Lyapunov-type inequalities for the first-order nonlinear Hamiltonian systems

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Abstract

In this paper, we establish several new Lyapunov-type inequalities for the first-order nonlinear Hamiltonian system

\[ \begin{align*}
    x'(t) &= \alpha(t)x(t) + \beta(t)|y(t)|^{\mu-2}y(t), \\
    y'(t) &= -\gamma(t)|x(t)|^{\nu-2}x(t) - \alpha(t)y(t),
\end{align*} \]

which generalize or improve all related existing ones.

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1. Introduction

Consider the first-order Hamiltonian system

\[ \begin{align*}
    x'(t) &= \alpha(t)x(t) + \beta(t)|y(t)|^{\mu-2}y(t), \\
    y'(t) &= -\gamma(t)|x(t)|^{\nu-2}x(t) - \alpha(t)y(t),
\end{align*} \]

where \( \mu, \nu > 1 \) and \( \frac{1}{\mu} + \frac{1}{\nu} = 1 \). \( \alpha(t), \beta(t) \) and \( \gamma(t) \) are locally Lebesgue integrable real-valued functions defined on \( \mathbb{R} \).

Let \( u(t) = (x(t), y(t))^T \), \( H(t, x, y) = v^{-1}\gamma(t)|x|^\nu + \alpha(t)xy + \mu^{-1}\beta(t)|y|^\mu \) and

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Then we can rewrite (1.1) as a standard first-order Hamiltonian system

\[ u'(t) = J\nabla H(t, u(t)). \]

When \( \mu = \nu = 2 \), system (1.1) reduces to the first-order linear Hamiltonian system

\[ \begin{align*}
    x'(t) &= \alpha(t)x(t) + \beta(t)y(t), \\
    y'(t) &= -\gamma(t)x(t) - \alpha(t)y(t).
\end{align*} \]

In addition, the special cases of system (1.1) contain many well-known differential equations which have been studied extensively and have many applications in the literature, such as the second-order linear differential equations

\[ \begin{align*}
    x''(t) + q(t)x(t) &= 0, \\
    x''(t) + p(t)x'(t) + q(t)x(t) &= 0,
\end{align*} \]
and the second-order half-linear differential equations
\[ \rho(t) |x'(t)|^{r-2}x''(t) + q(t) |x(t)|^{r-2}x(t) = 0 \] (1.5)
and
\[ (\rho(t)|x'(t)|^{r-2}x'(t)') + p(t) |x(t)|^{r-2}x'(t) + q(t) |x(t)|^{r-2}x(t) = 0, \] (1.6)
where \( r > 1, p(t), q(t) \) and \( \rho(t) \) are locally Lebesgue integrable real-valued functions defined on \( \mathbb{R} \) and \( \rho(t) > 0 \). For example, let
\[ y(t) = \exp \left( \int_0^t p(s) ds \right) |x'(t)|^{r-2}x'(t), \]
then (1.6) can be written as the form of (1.1):
\[
\begin{cases}
  x'(t) = \exp \left( -(r-1)^{-1} \int_0^t p(s) ds \right) |y(t)|^{(2-r)/(r-1)}y(t), \\
y'(t) = -q(t) \exp \left( \int_0^t p(s) ds \right) |x(t)|^{r-2}x(t),
\end{cases}
\] (1.7)
where \( \mu = r/(r-1), \nu = r \) and
\[ \alpha(t) = 0, \quad \beta(t) = \exp \left( -(r-1)^{-1} \int_0^t p(s) ds \right), \quad y(t) = q(t) \exp \left( \int_0^t p(s) ds \right). \]

In 1907, Lyapunov [1] established the first so-called Lyapunov inequality for Hill’s equation (1.3), which was improved to the following classical form
\[ (b-a) \int_a^b q^+(t) dt > 4 \] (1.8)
by Wintner [2] in 1951, and the constant 4 cannot be replaced by a larger number, where and in the sequel \( q^+(t) = \max \{q(t), 0\} \), if (1.3) has a real solution \( x(t) \) such that
\[ x(a) = x(b) = 0, \quad x(t) \neq 0, \quad t \in [a, b], \] (1.9)
where and in the sequel \( a, b \in \mathbb{R} \) with \( a < b \).

There are several proofs and generalizations of (1.8) in the literature. Several authors including Cheng [3], Eliason [4], Hartman and Wintner [5], Hochstadt [6], Kwong [7], Nehari [8,9], Reid [10,11] and Singh [12] have contributed to the above results. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey paper [13] by Cheng and the references quoted therein.

