  \hspace{1cm} (2.1)

where $a, b$ are $T$-periodic continuous functions. The Poincaré map of (2.1) is a planar linear transformation, with $(0, 0)$ a fixed point, and the coefficients matrix
(Poincaré matrix) is given by

\[ M = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \psi_1(T) & \psi_2(T) \end{pmatrix}, \]

where \((\phi_1(t), \psi_1(t))^T\) and \((\phi_2(t), \psi_2(t))^T\) are real-valued solutions of (2.1) satisfying 
\((\phi_1(0), \psi_1(0)) = (1, 0)\) and 
\((\phi_2(0), \psi_2(0)) = (0, 1)\), respectively. The eigenvalues \(\lambda_{1,2}\) of \(M\) are called the Floquet multipliers of (2.1). It is obvious that \(\lambda_1 \cdot \lambda_2 = 1\). We can classify (2.1) into three types, according to the Floquet multipliers, as either

- hyperbolic when \(|\lambda_{1,2}| \neq 1\), or
- elliptic when \(|\lambda_{1,2}| = 1\) but \(\lambda_{1,2} \neq \pm 1\), or
- parabolic when \(\lambda_{1,2} = \pm 1\), respectively.

In this survey, we are mainly interested in the elliptic case and assume that the Floquet multipliers of (2.1) are \(\lambda\) and \(\bar{\lambda}\) with \(\lambda = \exp(i\theta), \theta > 0\). For each \(t_0 \in \mathbb{R}\), let \(\Phi(t_0,t_0)\) denote the matrix solution of (2.1) with initial condition

\[ \Phi(t_0,t_0) = I. \]

Then the matrix \(\Phi(t_0 + T, t_0)\) is a monodromy matrix of (2.1) and belongs to the class of matrices which are similar to \(R_{\pm \theta}\), here

\[ R_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

is the rigid rotation. Take the temporal-spatial change of variables

\[ s = \alpha(t - t_0), \quad x^*(s) = x(t_0 + s/\alpha), \quad y^*(s) = \alpha^{-1}y(t_0 + s/\alpha), \]

where \(t_0 \in \mathbb{R}, \alpha > 0\). Then (2.1) is transformed into a linear system of the same type

\[ \begin{cases} \dot{x}^* = b^*(s)x^*, \\ \dot{y}^* = -a^*(s)x^*, \end{cases} \]

where \(a^*(s) = \alpha^{-2}a(t_0 + s/\alpha), b^*(s) = b(t_0 + s/\alpha)\) are now \(T^* := \alpha T\) periodic in \(s\).

This class of variable changes has a remarkable property such that the resulting monodromy matrix can has the canonical form after some appropriate transformation. More precisely, we have

**Theorem 2.1.** [2, Theorem 2.5] Assume that (2.1) is elliptic with the Floquet multipliers \(\lambda\) and \(\bar{\lambda}\) with \(\lambda = \exp(i\theta), \theta > 0, \theta \neq n\pi, n = 1,2,\ldots\). Moreover, assume that \(b(t)\) satisfies

\[ b(t) \neq 0 \quad \text{for all} \ t \in \mathbb{R}. \]

Then there always exist some \(\alpha > 0\) and \(t_0 \in \mathbb{R}\) such that under the transformation (2.3), the Poincaré matrix of the transformed system (2.4) is a rigid rotation (2.2).

**Remark 2.2.** An equation (2.1) will be called \(R\)-elliptic if the Poincaré matrix is a rigid rotation. Theorem 2.1 is crucial in our analysis because we can compute the twist coefficients for the nonlinear systems in a much simpler way assuming the linear system is \(R\)-elliptic.

When \(b(t) \equiv 1\), the linear system (2.1) is reduced to Hill equation [20]

\[ x'' + a(t)x = 0, \]

The corresponding property of Theorem 2.1 for (2.5) was proved in [26], which plays an important role in establishing the third order approximation for Lagrangian equations [28].
2.2. Relation between linear systems and Ermakov-Pinney equations. We assume that one of the coefficients \( a(t) \) or \( b(t) \) in (2.1) does not change sign for all \( t \in \mathbb{R} \). Without loss of generality, we assume that

\[
b \in L^1(\mathbb{S}_T) \quad \text{and} \quad \text{essinf}_t b(t) > 0,
\]

where \( \mathbb{S}_T = \mathbb{R}/T\mathbb{Z} \). Define

\[
B(t) = \int_0^t b(\tau)d\tau, \quad T_b = \int_0^T b(t)dt > 0.
\]

Then \( B(t) \) is increasing with \( t \) according to the condition (2.6), and therefore \( B^{-1}(s) \) is well defined. Moreover, by the \( T \)-periodicity of \( b(t) \), we have

\[
B(t + T) = B(T) + T_b.
\]

We change the time \( t \) to \( s = B(t) \), and define

\[
q(s) = \frac{a(B^{-1}(s))}{b(B^{-1}(s))}.
\]

Then \( q(s) \) is a \( T_b \)-periodic function.

Next, we can establish a one-to-one correspondence between the linear system (2.1) and the Ermakov-Pinney equation of form

\[
\ddot{r}(s) + q(s)r(s) = \frac{1}{r^3(s)}, \quad (2.7)
\]

Let \( (\phi_1(t), \psi_1(t))^T \) and \( (\phi_2(t), \psi_2(t))^T \) be as before. Then they are linearly independent real-valued solutions of (2.1). Let \( \phi(t) = \phi_1(t) + i\phi_2(t), \quad \psi(t) = \psi_1(t) + i\psi_2(t) \), and set

\[
\phi(t) = R(t)e^{i\varphi(t)},
\]

where \( R, \varphi \in C^2(\mathbb{R}) \). Then \( R(t) \) and \( \varphi(t) \) are real-valued functions and \( R(t) > 0 \) for all \( t \in \mathbb{R} \). Moreover, they have initial conditions

\[
R(0) = 1, \quad \varphi(0) = 0, \quad \dot{R}(0) = 0, \quad \dot{\varphi}(0) = b(0). \quad (2.8)
\]

Using the facts

\[
\psi(t) = \frac{\dot{\phi}(t)}{b(t)}, \quad \dot{\psi}(t) = -a(t)\phi(t),
\]

we can obtain that \( R(t) \) and \( \varphi(t) \) satisfy the equations

\[
b\ddot{R} + b^2aR - bR\dot{\varphi}^2 - b\dot{R} = 0, \quad 2bR\ddot{\varphi} + bR\dot{\varphi} - b\dot{R} = 0. \quad (2.9)
\]

From the second equation and initial conditions (2.8), we have

\[
\ddot{\varphi} = \frac{b(0)}{R^2} \frac{R^2(0)\dot{\varphi}(0)}{b(0)} = \frac{b(t)}{R^2}. \quad (2.10)
\]

Substituting (2.10) into the first equation in (2.9), we obtain following equation for \( R(t) \),

\[
\dddot{R} + a(t)b(t)R = \frac{b^2(t)}{R^3} + \frac{\dot{b}(t)}{b(t)}\dot{R}. \quad (2.11)
\]

Let

\[
r(s) = r(B(t)) = R(t).
\]

Then

\[
\frac{dR}{dt} = \frac{br}{ds}, \quad \frac{d^2R}{dt^2} = b^2\frac{d^2r}{ds^2} + i\frac{dr}{ds}.
\]
Taking the above two equalities into (2.11), it is easy to verify that \( r(s) \) satisfies (2.7).

Conversely, if \( r(s) \) is a positive solution of (2.7), it is not difficult to construct general solutions of (2.1) by reversing the above procedure.

