ROTATION NUMBER, PERIODIC FUČÍK SPECTRUM AND MULTIPLE PERIODIC SOLUTIONS

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In this paper, we will introduce the rotation number for the one-dimensional asymmetric $p$-Laplacian with a pair of periodic potentials. Two applications of this notion will be given. One is a clear characterization of two unbounded sequences of Fučík curves of the periodic Fučík spectrum of the $p$-Laplacian with potentials. With the help of the Poincaré–Birkhoff fixed point theorem, the other application is some existence result of multiple periodic solutions of nonlinear ordinary differential equations concerning with the $p$-Laplacian.

Keywords: Rotation number; $p$-Laplacian; Fučík spectrum; periodic solution.

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1. Introduction

Rotation number is a fundamental notion introduced by Poincaré in describing the dynamics of homeomorphisms of the circle and differential equations of the 2-torus. The extension to systems of many degrees of freedom with certain recurrence is also very important in many problems such as spectrum of Schrödinger operators.

In this paper, we will develop some interesting applications of rotation numbers to the asymmetric $p$-Laplacian oscillators with potentials. More precisely, given two functions $w, v \in L^1(\mathbb{R}/T\mathbb{Z})$ called potentials, we will consider the following scalar differential equation

$$\left(\phi_p(x')\right)' + w(t)\phi_p(x_+) + v(t)\phi_p(x_-) = 0.$$  \hspace{1cm} (1.1)

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Here $\phi : \mathbb{R} \to \mathbb{R}$ is defined by $\phi_p(x) = |x|^{p-2}x$, $1 < p < \infty$, and $x_+ = \max(x, 0)$ and $x_- = \min(x, 0)$ for $x \in \mathbb{R}$. By introducing the $p$-polar coordinates in the phase plane of (1.1), Eq. (1.1) induces a differential system on 2-torus. Hence one can use such an induced system to introduce the rotation number of (1.1), denoted by $\rho(w, v)$. This can be viewed as a functional of potentials $(w, v) \in (L^1(\mathbb{R}/T\mathbb{Z}))^2$. We will develop several important properties of $\rho(w, v)$. For details, see Sec. 2.

The first application of rotation numbers $\rho(w, v)$ is to the periodic Fučík spectrum. That is, let us introduce two eigen-parameters $(\lambda, \mu) \in \mathbb{R}^2$ and consider the following equation

$$
(\phi_p(x'))' + (\lambda + w(t))\phi_p(x_+) + (\mu + v(t))\phi_p(x_-) = 0,
$$

(1.2)

with the $T$-periodic boundary conditions

$$
x(0) = x(T), \quad x'(0) = x'(T).
$$

(1.3)

The (T-periodic) Fučík spectrum $\mathcal{F}_{w,v}$ of (1.2) (with potentials $w$ and $v$) is defined as the set of those parameters $(\lambda, \mu) \in \mathbb{R}^2$ such that problem (1.2)–(1.3) has non-trivial solutions.

The Fučík spectrum $\mathcal{F}_{w,v}$ is a generalization of periodic eigenvalues and plays an important role in many problems concerning with the $p$-Laplacian oscillators. Thus the structure of eigenvalues and Fučík spectrum of the $p$-Laplacian with or without potentials is a basic problem. Let us briefly review some important progress on the spectrum of the $p$-Laplacian.

Notice that when $w = v$ and $\lambda = \mu$, problem (1.2) is reduced to the eigenvalue problem

$$
(\phi_p(x'))' + (\lambda + w(t))\phi_p(x) = 0,
$$

(1.4)

called the one dimensional $p$-Laplacian with the potential $w(t)$. In case $p = 2$, (1.4) is the Hill’s equation. With the periodic boundary condition (1.3), or with the anti-periodic, or other separated boundary conditions, the eigenvalues of (1.4) are completely clear. In case $p \neq 2$, it was shown only in very recent years that the structure of eigenvalues of (1.4)–(1.3) is completely different from that for Hill’s equation. That is, besides the variational eigenvalues of (1.4)–(1.3) which correspond to those for the case $p = 2$, Binding and Rynne [2] proved that when the potential $w(t)$ is not constant, problem (1.4)–(1.3) has also some non-variational eigenvalues. But for separated (e.g. Dirichlet or Neumann) boundary conditions, (1.4) has only variational eigenvalues. Eigenvalue problems of the one-dimensional $p$-Laplacian with singular, indefinite weights have been extensively studied recently. Some interesting applications of these new results to the solvability of boundary value problems of nonlinear equations have been obtained. See [7, 8].