In particular, Hartman and Wintner [5] obtained a better Lyapunov inequality than (1.8)
\[ \int_a^b q^+(t)(t-a)(b-t) dt > b-a. \] (1.10)

In 1969, Fink and Mary [14] extended Lyapunov inequality (1.8) to Eq. (1.4) and obtained the following Lyapunov-type inequality
\[ (b-a) \int_a^b q^+(t) dt > 4 \exp \left( -\frac{1}{2} \int_a^b |p(s)| ds \right). \] (1.11)

In 2003, Guseinov and Kaymakcalan [15] further generalized (1.8) to the planar linear Hamiltonian system (1.2), and proved the following Lyapunov-type inequality
\[ \int_a^b |\alpha(t)| dt + \left[ \int_a^b \beta(t) dt \int_a^b y^+(t) dt \right]^{1/2} \geq 2, \] (1.12)
if system (1.2) has a solution \((x(t), \gamma(t))\) satisfying (1.9) and \( \beta(t) \) satisfies the following condition:
\[ (B0) \quad \beta(t) \geq 0, \quad \forall t \in [a, b]. \]

In 2010, Wang [16] obtained a new Lyapunov-type inequality different from (1.12)
\[ \int_a^b \beta(t) dt \int_a^b y^+(t) dt \geq 4 \exp \left( -2 \int_a^b |\alpha(t)| dt \right). \] (1.13)
We remark that both (1.12) and (1.13) reduce to
\[
(b - a) \int_a^b \gamma^+ (t) dt \geq 4,
\]
when \(\alpha(t) = 0\) and \(\beta(t) = 1\), which is almost the same as (1.8) except “\(>\)” being replaced by “\(\geq\)”. However, (1.12) and (1.13) are complements.

On the other hand, Yang [17] extended Lyapunov inequality (1.8) to the second-order half-linear differential equation (1.5) and obtained the following Lyapunov-type inequality
\[
\int_a^b q^+ (t) dt \left( \int_a^b [\rho(t)]^{-1/(r-1)} dt \right)^{r-1} > 2^r,
\]
if Eq. (1.5) has a solution \(x(t)\) satisfying (1.9).

Recently Lee et al. [18] proved that if Eq. (1.6) has a solution \(x(t)\) satisfying (1.9), then
\[
(b - a)^{r-1} \int_a^b q^+ (t) dt > 2^r \exp \left( -\int_a^b |p(s)| ds \right), \quad 1 < r < 2;
\]
\[
4 \exp \left( -\int_a^b |p(s)| ds \right), \quad r \geq 2.
\]
Later, the above inequality was further improved to
\[
(b - a)^{r-1} \int_a^b q^+ (t) dt > 2^r \exp \left( -\int_a^b |p(s)| ds \right)
\]
by Tiryaki et al. [19].

We remark that both inequalities (1.16) and (1.17) reduce to
\[
(b - a) \int_a^b q^+ (t) dt > 4 \exp \left( -\int_a^b |p(s)| ds \right),
\]
when \(r = 2\), which is worse than (1.11). Thus, both (1.16) and (1.17) are not really the generalization of (1.11) to Eq. (1.6).

In 2007, Tiryaki et al. [19] further generalized (1.12) to the nonlinear system (1.1), and proved the following Lyapunov-type inequality
\[
\int_a^b |\alpha(t)| dt + \left( \int_a^b \beta(t) dt \right)^{1/\mu} \left( \int_a^b \gamma^+ (t) dt \right)^{1/\nu} \geq 2.
\]

The other related results can be found in [20–26].

Motivated by Yang [17], Lee et al. [18] and Tiryaki et al. [19], the purpose of this paper is to generalize Lyapunov-type inequalities (1.10) and (1.11) for the second-order linear differential equations (1.3) and (1.4), respectively, to the nonlinear system (1.1). Our Lyapunov-type inequalities obtained in this paper improve or generalize all results mentioned above.