Besides the above one-to-one correspondence, we can also establish the following further result. Here we omit the proof as it is similar to that of [39, Theorems 2.1 and 2.2].

**Theorem 2.3.** Linear system (2.1) is stable in the sense of Lyapunov if and only if the Ermakov-Pinney equation (2.7) has a positive \( T_b \)-periodic solution. Moreover, if the linear system (2.1) is elliptic, then (2.7) has a unique positive \( T_b \)-periodic solution.

**Remark 2.4.** When \( b(t) \equiv 1 \), as a consequence of Theorem 2.3, we can re-obtain the following result that has been proved in [39]. This relation plays an important role in following discussions.

**Theorem 2.5.** [39, Theorems 2.1 and 2.2] Hill equation (2.5) is stable in the sense of Lyapunov if and only if the Ermakov-Pinney equation

\[
\ddot{r} + a(t)r = \frac{1}{r^3}
\]

has a positive \( T \)-periodic solution. Moreover, if equation (2.5) is elliptic, then (2.12) has a unique positive \( T \)-periodic solution.

### 2.3. Birkhoff normal form and twist type

Here we will only consider the nonlinear scalar Lagrangian equation (1.1), where \( g(t, x) : \mathbb{R} \rightarrow \mathbb{R} \) is \( T \)-periodic in \( t \) and of class \( C^{0,5} \) in \((t, x)\). The results presented here are also applicable to the planar nonlinear systems (see [1]).

Let \( x = u(t) \) be a \( T \)-periodic solution of (1.1). Associated with equation (1.1), we have the Poincaré map \( \mathcal{P} \) which is defined in \( \mathbb{R}^2 \approx \mathbb{C} \), and the \( T \)-periodic solution \( u(t) \) corresponds to a fixed point of \( \mathcal{P} \). After translating it to the origin, we get an area-preserving map \( \mathcal{P} : D \subset \mathbb{C} \rightarrow \mathbb{C} \) with \( D \) a disk centered at the fixed point 0.

Therefore, the stability of the periodic solution \( u(t) \) of (1.1) is equivalent to that of the fixed point \((0, 0)\) of the map \( \mathcal{P} \).

When the first order linear equation (2.5) is hyperbolic, the periodic solution \( x = u(t) \) is unstable for the nonlinear equation (1.1). However, when (2.5) is elliptic or parabolic, the stability or the instability of the periodic solution \( x = u(t) \) depends on higher order terms. Thus, higher order approximations must be considered. In the case that the linearized equation is elliptic, the stability of the periodic solution \( u(t) \) of the original nonlinear equation can be studied through the Birkhoff normal form and the Moser’s twist theorem.

When \( \mathcal{P} \in C^3(\mathcal{D}, \mathbb{C}) \) and the differential \( d\mathcal{P}(0) \) of \( \mathcal{P} \) at the origin is elliptic, \( \mathcal{P} \) is conjugate to the corresponding Birkhoff normal form [28, 31, 32]:

**Case I:** \( \lambda^n \neq 1, n = 1, 2, 3, 4 \), \( N(z, \bar{z}) = \lambda[z + i\beta|z|^2z + \cdots] \).

**Case II:** \( \lambda = \pm i \), \( N(z, \bar{z}) = \lambda[z + i\beta|z|^2z + \gamma\bar{z}^3 + \cdots] \).

Here the dots denote a remaining term of order \( o(|z|^3) \) as \( z \rightarrow 0 \), and \( \lambda = \lambda_1, \beta, \gamma \) are constants with \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{C} \). The quantities \( \beta \) and \( |\gamma| \) are uniquely determined by \( \mathcal{P} \) and are called the first and the second twist coefficient, respectively. From Moser’s twist theorem [31], when \( \mathcal{P} \) is sufficiently smooth, the twist coefficients can determine in most case the stability of the origin.
Proposition 2.1. [28, Proposition 2.2] Let \( P \in C^5(D, C) \) be an area-preserving map with a fixed point \( z = 0 \), and assume that the differential \( dP(0) \) is elliptic. The fixed point \( z = 0 \) is Lyapunov stable if one of the following alternatives holds
\[
\lambda^n \neq 1, \quad n = 1, 2, 3, 4 \quad \text{and} \quad \beta \neq 0, \quad (2.13)
\]
or
\[
\lambda = \pm i, \quad \text{and} \quad |\beta| > |\gamma|. \quad (2.14)
\]

Definition 2.6. Let \( P \in C^3(D, C) \) be a Poincaré map given as previous, and with fixed point \( z = 0 \) corresponding to the periodic solution \( x = u(t) \). The fixed point is said to be of twist type if it is elliptic and either (2.13) or (2.14) holds. Accordingly, we say that the periodic solution \( u(t) \) is of twist type if \( z = 0 \) is of twist type as a fixed point of the Poincaré map \( P \).

3. Twist coefficients and twist criteria.

3.1. Twist coefficients. Hereinafter, we will set \( T = 2\pi \). For simplicity, we assume \( u(t) \equiv 0 \) and (1.1) can be rewritten in the form
\[
x'' + a(t)x + b(t)x^2 + c(t)x^3 + \cdots = 0, \quad (3.1)
\]
where \( a, b, c \in C(\mathbb{R}/2\pi\mathbb{Z}) \) are given by
\[
a(t) = g_x(t, u(t)), \quad b(t) = \frac{1}{2}g_{xx}(t, u(t)), \quad c(t) = \frac{1}{6}g_{xxx}(t, u(t)). \quad (3.2)
\]

Following Theorem 2.1, we can further assume that the linearization (2.5) of (3.1) at \( x = 0 \) is \( R \)-elliptic, which is equivalent to
\[
\Psi(t + 2\pi) \equiv \lambda \Psi(t),
\]
for some \( \lambda \in S^1, \lambda \neq \pm 1 \). Here \( \Psi(t) = \phi_1(t) + i\phi_2(t) \) is the (complex-valued) solution of (2.5) satisfying
\[
\Psi(0) = 1, \quad \Psi'(0) = i.
\]

When (2.5) is \( R \)-elliptic, the corresponding Poincaré map \( P \) has the expansion of form
\[
P(z, \bar{z}) = \lambda z + P_2(z, \bar{z}) + P_3(z, \bar{z}) + \cdots
\]
where \( \lambda \) is given by the definition of \( R \)-ellipticity, and \( P_2, P_3 \) are homogenous polynomials of degree 2 and 3, respectively. Let
\[
\lambda = e^{-i\theta}, \quad P_2(z, \bar{z}) = Az^2 + Bz\bar{z} + Cz^2,
\]
\[
P_3(z, \bar{z}) = Mz^3 + Nz^2\bar{z} + Pz\bar{z}^2 + Qz^3.
\]
The coefficients are as follows
\[
A = -\frac{i\lambda}{4} \int_0^{2\pi} b(t)\Psi(t)^2\Psi(t)dt, \quad C = -\frac{i\lambda}{4} \int_0^{2\pi} b(t)\Psi(t)^3dt, \quad (3.3)
\]
\[
N = -\frac{3i\lambda}{8} \int_0^{2\pi} c(t)|\Psi(t)|^4dt - \frac{i\lambda}{4} \int \int G(t, s)b(t)b(s)[|\Psi(t)||\Psi(s)|]^2 + \Psi(t)^2\bar{\Psi}(s)^2]dsdt \quad (3.4)
\]
\[
Q = -\frac{i\lambda}{8} \int_0^{2\pi} c(t)|\Psi(t)|^4dt - \frac{i\lambda}{4} \int \int G(t, s)b(t)b(s)\Psi(t)^2\bar{\Psi}(s)^2]dsdt, \quad (3.5)
\]
where \( \Delta = \{(t, s) \in \mathbb{R}^2|0 < s < t < 2\pi\} \) and \( G(t, s) = \phi_1(t)\phi_2(s) - \phi_2(t)\phi_1(s) \). Therewith, the twist coefficients are [28].
If $\lambda^n \neq 1$, $n = 1, 2, 3, 4$,

$$
\beta = \text{Im}(\bar{\lambda}N) + \frac{3\sin \theta}{1 - \cos \theta} |A|^3 + \frac{\sin 3\theta}{1 - \cos 3\theta} |C|^2. \tag{3.6}
$$

If $\lambda = \pm i$,

$$
\beta = \mp \text{Im}(iN) \mp 3|A|^2 \pm |C|^2, \quad |\gamma| = |Q - 2\bar{A}C|. \tag{3.7}
$$

Here (3.3)–(3.5) are obtained by expanding the solutions $x(t)$ of (3.1) as Taylor expansions with respect to the initial condition $x(0) = q, x'(0) = p, z = q + ip$, up to the third order.