Let us go back the Fučík spectrum of (1.2)–(1.3). It is obvious that $\mathcal{F}_{w,v}$ contains the following two lines

$$
\mathcal{L}_w^+ = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda = \bar{\lambda}_0(w), \ \mu \in \mathbb{R}\},
$$

$$
\mathcal{L}_v^- = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda \in \mathbb{R}, \ \mu = \bar{\lambda}_0(v)\}.
$$
Rotation Number, Periodic Fučík Spectrum and Multiple Periodic Solutions

Here $\lambda_0(w)$ is the smallest $T$-periodic eigenvalue of (1.4) [15]. In case $p = 2$, when $w(t)$ and $v(t)$ are constant, e.g., $w = v = 0$, the $T$-periodic Fučík spectrum $F_{0,0}$ of

$$
(\phi_p(x'))' + \lambda \phi_p(x_+) + \mu \phi_p(x_-) = 0
$$

is completely clear. That is, besides the trivial lines $L_{0}^{+}$, $F_{0,0}$ consists of a double-sequence of hyperbolic-like curves passing through points $((2n\pi/T)^2, (2n\pi/T)^2)$, $n \in \mathbb{N}$. However, it remains an open problem that what is the structure of $F_{w,v}$ when $p \neq 2$. In this paper, we use rotation numbers to give a very simple and geometrical, but partial understanding to the periodic Fučík spectrum $F_{w,v}$ of (1.2). The idea is as follows. Given $(w, v) \in (L^1(\mathbb{R}/T\mathbb{Z}))^2$, let us introduce

$$
\rho_{w,v}(\lambda, \mu) := \rho(\lambda + w, \mu + v), \quad (\lambda, \mu) \in \mathbb{R}^2,
$$

called the rotation number function of (1.2). After developing some properties of $\rho_{w,v}(\lambda, \mu)$, we can obtain the following results.

Besides the trivial Fučík lines $L_w^+$ and $L_v^-$, for any $n \in \mathbb{N}$, the preimage

$$
\rho_{w,v}^{-1}(n) \subset \mathbb{R}^2
$$

is a connected domain, whose boundary curves belong to the $T$-periodic Fučík spectrum $F_{w,v}$. Especially, when $w = v \in L^1(\mathbb{R}/T\mathbb{Z})$, $F_{w,w}$ contains two sequences of curves emanating from $(\lambda_n(w), \mu_n(w))$ and $(\overline{\lambda}_n(w), \overline{\mu}_n(w))$ respectively, where $\lambda_n(w) \leq \overline{\lambda}_n(w)$ are the $n$th variational $T$-periodic eigenvalues of

$$
(\phi_p(x'))' + (\lambda + w(t))\phi_p(x) = 0
$$

in Zhang [15].

For precise statements, see Theorem 3.1 of Sec. 3. This result is an extension of Zhang [16] where the case $p = 2$ of (1.2) is studied. Compared with the variational approach to the Fučík spectrum of the one-dimensional $p$-Laplacian with weight

$$
(\phi_p(x'))' + \lambda w(t)\phi_p(x_+) + \mu v(t)\phi_p(x_-) = 0
$$

with respect to separated boundary conditions, as in Leadi and Marcos [9] and in Pinasco [13], our results can yield more Fučík curves and are more direct. Alif [1] also obtained the structure of the double-sequence curves for the Fučík spectrum of (1.6) with respect to Dirichlet and Neumann boundary conditions.

In Sec. 4, we will use rotation numbers $\rho(w, v)$ to give some interesting results of multiple periodic solutions of the following nonlinear scalar differential equation

$$
(\phi_p(x'))' + f(t, x) = 0,
$$

where $f(t, x) \equiv f(t + T, x)$ admits some asymptotic properties at the origin and near infinity. See (4.1) and (4.2). By exploiting the Poincaré–Birkhoff fixed point theorem [3], we can use the difference between the rotation numbers of the asymptotical equations of (1.7) at the origin and at infinity to give very precise multiple $T$-periodic solutions of (1.7), together with a detailed description on the node
structure of these periodic solutions. For precise statements of these results, see Theorem 4.1 of Sec. 4.

2. Rotation Numbers

Given an exponent $p \in (1, \infty)$, denote by $p^* \in (1, \infty)$ the conjugate number of $p$: $1/p + 1/p^* = 1$. We introduce a $p$-polar coordinate transformation of the plane $\mathbb{R}^2$

$$
\begin{align*}
    x &= r^{2/p} C_p(\theta) \\
    y &= r^{2/p^*} S_p(\theta),
\end{align*}
$$

(2.1)

where $(C_p(\cdot), S_p(\cdot))$ is the unique solution of

$$
\begin{align*}
    x' &= -\phi_p(y) \\
    y' &= \phi_p(x)
\end{align*}
$$

satisfying $(x(0), y(0)) = (1, 0)$. These functions $C_p$ and $S_p$ are called $p$-cosine and $p$-sine because they possess properties similar to those of cosine and sine as shown in the following lemma.

Lemma 2.1.