2. Lyapunov-type inequalities for system (1.1)

In this section, we establish some new Lyapunov-type inequalities for system (1.1). Denote
\[
\zeta(t) := \left[ \int_a^t \beta(\tau) \exp \left( \mu \int_\tau^t \alpha(s) ds \right) d\tau \right]^{\nu/\mu},
\]
\[
\eta(t) := \left[ \int_t^b \beta(\tau) \exp \left( -\mu \int_\tau^t \alpha(s) ds \right) d\tau \right]^{\nu/\mu}.
\]

**Theorem 2.1.** Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then one has the following inequality
\[
\int_a^b \frac{\zeta(t) \eta(t)}{\zeta(t) + \eta(t)} \gamma^+ (t) dt \geq 1.
\]
Proof. By (1.1), we obtain
\[ (x(t)y(t))' = \beta(t) |y(t)|^{\mu} - \gamma(t) |x(t)|^{\nu}. \]
Integrating the above equation from \(a\) to \(b\) and taking into account that \(x(a) = x(b) = 0\), we obtain
\[ \int_a^b y(t) |x(t)|^{\nu} dt = \int_a^b \beta(t) |y(t)|^{\mu} dt. \tag{2.4} \]
From the first equation of (1.1) and the fact that \(x(a) = x(b) = 0\), we have
\[ x(t) = \int_a^t \beta(\tau) |y(\tau)|^{\mu-2} y(\tau) \exp \left( \int_{\tau}^t \alpha(s) ds \right) d\tau, \quad t \geq a, \tag{2.5} \]
and
\[ x(t) = -\int_t^b \beta(\tau) |y(\tau)|^{\mu-2} y(\tau) \exp \left( -\int_t^\tau \alpha(s) ds \right) d\tau, \quad t \leq b. \tag{2.6} \]
It follows from (2.1), (2.5) and the Hölder inequality that
\[ |x(t)|^{\nu} = \left[ \int_a^t \beta(\tau) |y(\tau)|^{\mu-2} y(\tau) \exp \left( \int_{\tau}^t \alpha(s) ds \right) d\tau \right]^{\nu/\mu} \]
\[ \leq \left[ \int_a^t \beta(\tau) \exp \left( \mu \int_{\tau}^t \alpha(s) ds \right) d\tau \right]^{\nu/\mu} \int_a^t \beta(\tau) |y(\tau)|^{\mu} d\tau \]
\[ = \zeta(t) \int_a^t \beta(\tau) |y(\tau)|^{\mu} d\tau, \quad a \leq t \leq b. \tag{2.7} \]
Similarly, it follows from (2.2), (2.6) and the Hölder inequality that
\[ |x(t)|^{\nu} = \left[ \int_t^b \beta(\tau) |y(\tau)|^{\mu-2} y(\tau) \exp \left( -\int_t^\tau \alpha(s) ds \right) d\tau \right]^{\nu/\mu} \]
\[ \leq \left[ \int_t^b \beta(\tau) \exp \left( -\mu \int_t^\tau \alpha(s) ds \right) d\tau \right]^{\nu/\mu} \int_t^b \beta(\tau) |y(\tau)|^{\mu} d\tau \]
\[ = \eta(t) \int_t^b \beta(\tau) |y(\tau)|^{\mu} d\tau, \quad a \leq t \leq b. \tag{2.8} \]
From (2.7) and (2.8), we have
\[ |x(t)|^{\nu} \leq \frac{\zeta(t) \eta(t)}{\zeta(t) + \eta(t)} \int_a^b \beta(\tau) |y(\tau)|^{\mu} d\tau, \quad a \leq t \leq b. \tag{2.9} \]
Now, it follows from (2.4) and (2.9) that
\[ \int_a^b \gamma^{+}(t) |x(t)|^{\nu} dt \leq \int_a^b \frac{\zeta(t) \eta(t)}{\zeta(t) + \eta(t)} \gamma^{+}(t) dt \int_a^b \beta(t) |y(t)|^{\mu} dt \]
\[ = \int_a^b \frac{\zeta(t) \eta(t)}{\zeta(t) + \eta(t)} \gamma^{+}(t) dt \int_a^b \beta(t) |y(t)|^{\mu} dt \]
\[ \leq \int_a^b \frac{\zeta(t) \eta(t)}{\zeta(t) + \eta(t)} \gamma^{+}(t) dt \int_a^b \gamma^{+}(t) |x(t)|^{\nu} dt. \tag{2.10} \]
We claim that
\[ \int_a^b \gamma^{+}(t) |x(t)|^{\nu} dt > 0. \tag{2.11} \]
If (2.11) is not true, then
\[ \int_a^b \gamma^{+}(t) |x(t)|^{\nu} dt = 0. \tag{2.12} \]
From (2.4) and (2.12), we have
\[
0 \leq \int_a^b \beta(t) |y(t)|^\mu \, dt = \int_a^b \gamma(t) |x(t)|^\nu \, dt \\
\leq \int_a^b \gamma^+(t) |x(t)|^\nu \, dt = 0.
\]
It follows that
\[
\beta(t) |y(t)|^{\mu-2} y(t) \equiv 0, \quad a \leq t \leq b. \tag{2.13}
\]
Combining (2.5) with (2.13), we obtain that \(x(t) \equiv 0\) for \(a \leq t \leq b\), which contradicts (1.9). Therefore, (2.11) holds. Hence, it follows from (2.10) and (2.11) that (2.3) holds. \(\square\)