From (3.3)–(3.7), when (2.5) is $R$-elliptic, the twist coefficients $\beta$ and $\gamma$ are functionals of the coefficients $b$ and $c$ in forms of integrals, while integration kernels depending on $\Psi$ and $\theta$, which are in turn determined by the linear equation (2.5). The forms (3.6) and (3.7) are not convenient for further analysis, and we will give alternative forms for the twist coefficients.

From Theorem 2.5, let $r(t)$ be the unique positive $2\pi$-periodic solution of the Ermakov-Pinney equation (2.12) with $a(t)$ given by (3.2), and write $\Psi(t) = r(t)e^{i\varphi(t)}$.

Following result gives the explicit formula for the twist coefficients.

**Theorem 3.1.** Let $T = 2\pi$. Define

$$
L_1(r, \varphi, c) = -\frac{3}{8} \int_0^T c(\tau) r^4(\tau) d\tau,
$$

$$
J_1(r, \varphi, b) = \int_0^T b(\tau) b(\zeta) r^3(\tau) r^3(\zeta) \chi_1(|\varphi(\tau) - \varphi(\zeta)|) d\tau d\zeta
$$

$$
+ \frac{3}{16} \cot \frac{\theta}{2} \left| \int_0^T b(\tau) r^3(\tau) e^{-i\varphi(\tau)} d\tau \right|^2,
$$

$$
+ \frac{1}{16} \cot \frac{3\theta}{2} \left| \int_0^T b(\tau) r^3(\tau) e^{3i\varphi(\tau)} d\tau \right|^2,
$$

$$
L_2(r, \varphi, c) = \frac{1}{8} \int_0^T c(\tau) r^4(\tau) e^{4i\varphi(\tau)} d\tau,
$$

$$
J_2(r, \varphi, b) = -\frac{1}{8} \int_0^T b(\tau) b(s) r^3(\tau) r^3(s) \chi_3(t, s) ds d\tau,
$$

where

$$
\chi_1(x) = \frac{3\sin x - 2\sin^3 x}{8}, \quad x \in [0, T],
$$

and

$$
\chi_3(t, s) = e^{i(2\varphi(t) + \varphi(s))} \left( \sin |\varphi(t) - \varphi(s)| \pm e^{-(\varphi(t) - \varphi(s))} \right), \quad t, s \in [0, T].
$$

Then there exists a constant $\sigma > 0$ such that

$$
\beta = \sigma^{-1}(L_1(r, \varphi, c) + J_1(r, \varphi, b)), \tag{3.8}
$$

$$
|\gamma| = \sigma^{-1}|L_2(r, \varphi, c) + J_2(r, \varphi, b)|. \tag{3.9}
$$
Proof. Let $t_0 \in [0, 2\pi]$ be a critical point of $r(t)$ and
\[ \sigma = 1/(r(t_0))^2. \]
This is always possible since $r(t)$ is $2\pi$-periodic. Then the change of variables
\[ \xi = x, \quad \tau = \sigma(t - t_0) \] transforms (3.1) to
\[ \xi'' + a^*(\tau)\xi + b^*(\tau)\xi^2 + c^*(\tau)\xi^3 + \cdots = 0. \] (3.11)
Here $a^*, b^*, c^* \in C(\mathbb{R}/T^*\mathbb{Z})$ with $T^* = 2\pi \sigma$, and
\[ a^*(\tau) = \sigma^{-2}a(t_0 + \sigma^{-1}\tau), \quad b^*(\tau) = \sigma^{-2}b(t_0 + \sigma^{-1}\tau), \quad c^*(\tau) = \sigma^{-2}c(t_0 + \sigma^{-1}\tau). \]
Correspondingly, the linearization equation (2.5) is transformed into
\[ \xi'' + a^*(\tau)\xi = 0 \] (3.12)
which is $R$-elliptic. This can be seen by the fact that the function
\[ \Phi(\tau) = \Psi(t_0 + \sigma^{-1}\tau)\Psi(t_0)^{-1} \]
satisfies (3.12) and the initial condition
\[ \Phi(0) = 1, \quad \Phi'(0) = i, \]
and
\[ \Phi(\tau + 2\pi \sigma) = e^{i\theta}\Phi(\tau). \]

Let
\[ r^*(\tau) = \sigma^{-1/2}r(t_0 + \sigma^{-1}\tau), \quad \varphi^*(\tau) = \varphi(t_0 + \sigma^{-1}\tau) \]
and apply (3.3)–(3.7) to (3.11), the twist coefficients of (3.1), which equal to those for (3.11), can be written as follows
\[ \beta = L_1(r^*, \varphi^*, c^*) + J_1(r^*, \varphi^*, b^*), \]
\[ |\gamma| = |L_2(r^*, \varphi^*, c^*) + J_2(r^*, \varphi^*, b^*)|. \]
Here the period $T$ is replaced by $T^*$. Now (3.8) and (3.9) follow from the transformation (3.10).

It is sometimes more convenient to use the alternative form of $J_1(r, \varphi, b)$ given below
\[ J_1(r, \varphi, b) = \int_0^T b(t)b(s)r^3(t)r^3(s)\chi_2(|\varphi(t) - \varphi(s)|)dt ds, \]
where
\[ \chi_2(x) = \frac{3}{16}\cos(x - \theta/2) + \frac{1}{16}\cos(3\theta/2), \quad x \in [0, \theta]. \]

Since we are only interested in the sign of $\beta$ and $|\gamma| - |\gamma|$, let us introduce
\[ \beta = L_1(r, \varphi, c) + J_1(r, \varphi, b) \] (3.13)
and
\[ |\gamma| = |L_2(r, \varphi, c) + J_2(r, \varphi, b)|. \] (3.14)
From Theorem 3.1, they differ from the original twist coefficients by a constant. Thus, the twist character of the periodic solution $x = 0$ of (3.1) is decided by $\beta$ and $|\gamma|$ defined as (3.13)–(3.14), which are in turn determined through the coefficients $a(t)$, $b(t)$ and $c(t)$. However, in calculating the twist coefficients, one need to know the function $r(t)$ which is not always possible.
In fact, when we want to establish the twist criteria for concrete examples, we have to work out the estimate of $r(t)$. Sometimes the bounds of the rotation number $[6]$ of Hill equation (2.5) are also necessary (see [5]).

### 3.2. Twist criteria for scalar Lagrangian equation.

#### 3.2.1. General cases.

First, we give some twist criteria for general scalar Lagrangian equations of which the periodic solution is elliptic and without resonances up to order 3.

For any $0 < \sigma_1 \leq \sigma_2$, let

$$C_{\sigma_1, \sigma_2} = \{a(t) \in C(\mathbb{R}/2\pi \mathbb{Z}, \mathbb{R}) \mid a(t) \text{ satisfies } \sigma_1^2 \leq a(t) \leq \sigma_2^2 \text{ for all } t\}.$$

For any $1 \leq k \leq 4$, define the set

$$\Omega_k = \{\omega > 0 \mid \omega \neq p/q \text{ for all } p, q \in \mathbb{N} \text{ with } 1 \leq q \leq k\}.$$

Let $e^{\pm i\theta}$ be the Floquet multiples of (2.5). We will instead use the rotation number $\rho$ in the discussion below, with $\theta = 2\pi \rho$.

The following result gives a sufficient condition for the periodic solution $x = 0$ to be of twist type.

**Theorem 3.2.** [12, Theorem 3.1] Assume that $a(t) \in C(\mathbb{R}/2\pi \mathbb{Z}, \mathbb{R})$ satisfies $a(t) \geq 0$ for all $t$ and (2.5) is elliptic without resonances up to order 3 ($\lambda^n \neq 1, n = 1, 2, 3$). Then there exists a constant $\mu = \mu(a) > 0$, depending on $\rho = \rho(a)$, such that

$$x(t) = 0 \text{ (as a } 2\pi-\text{periodic solution of (3.1))}$$

is of twist type provided that $b(t)$ and $c(t)$ satisfy

$$\max_{t \in \mathbb{R}} c(t) < -\mu \|r\|_4^2 \|b\|_4^2.$$  

(3.15)

**Proof.** Let $\beta$ and $\gamma$ be defined as (3.13) and (3.14). From the above discussions, $x(t) = 0$ is of twist type if $\beta \neq 0$, when $\rho \in \Omega_4$ or

$$|\beta| > |\gamma|, \text{ when } \rho = \frac{2k - 1}{4}, \, k \in \mathbb{N}.$$

Thus, it is sufficient to show that

$$|L_1(r, \varphi, c)| > |J_1(r, \varphi, b)|, \text{ if } \rho \in \Omega_4,$$

and

$$|L_1(r, \varphi, c)| - |L_2(r, \varphi, c)| > |J_1(r, \varphi, b)| + |J_2(r, \varphi, b)|, \text{ if } \rho = \frac{2k - 1}{4}, \, k \in \mathbb{N}. \quad (3.16)$$

When $\rho \in \Omega_3$, there exists a positive function $K(2\pi \rho)$ such that

$$|J_1(r, \varphi, b)| \leq K(2\pi \rho) \left(\int_0^{2\pi} |b(t)|r^3(t)dt\right)^2.$$

Furthermore, it is easy to have

$$\chi_3(t, s) \leq \left(\frac{3 + \sqrt{5}}{2}\right)^{1/2} =: c_0$$

and hence

$$|J_2(r, \varphi, b)| \leq \frac{c_0}{8} \left(\int_0^{2\pi} |b(t)|r^3(t)dt\right)^2.$$
Under the condition (3.15), we have
\[
|L_1(r, \varphi, c)| = \frac{3}{8} \int_0^{2\pi} |c(t)| r^4(t) dt, \quad |L_2(r, \varphi, c)| \leq \frac{1}{8} \int_0^{2\pi} |c(t)| r^4(t) dt.
\]
Thus, (3.16) and (3.17) are satisfied when
\[
\frac{3}{8} \int_0^{2\pi} |c(t)| r^4(t) dt > K(2\pi \rho) \left( \int_0^{2\pi} |b(t)| r^3(t) dt \right)^2, \quad \text{if} \ \theta \in \Omega_4, \tag{3.18}
\]
or
\[
\frac{1}{4} \int_0^{2\pi} |c(t)| r^4(t) dt > (K(2\pi \rho) + \frac{c_0}{8}) \left( \int_0^{2\pi} |b(t)| r^3(t) dt \right)^2, \quad \text{if} \ \theta = \frac{2k-1}{4}, \quad k \in \mathbb{N}.
\]
Note that
\[
\left( \int_0^{2\pi} |b(t)| r^3(t) dt \right)^2 \leq \|r\|_4^2 \|b\|_4^2
\]
and
\[
\int_0^{2\pi} |c(t)| r^4(t) dt \geq (-\max_{\tau} c(t)) \|r\|_4^2.
\]
The conditions (3.18) and (3.19) can be guaranteed, respectively, by
\[
\max_{\tau} c(t) < -\mu \|r\|_4^2 \|b\|_4^2,
\]
where \(\mu\) is given by
\[
\mu = \begin{cases} 
\frac{8}{3} K(2\pi \rho), & \text{if} \ \rho \in \Omega_4, \\
4K(2\pi \rho) + \left( \frac{3+\sqrt{5}}{8} \right)^{1/2}, & \text{if} \ \rho = \frac{2k-1}{4}, \quad k \in \mathbb{N}.
\end{cases}
\]

To obtain the explicit criteria for twist type, we will estimate \(r\) and \(\|r\|_4\) using \(a(t)\) as follows.

At first, assuming that \(a(t) \in C_{\sigma_1, \sigma_2}\). Then \(\sigma_1 \leq \rho \leq \sigma_2\). We will estimate \(\|r\|_4\) by two methods for the cases \(\rho \in \Omega_4\) and \(\rho \in \Omega_3\), respectively. Note that \(\Omega_3\) and \(\Omega_4\) consist of disjoint intervals. Let \(I_n = (a_n, b_n)\) and \(J_n = (c_n, d_n)\) be intervals of \(\Omega_3, \ \Omega_4\), respectively, i.e.,
\[
\Omega_3 = \bigcup_{n \in \mathbb{N}} I_n, \quad \Omega_4 = \bigcup_{n \in \mathbb{N}} J_n.
\]
In fact, we have
\[
a_n = (n-1)/3, \quad b_n = n/3, \quad c_n = (n-1)/4, \quad d_n = n/4.
\]
Define
\[
D_3 = \{ (\sigma_1, \sigma_2) \mid a_n < \sigma_1 \leq \sigma_2 < b_n, \quad n \in \mathbb{N} \},
\]
and
\[
D_4 = \{ (\sigma_1, \sigma_2) \mid c_n < \sigma_1 \leq \sigma_2 < d_n, \quad n \in \mathbb{N} \}.
\]
Let \(a(t) \in C_{\sigma_1, \sigma_2}\). Then \(\rho \in \Omega_3\) if \((\sigma_1, \sigma_2) \in D_3\), and \(\rho \in \Omega_4\) if \((\sigma_1, \sigma_2) \in D_4\).