**Theorem 2.2.** Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then there exists \(c \in (a, b)\) such that
\[
\int_c^b \xi(t) \gamma^+(t) \, dt = \int_c^b \eta(t) \gamma^+(t) \, dt \geq 1. \tag{2.14}
\]

**Proof.** Choose \(c \in (a, b)\) such that
\[
\int_c^b \xi(t) \gamma^+(t) \, dt = \int_c^b \eta(t) \gamma^+(t) \, dt \equiv M. \tag{2.15}
\]
From (2.7) and (2.15), we have
\[
\int_a^c \gamma^+(t) |x(t)|^\nu \, dt \leq \int_a^c \gamma^+(t) \xi(t) \left( \int_a^t \beta(\tau) |y(\tau)|^\mu \, d\tau \right) \, dt \\
\leq M \int_a^c \beta(t) |y(t)|^\mu \, dt. \tag{2.16}
\]
Similarly, we can obtain from (2.8) and (2.15)
\[
\int_c^b \gamma^+(t) |x(t)|^\nu \, dt \leq M \int_c^b \beta(t) |y(t)|^\mu \, dt. \tag{2.17}
\]
Adding (2.16) and (2.17) and using (2.4), we have
\[
\int_a^b \gamma^+(t) |x(t)|^\nu \, dt \leq M \int_a^b \beta(t) |y(t)|^\mu \, dt \\
= M \int_a^b \gamma(t) |x(t)|^\nu \, dt \\
\leq M \int_a^b \gamma^+(t) |x(t)|^\nu \, dt. \tag{2.18}
\]
Hence, it follows from (2.11) and (2.18) that \(M \geq 1\), i.e. (2.14) holds. \(\square\)

**Corollary 2.3.** Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then one has the following inequality
\[
\int_a^b \tilde{\gamma}^+(t) \left( \int_a^t \tilde{\beta}(\tau) \, d\tau \right) \int_t^b \tilde{\beta}(\tau) \, d\tau \frac{v}{2\mu} \, dt \geq 2 \exp \left( -\frac{v}{2} \int_a^b (|x(s)| + \mu^{-1} |\omega(s)|) \, ds \right), \tag{2.19}
\]
where
\[
\tilde{\beta}(t) = \beta(t) \exp \left( \int_{t_0}^t \omega(s) \, ds \right), \quad \tilde{\gamma}(t) = \gamma(t) \exp \left( -\frac{v}{\mu} \int_{t_0}^t \omega(s) \, ds \right) \tag{2.20}
\]
for some \(t_0 \in \mathbb{R}\).
**Proof.** Since
\[ \zeta(t) + \eta(t) \geq 2[\zeta(t)\eta(t)]^{1/2}, \]
it follows from (2.1)–(2.3) and (2.20) that
\[
1 \leq \int_a^b \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} \gamma^+(t) dt \\
\leq \frac{1}{2} \int_a^b [\zeta(t)\eta(t)]^{1/2} \gamma^+(t) dt \\
= \frac{1}{2} \int_a^b \gamma^+(t) \left[ \int_a^t \beta(t \exp \left( \mu \int \alpha(s) ds \right) \right] dt \\
\leq \frac{1}{2} \left[ \int_a^b \gamma^+(t) \left( \int_a^t \beta(t \exp \left( \mu \int \alpha(s) ds \right) \right) dt \right]^{v/\mu} \\
\leq \frac{1}{2} \left[ \int_a^b \tilde{\gamma}^+(t) \left( \int_a^t \tilde{\beta}(t) dt \right) \left( \int_a^t \tilde{\beta}(t) dt \right) \right]^{v/\mu} \exp \left( \frac{v}{2} \int_a^b (|\alpha(s)| + \mu^{-1} |\omega(s)|) ds \right),
\]
which implies that (2.19) holds. \[ \square \]

**Corollary 2.4.** Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \tilde{\gamma}^+(t) dt \int_a^b \tilde{\gamma}^+(t) \left( \int_a^t \tilde{\beta}(t) dt \right) \left( \int_a^t \tilde{\beta}(t) dt \right) \right]^{v/\mu} dt \\
\geq 4 \exp \left( -v \int_a^b (|\alpha(s)| + \mu^{-1} |\omega(s)|) ds \right), \quad (2.22)
\]
where \(\tilde{\beta}(t)\) and \(\tilde{\gamma}(t)\) are defined by (2.20).

**Proof.** By Theorem 2.2, we have
\[
1 \leq \int_a^c \gamma^+(t) \left[ \int_a^t \beta(t \exp \left( \mu \int \alpha(s) ds \right) \right]^{v/\mu} dt \\
= \int_a^c \gamma^+(t) \exp \left( \frac{v}{\mu} \int_a^t \omega(s) ds \right) \left[ \int_a^t \tilde{\beta}(t) \exp \left( \mu \int \alpha(s) ds - \int_a^t \omega(s) ds \right) \right]^{v/\mu} dt \\
\leq \int_a^c \gamma^+(t) \left[ \int_a^t \tilde{\beta}(t) \exp \left( \mu |\alpha(s)| + |\omega(s)| ds \right) \left( \mu |\alpha(s)| + |\omega(s)| ds \right) \right]^{v/\mu} dt \\
\leq \left[ \int_a^c \gamma^+(t) dt \right] \left[ \int_a^c \tilde{\beta}(t) dt \right]^{v/\mu} \exp \left( v \int_a^c (|\alpha(s)| + \mu^{-1} |\omega(s)|) ds \right), \quad (2.23)
\]
and
\[
1 \leq \int_c^b \gamma^+(t) \left[ \int_t^b \beta(t \exp \left( -\mu \int_a^t \alpha(s) ds \right) \right]^{v/\mu} dt \\
= \int_c^b \gamma^+(t) \exp \left( \frac{v}{\mu} \int_a^t \omega(s) ds \right) \left[ \int_t^b \tilde{\beta}(t) \exp \left( -\mu \int_a^t \alpha(s) ds - \int_a^t \omega(s) ds \right) \right]^{v/\mu} dt \\
\leq \int_c^b \gamma^+(t) \left[ \int_t^b \tilde{\beta}(t) \exp \left( \mu |\alpha(s)| + |\omega(s)| ds \right) \right]^{v/\mu} dt \\
\leq \left[ \int_c^b \gamma^+(t) \left( \int_t^b \tilde{\beta}(t) dt \right)^{v/\mu} \right] \exp \left( v \int_c^b (|\alpha(s)| + \mu^{-1} |\omega(s)|) ds \right). \quad (2.25)
\]
\[
\left\lbrack \int_{c}^{b} \tilde{\nu}^+ (t) dt \right\rbrack \left( \int_{c}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} \exp \left( \nu \int_{c}^{b} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right).
\]

From (2.23) and (2.25), we have
\[
\int_{a}^{b} \tilde{\nu}^+ (t) \left( \int_{a}^{c} \tilde{\beta} (t) dt \int_{t}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} dt \\
\geq \left( \int_{a}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} \int_{a}^{c} \tilde{\nu}^+ (t) \left( \int_{a}^{t} \tilde{\beta} (t) dt \right)^{v/\mu} dt \\
\geq \left( \int_{a}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} \exp \left( -\nu \int_{a}^{c} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right).
\]

and
\[
\int_{c}^{b} \tilde{\nu}^+ (t) \left( \int_{a}^{c} \tilde{\beta} (t) dt \int_{t}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} dt \\
\geq \left( \int_{c}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} \int_{c}^{b} \tilde{\nu}^+ (t) \left( \int_{t}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} dt \\
\geq \left( \int_{c}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} \exp \left( -\nu \int_{c}^{b} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right).
\]

From (2.24), (2.26), (2.27) and (2.28), we have
\[
\int_{a}^{b} \tilde{\nu}^+ (t) \left( \int_{a}^{t} \tilde{\beta} (t) dt \int_{t}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} dt \\
\geq \left( \int_{c}^{b} \tilde{\beta} (t) dt \right)^{v/\mu} \exp \left( -\nu \int_{a}^{c} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right) + \exp \left( -\nu \int_{c}^{b} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right) \\
= \frac{\int_{a}^{b} \tilde{\nu}^+ (t) dt}{\int_{a}^{c} \tilde{\nu}^+ (t) dt \int_{c}^{b} \tilde{\nu}^+ (t) dt} \exp \left( -\nu \int_{a}^{b} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right) \\
\geq \frac{4}{\int_{a}^{b} \tilde{\nu}^+ (t) dt} \exp \left( -\nu \int_{a}^{b} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right).
\]

It follows that (2.22) holds. \( \square \)

Since
\[
\int_{a}^{t} \tilde{\beta} (t) dt \int_{t}^{b} \tilde{\beta} (t) dt \leq \frac{1}{4} \left( \int_{a}^{b} \tilde{\beta} (t) dt \right)^{2},
\]
Corollary 2.5 follows from Corollary 2.3 or Corollary 2.4.