Let
\[
z(t) = -\frac{r'(t)}{r(t)} + \frac{i}{r^2(t)}.
\]
Then \(z(t)\) satisfies the Riccati equation
\[
z' = z^2 + a(t). \tag{3.20}
\]
The estimate of the range of \( r(t) \) can be reduced to estimating the critical values \( r_0 = r(t_0) > 0 \) of \( r(t) \), which corresponds to \( z_0 = -i/r_0^2 \) of the function \( z(t) \). On the other hand, it is known that the Poincaré map of the Riccati equation (3.20) is a Möbius transformation

\[
T(w) = \frac{aw + b}{cw + d},
\]

(3.21)

where \( a, b, c, d \) are real numbers. Furthermore, \( z_0 = -i/r_0^2 \) is a fixed point of \( T \) and purely imaginary. Thus, we have \( a = d \) and \( b/c < 0 \). Assuming that \( a = d = 1 \), we have \( r_0 = (-c/b)^{1/4} \). Therefore \( r_0 \) can be estimated by the coefficients \( b \) and \( c \) in the Möbius transformation (3.21), which are given by \( b = T(0) \) and \( c = 1/T^{-1}(\infty) \), respectively. Taking into account that \( T(w) = z(2\pi; w) \), where \( z(t; w) \) is the solution of (3.20) satisfying \( z(0; w) = w \), the coefficients \( b \) and \( c \) can be estimated by comparing the solution of (3.20) with those of the equations

\[
z' = z^2 + \sigma_1, \quad z' = z^2 + \sigma_2.
\]

We omit the detail here and refer the readers to [10]. Explicitly, when \( (\sigma_1, \sigma_2) \in D_4 \), i.e., \( \rho \in \Omega_4 \), we have

\[
b \in [\sigma_1 \tan 2\pi\sigma_1, \sigma_2 \tan 2\pi\sigma_2]
\]

and

\[-c \in [\sigma_1^{-1} \tan 2\pi\sigma_1, \sigma_2^{-1} \tan 2\pi\sigma_2].
\]

Thus, let

\[
N_4(\sigma_1, \sigma_2) = \max \left\{ \left( \frac{2\pi}{\sigma_1 \sigma_2 \tan 2\pi\sigma_1} \right)^{1/2}, \left( \frac{2\pi}{\sigma_1 \sigma_2 \tan 2\pi\sigma_2} \right)^{1/2} \right\},
\]

we have

\[
||r||^2_t \leq N_4(\sigma_1, \sigma_2)
\]

when \( a(t) \in C_{\sigma_1, \sigma_2} \) with \( (\sigma_1, \sigma_2) \in D_4 \).

When \( (\sigma_1, \sigma_2) \in D_3 \), the above comparison results will fail because of solution blow up. In this case, we have to estimate the range of the periodic solution \( z(t) \) of the Riccati equation (3.20) by an alternative way. Let

\[
\sigma(a) = \frac{1}{2\pi} \int_0^{2\pi} a(t) dt.
\]

Then \( z(t) = i\sigma(a) \) if \( a(t) \) is a constant, i.e., \( \sigma_1 = \sigma_2 \). When \( \sigma_1 < \sigma_2 \), we have following result.

**Proposition 3.2.** [12, Proposition 2.2] Let \( \sigma \) be defined as above. If

\[
\sigma \in \Omega_2 \quad \text{and} \quad ||\tilde{a}||_1 = \int_0^{2\pi} |a(t) - \sigma^2| dt \leq \frac{\sin^2 2\pi \sigma}{2\pi},
\]

(3.22)

then the periodic solution \( z(t) \) (with \( \Re \{z(t)\} > 0 \)) of the Riccati equation (3.20) satisfies

\[
|z(t) - i\sigma(a)| \leq \tau(a) = \frac{\sin 2\pi \sigma}{2\pi} - \frac{(\sin^2 2\pi \sigma - 2\pi ||\tilde{a}||_1)^2}{2\pi} \leq \frac{||\tilde{a}||_1}{\sin 2\pi \sigma}
\]

(3.23)

for all \( t \).
From Proposition 3.2, we have
\[
\left| \frac{1}{r^2(t)} - \sigma \right| \leq |z(t) - i\sigma| \leq \tau(a).
\]
Thus, we have the inequality
\[
r^2 \leq \frac{\sigma r^2 - 1 - (\sigma^2 - \tau^2)\nu^4}{\sigma}.
\]
This inequality implies [12]
\[
\frac{(2\pi)^{1/2}}{\sigma} \frac{(1 + 16\sigma^2)^{1/2}}{(1 + \sigma^2)^{1/2} + 4\tau} \leq \sigma \leq \frac{(2\pi)^{1/2}}{\sigma} \frac{(1 + 16\sigma^2)^{1/2}}{(1 + \sigma^2)^{1/2} - 4\tau}.
\]
Furthermore, when \(a(t) \in C_{\sigma_1, \sigma_2}\) for some \(0 < \sigma_1 \leq \sigma_2\), \(\sigma\) and \(\|\tilde{a}\|_1\) can be estimated as [12]
\[
\sigma_1 \leq \sigma \leq \sigma_2, \quad \|\tilde{a}\|_1 \leq \pi(\sigma_2^2 - \sigma_1^2).
\]
Thus, there is a positive function \(G_2(\sigma)\) defined on \(\Omega_2\), such that when \(\sigma_2 \in \Omega_2\), and \(G_2(\sigma_2) \leq \sigma_1 \leq \sigma_2\), (3.22) is satisfied for all \(a(t) \in C_{\sigma_1, \sigma_2}\). Let
\[
D_2 = \{(\sigma_1, \sigma_2) | G_2(\sigma_2) \leq \sigma_1 \leq \sigma_2, \sigma_2 \in \Omega_2\}.
\]
From (3.23)-(3.25), there exists a positive function \(N_2(\sigma_1, \sigma_2)\) defined on \(D_2\) such that if \(a(t) \in C_{\sigma_1, \sigma_2}\) for some \((\sigma_1, \sigma_2) \in D_2\), the solution \(r(t)\) satisfies
\[
\|r\|_2^2 \leq N_2(\sigma_1, \sigma_2).
\]
The explicit expression for \(G_2(\sigma)\) and \(N_2(\sigma_1, \sigma_2)\) are given in [12].

Thus, together with previous results, define
\[
N(\sigma_1, \sigma_2) = \begin{cases} 
N_2(\sigma_1, \sigma_2), & \text{if } (\sigma_1, \sigma_2) \in D_2 \setminus D_4, \\
N_4(\sigma_1, \sigma_2), & \text{if } (\sigma_1, \sigma_2) \in D_4 \setminus D_2, \\
\min\{N_2(\sigma_1, \sigma_2), N_4(\sigma_1, \sigma_2)\}, & \text{if } (\sigma_1, \sigma_2) \in D_2 \cap D_4.
\end{cases}
\]
Then for \(a(t) \in C_{\sigma_1, \sigma_2}\) with \((\sigma_1, \sigma_2) \in D_2 \cup D_4\),
\[
\|r\|_2^2 \leq N(\sigma_1, \sigma_2).
\]
(3.26)

If we restrict (3.26) to the case with \(\rho \in \Omega_3\) and let
\[
D = D_3 \cap (D_2 \cup D_4),
\]
then for any \(a(t) \in C_{\sigma_1, \sigma_2}\) with \((\sigma_1, \sigma_2) \in D\), we have \(\rho \in \Omega_3\) and the estimate (3.26) holds. Explicitly, the domain \(D\) is given as
\[
D = \{(\sigma_1, \sigma_2) | G(\sigma_2) < \sigma_1 \leq \sigma_2, \quad \sigma_2 \in \Omega_3\},
\]
where
\[
G(\sigma_2) = \max\{3\sigma_2^2/3, \min\{G_2(\sigma_2), 4\sigma_2^2/4\}\}.
\]

Thus, together with Theorem 3.2, we have the following result.