**Corollary 2.5.** Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\left( \int_{a}^{b} \tilde{\beta} (t) dt \right)^{1/\mu} \left( \int_{a}^{b} \tilde{\nu}^+ (t) dt \right)^{1/v} \geq 2 \exp \left( -\frac{1}{2} \int_{a}^{b} \left( |\alpha (s)| + \mu^{-1} |\omega (s)| \right) ds \right).
\]

where \(\tilde{\beta} (t)\) and \(\tilde{\nu}^+ (t)\) are defined by (2.20).

Let \(\omega (t) = 0\) in Corollaries 2.3–2.5, we have immediately the following three corollaries.
Corollary 2.6. Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b y(t) \left( \int_a^t \beta(\tau) \int_t^b \beta(\tau) d\tau \right)^{\nu/2} dt \geq 2 \exp \left( -\frac{v}{2} \int_a^b |\alpha(s)| ds \right).
\] (2.30)

Corollary 2.7. Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b y(t) dt \int_t^b y(t) \left( \int_a^t \beta(\tau) \int_t^b \beta(\tau) d\tau \right)^{\nu/2} dt \geq 4 \exp \left( -v \int_a^b |\alpha(s)| ds \right).
\] (2.31)

Corollary 2.8. Suppose that hypothesis (B0) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\left( \int_a^b \beta(t) dt \right)^{1/\mu} \left( \int_a^b y^+(t) dt \right)^{1/\nu} \geq 2 \exp \left( -\frac{1}{2} \int_a^b |\alpha(s)| ds \right).
\] (2.32)

Note that inequalities (2.3), (2.14), (2.19), (2.22) and (2.29)–(2.32) are, in general, not strict. In fact, Example 2.6 in [26] shows the equation may hold in (2.22). Let us impose the following stronger hypothesis
\[(B1) \quad \beta(t) > 0, \quad \forall \ t \in [a, b].\]

In case hypothesis (B0) is replaced by (B1) in the proof of Theorem 2.1, then (2.9) is strict except end-points \(a\) and \(b\), i.e.
\[
|x(t)|^\mu < \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} \int_a^b \beta(\tau)|y(\tau)|^\mu d\tau, \quad a < t < b.
\] (2.33)

In fact, if (2.33) is not true, then there exists a \(t_* \in (a, b)\) such that
\[
|x(t_*)|^\mu = \frac{\zeta(t_*)\eta(t_*)}{\zeta(t_*) + \eta(t_*)} \int_a^{t_*} \beta(\tau)|y(\tau)|^\mu d\tau.
\] (2.34)

Hence, from (2.7), (2.8) and (2.34), we obtain
\[
|x(t_*)|^\mu = \zeta(t_*) \int_a^{t_*} \beta(\tau)|y(\tau)|^\mu d\tau.
\] (2.35)

and
\[
|x(t_*)|^\mu = \eta(t_*) \int_{t_*}^b \beta(\tau)|y(\tau)|^\mu d\tau.
\] (2.36)

It follows from (2.7) and (2.35) that
\[
\left[ \int_a^{t_*} \beta(\tau)|y(\tau)|^{\mu-2} y(\tau) \exp \left( \int_{t_*}^{t_*} \alpha(s) ds \right) d\tau \right]^\nu = \left[ \int_a^{t_*} \beta(\tau) \exp \left( \mu \int_{t_*}^{t_*} \alpha(s) ds \right) d\tau \right]^{\nu/\mu} \int_a^{t_*} \beta(\tau)|y(\tau)|^\mu d\tau,
\]

which implies that there exists a constant \(c_1\) such that
\[
|y(\tau)| = c_1 \exp \left( \int_{t_*}^{t_*} \alpha(s) ds \right), \quad a \leq \tau \leq t_*.
\] (2.37)

Similarly, it follows from (2.8) and (2.36) that there exists a constant \(c_2\) such that
\[
|y(\tau)| = c_2 \exp \left( -\int_{t_*}^{t_*} \alpha(s) ds \right), \quad t_* \leq \tau \leq b.
\] (2.38)

Since \(y(\tau)\) is continuous at \(\tau = t_*\), (2.37) and (2.38) imply that \(c_1 = c_2\). If \(c_1 = c_2 = 0\), then \(y(\tau) = 0\) for \(\tau \in [a, b]\), it follows from (2.5) that \(x(t) = 0\) for \(t \in [a, b]\), which contradicts (1.9). If \(c_1 = c_2 \neq 0\), then \(|y(\tau)| > 0\) for \(\tau \in [a, b]\), it follows from (2.5) and (B1) that \(x(b) \neq 0\), which contradicts the fact that \(x(b) = 0\). Therefore, (2.33) holds. Hence, in view of the proof of Theorem 2.1, we have the following theorem.
Theorem 2.9. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then one has the following inequality
\[
\int_a^b \frac{\zeta(t)\eta(t)}{\zeta(t) + \eta(t)} Y^+(t) dt > 1,
\]
where \(\zeta(t)\) and \(\eta(t)\) are defined by (2.1) and (2.2), respectively.

In addition, in case hypothesis (B0) is replaced by (B1) in the proof of Corollary 2.4, then (2.24) and (2.26) are also strict. Thus, we shall arrive to the following results which are immediate consequences of Theorem 2.9 and Corollaries 2.3–2.8.

Corollary 2.10. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \tilde{Y}^+(t) \left( \int_a^t \tilde{\beta}(\tau) d\tau \right)^{\nu/\mu} dt > 2 \exp \left( -\frac{\nu}{2} \int_a^b (|\alpha(s)| + \mu^{-1}|\omega(s)|) ds \right),
\]
where \(\tilde{\beta}(t)\) and \(\tilde{\gamma}(t)\) are defined by (2.20).

Corollary 2.11. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \tilde{Y}^+(t) dt \int_a^b \tilde{Y}^+(t) \left( \int_a^t \tilde{\beta}(\tau) d\tau \right)^{\nu/\mu} dt > 4 \exp \left( -\nu \int_a^b (|\alpha(s)| + \mu^{-1}|\omega(s)|) ds \right),
\]
where \(\tilde{\beta}(t)\) and \(\tilde{\gamma}(t)\) are defined by (2.20).

Corollary 2.12. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\left( \int_a^b \tilde{\beta}(t) dt \right)^{1/\mu} \left( \int_a^b \tilde{Y}^+(t) dt \right)^{1/\nu} > 2 \exp \left( -\frac{1}{2} \int_a^b (|\alpha(s)| + \mu^{-1}|\omega(s)|) ds \right),
\]
where \(\tilde{\beta}(t)\) and \(\tilde{\gamma}(t)\) are defined by (2.20).

Corollary 2.13. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \gamma^+(t) \left( \int_a^t \beta(\tau) d\tau \right)^{\nu/2\mu} dt > 2 \exp \left( -\frac{\nu}{2} \int_a^b |\alpha(s)| ds \right).
\]

Corollary 2.14. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \gamma^+(t) dt \int_a^b \gamma^+(t) \left( \int_a^t \beta(\tau) d\tau \right)^{\nu/\mu} dt > 4 \exp \left( -\nu \int_a^b |\alpha(s)| ds \right).
\]

Corollary 2.15. Suppose that hypothesis (B1) is satisfied. If system (1.1) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\left( \int_a^b \beta(t) dt \right)^{1/\mu} \left( \int_a^b \gamma^+(t) dt \right)^{1/\nu} > 2 \exp \left( -\frac{1}{2} \int_a^b |\alpha(s)| ds \right).
\]

3. Comparisons with known results

In this section, we give a few comparisons with some known Lyapunov-type inequalities. First, applying Theorems 2.1 and 2.9, Corollaries 2.6 and 2.13 to the first-order linear Hamiltonian system (1.2), we have immediately the following four corollaries.

Corollary 3.1. Suppose that hypothesis (B0) is satisfied. If system (1.2) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \left[ \int_a^t \beta(\tau) \exp \left( 2 \int_t^\tau \alpha(s) ds \right) d\tau \right] \left[ \int_a^b \beta(\tau) \exp \left( -2 \int_t^\tau \alpha(s) ds \right) d\tau \right] \gamma^+(t) dt \geq 1.
\]

\[3.1\]
Corollary 3.2. Suppose that hypothesis (B1) is satisfied. If system (1.2) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b \left[ \int_t^b \beta(t) \exp \left( 2 \int_t^b \alpha(s) ds \right) d\tau \right] \left[ \int_t^b \beta(t) \exp \left( -2 \int_t^b \alpha(s) ds \right) d\tau \right] y^+(t) dt > 1. \tag{3.2}
\]

Corollary 3.3. Suppose that hypothesis (B0) is satisfied. If system (1.2) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b y^+(t) \left( \int_t^b \beta(t) d\tau \int_t^b \beta(t) d\tau \right)^{1/2} dt \geq 2 \exp \left( - \int_a^b |\alpha(s)| ds \right). \tag{3.3}
\]