**Theorem 3.3.** [12, Theorem 3.2] Assume that \(a(t) \in C(\mathbb{R}/2\pi \mathbb{Z}, \mathbb{R})\) is in \(C_{\sigma_1, \sigma_2}\) for some \((\sigma_1, \sigma_2) \in D\). Then there exists a constant \(\nu = \nu(\sigma_1, \sigma_2)\), depending only on \((\sigma_1, \sigma_2)\), such that \(x(t) = 0\) (as a 2\(\pi\)-periodic solution of (3.1)) is of twist type provided that \(b(t)\) and \(c(t)\) satisfy
\[
\max_{t \in \mathbb{R}} c(t) < -\nu(\sigma_1, \sigma_2)\|b\|_2^2.
\]
Explicitly, \( \nu(\sigma_1, \sigma_2) \) can be taken as

\[
\nu(\sigma_1, \sigma_2) = \begin{cases} 
(4K(2\pi\sigma_2) + \frac{3\sqrt{k}}{8})^{1/2} N(\sigma_1, \sigma_2), & \text{if } \frac{2k-1}{4} \in [\sigma_1, \sigma_2] \text{ for } k \in \mathbb{N}, \\
\frac{8}{3} K(2\pi\sigma_2), & \text{otherwise}.
\end{cases}
\]

3.2.2. \( L^1 \) criteria. It is sometimes interesting to find the \( L^1 \)-criteria for the stability of the periodic solution. This has been done in [11]. Main results are summarized here.

In fact, from Theorem 3.2, it is sufficient to obtain the estimates of the rotation number \( \rho \) and the norm of the periodic solution \( \|r\|^2 \) through the \( L^1 \) norm of \( a(t) \). Consider \( a(t) \in L^1(\mathbb{R}/2\pi\mathbb{Z}) \) such that its mean value \( \bar{a} = \frac{1}{2\pi} \int_0^{2\pi} a(t)dt \) is positive. Define positive constants \( \sigma, \delta, \sigma_1 \) and \( \sigma_2 \) by

\[
\sigma^2 = \bar{a}, \quad \delta = \int_0^{2\pi} |a(t) - \sigma^2|dt,
\]

and

\[
\sigma_1 = \sigma - \frac{\delta}{4\pi\sigma}, \quad \sigma_2 = \sigma + \frac{\delta}{4\pi\sigma}.
\]

A function \( a \in L^1(\mathbb{R}/2\pi\mathbb{Z}) \) is said to be admissible if \( [\sigma_1, \sigma_2] \subset \Omega_4 \). Lemma 3.2 in [11] proved that when \( a \) is admissible, then \( \rho \in [\sigma_1, \sigma_2] \subset \Omega_4 \). Let

\[
N(\sigma, \sigma_1, \sigma_2) = \frac{\sqrt{2\pi}}{\sigma} \max \left\{ \left( \frac{\tan 2\pi\sigma_1}{\tan 2\pi\sigma_2} \right)^{1/2}, \left( \frac{\tan 2\pi\sigma_2}{\tan 2\pi\sigma_1} \right)^{1/2} \right\}.
\]

Then Proposition 4.2 in [11] shown that

\[
\|r\|^2 \leq N(\sigma, \sigma_1, \sigma_2).
\]

Thus, we have the following result.

**Theorem 3.4.** [11, Theorem 2.3] Assume that \( a(t) \in L^1(\mathbb{R}/2\pi\mathbb{Z}) \) is admissible. Then, there exists a constant \( \mu = \mu(\sigma, \sigma_1, \sigma_2) > 0 \) such that \( x = 0 \) (as a periodic solution of (3.1)) is of twist type provide that \( b(t) \) and \( c(t) \) satisfy

\[
\max_{t \in \mathbb{R}} c(t) < -\mu \|b\|^2_4.
\]

Explicitly, \( \mu \) can be defined as

\[
\mu(\sigma, \sigma_1, \sigma_2) = \frac{8}{3} K(2\pi\sigma_2) N(\sigma, \sigma_1, \sigma_2).
\]

3.2.3. First stability zone. In above sections, we have reviewed some criteria when the linear equation (2.5) is not limited to the first stability zone. When (2.5) is in the first stability zone, more specific criteria can be obtained by a detailed comparison of the twist coefficients. For more results in this situation, we refer to [3, 11, 28, 38]. More specifically, we focus our attention on some criteria developed in [35] which are useful in the study of singular equations.

Let us denote

\[
a_* = \inf_{t \in [0, 2\pi]} a(t), \quad b_* = \inf_{t \in [0, 2\pi]} |b(t)|, \quad c_* = \inf_{t \in [0, 2\pi]} c(t),
\]

\[
a^* = \sup_{t \in [0, 2\pi]} a(t), \quad b^* = \sup_{t \in [0, 2\pi]} |b(t)|, \quad c^* = \sup_{t \in [0, 2\pi]} c(t).
\]

**Theorem 3.5.** [35, Theorem 3.1] Assume that for a periodic solution \( u(t) \) of equation (1.1),
(i) \(0 < a_s \leq a^* < \frac{1}{16}\),
(ii) \(c_s > 0\), and
(iii) \(10b_s^2a_s^{3/2} > 9c^*(a^*)^{5/2}\).

Then \(u(t)\) is of twist type.

**Proof.** Condition (i) implies that the solution is in the first stability region. Then, it is sufficient to prove that the twist coefficient \(\beta\) given by (3.8) is different from zero. To this aim, it is known that \(0 < 2\pi(a_s)^{1/2} \leq \theta \leq 2\pi(a^*)^{1/2} < \pi/2\) from [10, Lemma 3.6] and \((a^*)^{-1/4} \leq r(t) \leq (a_s)^{-1/4}\) from [21, Lemma 4.2]. As it was noted in [10], the function \(\chi_2(x)\) is positive if \(0 < \theta < \pi/2\). Hence

\[
\beta \geq -\frac{3\pi}{4} \frac{c^*}{a_s} + b_s^2(a^*)^{-5/2} \int_{[0,2\pi]^2} \frac{\chi_2(|\phi(t) - \phi(s)|)}{r^2(t)r^2(s)} dt ds.
\]

Taking into account that \(\phi' = \frac{1}{r^2}\), \(\int_{0}^{2\pi} \frac{dt}{r^2(t)} = \theta\),

the previous integral is

\[
\int_{[0,2\pi]^2} \frac{\chi_2(|\phi(t) - \phi(s)|)}{r^2(t)r^2(s)} dt ds = \int_{[0,\theta]^2} \chi_2(|u - v|) du dv = \frac{5\theta}{12}.
\]

Thus

\[
\beta \geq -\frac{3\pi}{4} \frac{c^*}{a_s} + b_s^2(a^*)^{-5/2}\frac{5\theta}{12} \geq -\frac{3\pi}{4} \frac{c^*}{a_s} + \frac{5\pi}{6} b_s^2(a^*)^{-5/2}(a_s)^{1/2}.
\]

Now, condition (iii) implies that \(\beta > 0\) and the proof is done. \(\square\)

Analogously, we have

**Theorem 3.6.** [35, Theorem 3.2] Assume for a periodic solution \(u(t)\) of equation (1.1) that

(i) \(0 < a_s \leq a^* < \frac{1}{16}\),
(ii) \(c_s > 0\), and
(iii) \(10b_s^2(a^*)^{3/2} < 9c^*a_s^{5/2}\).

Then \(u(t)\) is of twist type.

### 3.2.4. Parabolic case.

Though great progresses have been made for the elliptic case, there are only very few results when the linear equation (2.5) is parabolic.

The best results for stability criteria of parabolic fixed points, as far as authors know, were established by Ortega [27] and Núñez & Ortega [22] for the periodic differential equation

\[
x'' + a(t)x + c(t)x^{2n-1} + \cdots = 0, \quad n \geq 2,
\]

where \(a\) and \(c\) are \(2\pi\)-periodic. The results are given below.

**Theorem 3.7.** [27] Assume that \(\int_{0}^{2\pi} |c(t)| dt \neq 0\) and following conditions are satisfied

(i) The linearized equation (2.5) is stable;
(ii) \(c \geq 0\) or \(c \leq 0\).

Then \(x = 0\) is a stable solution of (3.27).
Theorem 3.8. [22, Theorem 5.1] Assume that the linear equation (2.5) is parabolic-unstable ($\nu \neq 0$). Then $x \equiv 0$ is stable for (3.27) if

$$\sigma \nu c(t) \geq 0, \text{ for all } t \in \mathbb{R},$$

and unstable if

$$\sigma \nu c(t) \leq 0, \text{ for all } t \in \mathbb{R}.$$  

Here $\sigma$ and $\nu$ equal $\pm 1$ or $0$, depending on the function $a(t)$ in the following way. In the parabolic case, the monodromy matrix for the linearized equation (2.5) belongs to $Sp(\mathbb{R}^2)$ and is conjugate to one of the six matrices $\pm P_+, \pm P_-, \pm I$, where

$$P_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P_- = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$  

The function $\sigma$ and $\nu$ of the conjugation class in $Sp(\mathbb{R}^2)$ are defined as

$$\sigma(P_+) = \sigma(P_-) = \sigma(I) = 1, \quad \sigma(-P_+) = \sigma(-P_-) = \sigma(-I) = -1,$$

$$\nu(P_+) = \nu(-P_-) = 1, \quad \nu(P_-) = \nu(-P_+) = -1, \quad \nu(I) = \nu(-I) = 0.$$  

While applying Theorems 3.7 and 3.8 to the stability of the equilibrium of the swing

$$x'' + \alpha(t) \sin x = 0,$$  

where $\alpha(t)$ is a positive $2\pi$-periodic function given by the swing length, one has the following result.

Corollary 3.9. [22, Corollary 5.2] The equilibrium $x \equiv 0$ is stable for (3.28) if and only if the equation

$$x'' + \alpha(t)x = 0$$

is elliptic or parabolic with $\sigma \nu \leq 0$.

3.3. Twist criteria for planar nonlinear systems. Since the twist criteria given in Section 2 are also applicable to planar systems, it is very interesting to establish the corresponding third order approximation for planar Hamiltonian systems. However, since planar nonlinear systems are more complicated than scalar Lagrangian equations, there are only very limited progresses toward the general method as far as we know. These includes some analysis result for the linear systems as given in Section 2.

Here we recall a twist criterion for a relatively simpler nonlinear planar system. Consider

$$\begin{cases}
\dot{x} = b(t)y + c(t)y^{2n-1} + \frac{\partial G}{\partial y}(t, x, y), \\
\dot{y} = -a(t)x - d(t)x^{2n-1} - \frac{\partial G}{\partial x}(t, x, y),
\end{cases}$$  

where $a, b, c, d$ are $T$-periodic functions and $n \geq 2$, and $G : \mathbb{R} \times B_{\epsilon}(0) \to \mathbb{R}$ is a continuous function with continuous derivatives of all orders with respect to the second variable, $T$-periodic in $t$, and

$$G(t, x, y) = O((x^2 + y^2)^{n+1/2}), \quad (x, y) \to (0, 0),$$

uniformly with respect to $t \in \mathbb{R}$. For such an example, one Lyapunov stability result for the trivial solution has been proved.
Theorem 3.10. [2, Theorem 3.5] Assume that (2.1) is stable, and
\[ \int_{0}^{T} |c(t)|dt \neq 0, \quad \int_{0}^{T} |d(t)|dt \neq 0. \]
Then the trivial solution \( x(t) \equiv 0 \) of (3.29) is stable if one of the following two conditions holds
(i) \( c(t) \geq 0, \quad d(t) \geq 0 \),
(ii) \( c(t) \leq 0, \quad d(t) \leq 0 \).

Remark 3.11. The first twist coefficient of (3.29) is given by
\[ \beta = -\frac{1}{2^{2n+1}n} \left( \frac{2n}{n} \right) \left( \int_{0}^{T} c(t)|\phi(t)|^{2n}dt + \int_{0}^{T} d(t)|\psi(t)|^{2n}dt \right), \]
where \( \phi, \psi \) are as in Section 2.

In Theorem 3.10, we have reduced the Lyapunov stability for the equilibrium \((x, y) = (0, 0)\) of nonlinear system (3.29) to that of the corresponding linear system (2.1). For linear systems (2.1), some stability criteria have been established in [36, 40]. The corresponding stability criteria for Hill equations were proved in [42].

4. Applications.

4.1. The forced pendulum. The equation of forced pendulum is a classical model that constitutes a significative touchstone for any theoretical advance in the knowledge of dynamical systems. Here, we can apply the above results to the stability of the least amplitude periodic solution \( x_\omega(t) \) of the forced pendulum
\[ x'' + \omega^2 \sin x = p(t), \tag{4.1} \]
where the frequency \( \omega > 0 \) and the forcing \( p \in C(\mathbb{R}/2\pi \mathbb{Z}) \). It follows from Theorem 2.1 in [10] that when
\[ \|p\|_1 \leq \frac{4\sqrt{2}}{3} \frac{\omega |\sin \omega \pi|^{3/2}}{(T/\varpi) \cos \varpi s |d\varpi|^1/2}, \]
the equation (4.1) has a unique 2\( \pi \)-periodic solution \( x = x_\omega(t) \) such that \( \|x_\omega\|_\infty \) is the smallest among all of 2\( \pi \)-periodic solutions of (4.1). Moreover, \( x_\omega(t) \) satisfies
\[ \|x_\omega\|_\infty \leq \frac{3\|p\|_1}{4\omega |\sin \omega \pi|}. \]
Applying Theorem 3.3 to \( x_\omega(t) \), we have the following stability result.

Theorem 4.1. [12, Theorem 4.1] These exists a function \( P(\omega) \) such that
(i) If \( \omega \in \Omega_3 \) and \( p(t) \in C(\mathbb{R}/2\pi \mathbb{Z}, \mathbb{R}) \) satisfy
\[ \|p\|_1 < P(\omega), \]
then the least amplitude 2\( \pi \)-periodic solution \( x_\omega(t) \) of (4.1) is of twist type.
(ii) At the fourth order resonances, we have \( \lim_{k \to \infty} P((2k - 1)/4) = \sqrt{2}/\pi \).
(iii) \( P(\omega) \) is of order \( O(\omega^{1/2}) \), when \( \omega \) is bounded away from the resonances of order \( \leq 4 \) and tends to \( +\infty \).
4.2. Superlinear equations. Consider the following superlinear equation
\[ x'' + e^x = \sigma + h(t), \] (4.2)
where \( h \in \tilde{L}^1(\mathbb{R}/TZ, \mathbb{R}) \), \( T > 0 \). Since \( e^x \) is increasing, equation (4.2) is of the Landesman-Lazer type. It is well known [37] that equation (4.2) has at least one \( T \)-periodic solution if and only if the mean value \( \sigma \) of the external forcing satisfies \( \sigma > 0 \). In order to describe further properties of the periodic solutions, let us define, for each \( h \in \tilde{L}^1(\mathbb{R}/TZ) \), a bifurcation value
\[ E_0(h) = \{ \sigma' > 0 \mid \text{for any } 0 < \sigma < \sigma', (4.2) \text{ has a unique } T \text{-periodic solution} \}. \]
Note that \( E_0(h) \) is well defined. In fact, it has been proved in [30] that
\[ \inf_h E_0(h) = \Sigma_0(T) = 16/T^2. \]
We use \( \psi_{\sigma,h}(t) \) to denote the unique \( T \)-periodic solution of (4.2) when \( \sigma \in (0, E_0(h)) \).
Next, we define
\[ E_1(h) = \inf \{ 0 < \sigma' < E_0(h) \mid \forall 0 < \sigma < \sigma', \text{the solution } \psi_{\sigma,h}(t) \text{ of (4.2) is elliptic} \}. \]
It was again proved in [30] that
\[ \inf_h E_1(h) = \Sigma_1(T) = 4/T^2. \]
Consider the bifurcation value
\[ E_2(h) = \inf \{ 0 < \sigma' < E_1(h) \mid \forall 0 < \sigma < \sigma', \text{the solution } \psi_{\sigma,h}(t) \text{ of (4.2) is twist} \}. \]
The following result shows that \( E_2(h) \) is well defined and \( E_2(h) \) has a positive lower bound
\[ \inf_h E_2(h) = \Sigma_2(T) > 0. \]