Corollary 3.4. Suppose that hypothesis (B1) is satisfied. If system (1.2) has a solution \((x(t), y(t))\) satisfying (1.9), then
\[
\int_a^b y^+(t) \left( \int_t^b \beta(t) d\tau \int_t^b \beta(t) d\tau \right)^{1/2} dt \geq 2 \exp \left( - \int_a^b |\alpha(s)| ds \right). \tag{3.4}
\]

Remark 3.5. It is easy to see that Lyapunov-type inequality (3.3) is better than (1.12) and (1.13). Furthermore, Lyapunov inequality (3.2) is just the best Lyapunov-type inequality (1.10) for Hill’s equation (1.3) where \(\alpha(t) = 0\) and \(\beta(t) = 1\).

For the second-order half-linear differential equation (1.5), let \(y(t) = \rho(t)|x'(t)|^{(r-2)}x'(t)\), then we can re-write (1.5) as the form of (1.1):
\[
\begin{align*}
\{x'(t) &= \frac{1}{|\rho(t)|^{1/(r-1)}} |y(t)|^{(2-r)/(r-1)}y(t), \\
y'(t) &= -q(t)|x(t)|^{r-2}x(t),
\end{align*} \tag{3.5}
\]

where \(\mu = r/(r-1), \nu = r\) and \(\alpha(t) = 0, \beta(t) = \frac{1}{|\rho(t)|^{1/(r-1)}}, \gamma(t) = q(t)\).

Applying Theorem 2.9, and Corollaries 2.13 and 2.14 to system (3.5) (i.e. (1.5)), we have immediately the following three corollaries.

Corollary 3.6. Suppose that \(r > 1\) and \(\rho(t) > 0\). If Eq. (1.5) has a solution \(x(t)\) satisfying (1.9), then
\[
\int_a^b \left( \int_a^t \rho(t) \left( \int_a^t \rho(t)^{-1/(r-1)} d\tau \right)^{r-1} \right)^{-1} \left( \int_a^b \rho(t) \left( \int_a^t \rho(t)^{-1/(r-1)} d\tau \right)^{r-1} \right)^{-1} q^+(t) dt > 1. \tag{3.6}
\]

Corollary 3.7. Suppose that \(r > 1\) and \(\rho(t) > 0\). If Eq. (1.5) has a solution \(x(t)\) satisfying (1.9), then
\[
\int_a^b q^+(t) \left( \int_a^t \rho(t)^{-1/(r-1)} d\tau \right)^{(r-1)/2} dt > 2. \tag{3.7}
\]

Corollary 3.8. Suppose that \(r > 1\) and \(\rho(t) > 0\). If Eq. (1.5) has a solution \(x(t)\) satisfying (1.9), then
\[
\int_a^b q^+(t) dt \int_a^b q^+(t) \left( \int_a^t \rho(t)^{-1/(r-1)} d\tau \right)^{r-1} dt > 4. \tag{3.8}
\]

Remark 3.9. It is easy to see that Lyapunov-type inequalities (3.7) and (3.8) are better than (1.15) which is the main result in [26]. Moreover, Lyapunov-type inequality (3.6) reproduces inequality (1.10) when \(r = 2\) and \(\rho(t) = 1\).

Finally, applying Corollary 2.10 to system (1.7) (i.e. (1.6)), where \(\alpha(t) = 0, \tilde{\beta}(t) = 1, \gamma(t) = q(t), \omega(t) = \frac{p(s)}{r-1}\), we have immediately the following corollary.
Corollary 3.10. Suppose that \( r > 1 \). If Eq. (1.6) has a solution \( x(t) \) satisfying (1.9), then
\[
\int_a^b q^+(t)(t-a)(b-t)^{(r-1)/2}dt > 2\exp\left(-\frac{1}{2} \int_a^b |p(s)|ds\right).
\] (3.9)

Remark 3.11. Obviously, Lyapunov-type inequality (3.9) is better than (1.17). Moreover, it is also better than Lyapunov-type inequality (1.11) when \( r = 2 \).

Remark 3.12. Since \( 2e^{-u/2} > 2 - u \) for \( u > 0 \), then Lyapunov-type inequality (2.32) is better than (1.19), while (2.32) is only a corollary (2.3) and (2.30). Moreover, (2.32) can be applied when \( \int_a^b |\alpha(s)|ds \geq 2 \), while (1.19) fails to apply.

References