**Theorem 4.2.** [5, Theorem 6.1] There exists a positive constant \( 0 < \Sigma_2 < \Sigma_1 \) (independent of \( h \in \tilde{L}^1(\mathbb{R}/2\pi\mathbb{Z}) \)) such that if \( 0 < \sigma \leq \Sigma_2 \), then, for each \( h \in \tilde{L}^1(\mathbb{R}/2\pi\mathbb{Z}) \), the solution \( \psi_{\sigma,h}(t) \) of (4.2) is of twist type.

4.3. Singular equations. In this section, we will review some stability results for the semilinear singular equation
\[ x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \] (4.3)
with \( a, b, c \in L^1(\mathbb{R}/2\pi\mathbb{Z}) \). This equation contains as particular cases the classical Lazer-Solimini equation
\[ x'' - \frac{1}{x^\alpha} = p(t) - s, \] (4.4)
with \( \int_0^{2\pi} p(t)dt = 0 \) and \( \alpha \geq 1 \), and the Brillouin equation
\[ x'' + \gamma(1 + \delta \cos t)x = \frac{1}{x}, \] (4.5)
with \( \gamma, \delta \) are positive constants and \( 0 < \delta \leq 1 \).

**Theorem 4.3.** [35, Proposition 4.1] Assume that \( s > 0 \) and that there exist constants \( m, M > 0 \) such that any \( 2\pi \)-periodic solution \( u(t) \) of (4.4) verifies \( m \leq u(t) \leq M \) for all \( t \). If
\[ m > (16\alpha)^{\frac{-1}{1-s}}, \quad \frac{M}{m} < \left( \frac{5(\alpha + 1)}{3(\alpha + 2)} \right)^{\frac{s}{2\alpha - s}}, \]
then (4.4) has a unique \( 2\pi \)-periodic solution \( u(t) \) which is of twist type.
The proof follows from Theorem 3.5. An explicit computation of the priori bounds $m, M$ involves classical techniques and tricks which are well known for those researchers with some experience in topological degree arguments. Such bounds are properly derived in [35]. As a consequence, the following result was proved.

**Corollary 4.4.** [35, Corollary 4.1] For any fixed $p \in L^1(\mathbb{R}/2\pi \mathbb{Z})$, $\int_0^{2\pi} p(t)dt = 0$ and $\alpha \geq 1$, there exist $s_1 > s_0 > 0$ such that if $0 < s < s_1$, equation (4.4) has a unique $2\pi$-periodic solution which is elliptic, whereas if $0 < s < s_0$, such a solution is of twist type.

It is important to remark that this is not a result of “small parameter” type, and in fact quantitative estimates of $s_0, s_1$ can be computed.

For the Brillouin equation, we have

**Theorem 4.5.** [35, Theorem 5.1] Assume the following condition holds:

$$\gamma(1 + \delta) < \frac{1}{32} \quad \text{and} \quad 800(1 + \delta)^9 < 81e^{-16\gamma\delta}(1 - \delta + e^{-4\gamma\delta})^5.$$  

Then, equation (4.5) has a $2\pi$-periodic solution $u(t)$ which is of twist type.

In the above results, the singularity is strong, that is, the potential is unbounded in the origin. In the general equation (4.3) the “strong force condition” means that $\lambda \geq 1$. Classically, such an assumption is important in order to get a priori bounds of the solutions. In the following two results, we can cover the strong singularity as well as the weak singularity.

**Theorem 4.6.** [34, Theorem 3.1] Assume that the Hill equation for (4.3) is elliptic without resonances up to order 4 and $b(t) > 0$ for a.e. $t$. Then, for any $\tilde{p} \in L^1(\mathbb{R}/2\pi \mathbb{Z})$, $\int_0^{2\pi} \tilde{p}(t)dt = 0$, there exists $S_1 > 0$ such that for any $s > S_1$, there exists a unique positive $T$-periodic solution of equation (4.3) which is of twist type.

**Theorem 4.7.** [5, Theorem 6.1] Assume that $\lambda > 0$, $\lambda \neq 3$ and $b(t) \equiv 1$. Then there exists a positive constant $L(\lambda) > 0$ such that for any $a \in L^1(\mathbb{R}/2\pi \mathbb{Z})$ with

$$a \geq 0 \quad \text{and} \quad 0 < \int_0^{2\pi} a(t)dt \leq L(\lambda),$$

the equation (4.3) has a unique twist positive $2\pi$-periodic solution.

4.4. **Relativistic oscillator.** Consider the relativistic pendulum

$$(\Phi(x'))' + a(t)\sin x = 0, \quad (4.6)$$

where $a(t)$ is a positive and continuous $2\pi$-periodic function and $\Phi$ is the relativistic operator given by

$$\Phi(s) = \frac{s}{\sqrt{1 - s^2}}.$$

Let $\Phi(x') = y$. Then (4.6) can be written in the equivalent form

$$\begin{cases} \dot{x} = \Phi^{-1}(y), \\ \dot{y} = -a(t)\sin x. \end{cases} \quad (4.7)$$

The stability of $x = 0$ of (4.6) is equivalent to the stability of the equilibrium of the planar system (4.7). To apply Theorem 3.10, we express (4.7) in the form

$$\begin{cases} \dot{x} = y - \frac{1}{2}y^3 + \cdots, \\ \dot{y} = -a(t)x + \frac{1}{6}a(t)x^3 + \cdots. \end{cases}$$
Theorem 4.8. [2, Corollary 4.3] If Hill equation (2.5) is stable, then the equilibrium \( x(t) \equiv 0 \) of (4.6) is stable.

5. Further problems. In spite of the amount of results reviewed in this paper, many fundamental problems still remain unsolved. In the case of scalar Lagrangian equations, despite the yet complete results for the case of elliptic, little is known for the case of parabolic. In particular, when \( b(t) \not\equiv 0 \), the stability problem for a parabolic fixed point of the nonlinear periodic differential equation (3.1) is open. Any progress along this topic will be exciting, because it relates the possibility to study the stability of periodic solution found in the restricted 3-body problem.

Up to now, we are only at the beginning towards the stability problem for planar nonlinear systems, as we can see from this survey. Only a special form of the planar systems (3.29) was studied. For the following planar nonlinear system

\[
\begin{align*}
\dot{x} &= a(t)y + c(t)y^2 + d(t)y^3 + \cdots, \\
\dot{y} &= -b(t)x - c(t)x^2 - h(t)x^3 - \cdots,
\end{align*}
\]

here \( a, b, c, d, e, h \) are \( T \)-periodic functions and the remaining terms in the system are periodic functions of the same period, the twist coefficients have been derived based on the basic facts for the planar linear systems given in this survey (refer [1] for detail). However, the stability criteria is still in its mystery because of the complicate dependence between the twist coefficients and the coefficients in the system. Moreover, it remains open on how to extend the general idea of the third order approximation to general cases of planar Hamiltonian systems.

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