(1) Both $C_p(\theta)$ and $S_p(\theta)$ are $2\pi_p$-periodic, where

$$
\pi_p = \frac{2\pi (p - 1)^{1/p}}{p \sin(\pi/p)};
$$

(2) $C_p(\theta)$ is even in $\theta$ and $S_p(\theta)$ is odd in $\theta$;

(3) $C_p(\theta + \pi_p) = -C_p(\theta)$, $S_p(\theta + \pi_p) = -S_p(\theta)$;

(4) $C_p(\theta) = 0$ if and only if $\theta = \pi_p/2 + n\pi_p, n \in \mathbb{Z}$; and $S_p(\theta) = 0$ if and only if $\theta = n\pi_p, n \in \mathbb{Z}$;

(5) $\frac{d}{d\theta} C_p(\theta) = -\phi_p'(S_p(\theta))$ and $\frac{d}{d\theta} S_p(\theta) = \phi_p'(C_p(\theta))$;

(6) $|C_p(\theta)|^{p^*}/p + |S_p(\theta)|^{p^*}/p^* \equiv 1/p$.

In the following, without loss of generality, let us take the period as $T = 2\pi_p$. That is, $w, v \in L^1(\mathbb{R}/2\pi_p \mathbb{Z})$. Let $y = -\phi_p(x')$. Then (1.2) becomes

$$
\begin{align*}
    x' &= -\phi_p(y), \\
    y' &= (\lambda + w(t))\phi_p(x_+) + (\mu + v(t))\phi_p(x_-).
\end{align*}
$$

(2.2)

In the polar coordinates (2.1), $r$ and $\theta$ satisfy the following equations

$$
\begin{align*}
    (\log r)' &= G(t, \theta; \lambda, \mu, w, v) \\
    &:= \begin{cases} 
        (p/2)(\lambda + w(t) - 1)\phi_p(C_p(\theta))\phi_p'(S_p(\theta)) & \text{if } C_p(\theta) \geq 0, \\
        (p/2)(\mu + v(t) - 1)\phi_p(C_p(\theta))\phi_p'(S_p(\theta)) & \text{if } C_p(\theta) < 0.
    \end{cases}
\end{align*}
$$

(2.3)

$$
\theta' = A(t, \theta; \lambda, \mu, w, v)
$$

$$
:= \begin{cases} 
        p(\lambda + w(t))|C_p(\theta)|^{p^*}/p + |S_p(\theta)|^{p^*}/p^* & \text{if } C_p(\theta) \geq 0, \\
        p(\mu + v(t))|C_p(\theta)|^{p^*}/p + |S_p(\theta)|^{p^*}/p^* & \text{if } C_p(\theta) < 0.
    \end{cases}
$$

(2.4)
Let \( A(t, \theta; \lambda, \mu, w, v) \) be the Poincaré map of (2.4) defined by

\[
\Theta_{\lambda, \mu}(\theta_0) := \theta(2\pi_p; \theta_0, \lambda, \mu).
\]

The rotation number of (2.4)

\[
\rho(\lambda + w, \mu + v) = \rho_{w,v}(\lambda, \mu) := \lim_{|t| \to \infty} \frac{\theta(t; \theta_0, \lambda, \mu) - \theta_0}{t}
\]

exists and is independent of \( \theta_0 \), see [6, Theorem 2.1, Chap. 2].

Let

\[ P = \{ w : \mathbb{R} \to \mathbb{R} : w(t) \text{ is } 2\pi_p\text{-periodic and } w \in L^1(0, 2\pi_p) \}. \]

For any \( w_1, w_2, v_1, v_2 \in P \), write \((w_1, v_1) \succ (w_2, v_2)\) if \( w_1 \geq w_2, v_1 \geq v_2, \) and both \( w_1(t) > w_2(t) \) and \( v_1(t) > v_2(t) \) hold for \( t \) in a common subset of \([0, 2\pi_p]\) of positive measure.

Denote the solution of (2.4) by \( \theta(t; \theta_0, w, v) \) when \( \lambda, \mu = 0 \). The following result follows from the comparison theorem for solutions.

**Lemma 2.2.** Let \( w_1, w_2, v_1, v_2 \in P \) and \((w_1, v_1) \succ (w_2, v_2)\). Then for any \( \theta_0 \in \mathbb{R} \),

1. \( \theta(t; \theta_0, w_1, v_1) \geq \theta(t; \theta_0, w_2, v_2) \) for all \( t \geq 0 \); and
2. \( \theta(t; \theta_0, w_1, v_1) > \theta(t; \theta_0, w_2, v_2) \) for all \( t > 2\pi_p \).

Some properties for the rotation number function \( \rho_{w,v}(\lambda, \mu) \), see (2.8), are collected in the following lemma.

**Lemma 2.3.** Let \( w, v \in P \). Then \( \rho_{w,v}(\lambda, \mu) \) has the following properties.

1. \( \rho_{w,v}(\lambda, \mu) \) is continuous in \((\lambda, \mu) \in \mathbb{R}^2\);
2. \( \rho_{w,v}(\lambda, \mu) \) does not decrease when either \( \lambda \) or \( \mu \) increases;
(3) \( \rho_{w,v}(\lambda, \mu) \geq 0 \) for all \((\lambda, \mu) \in \mathbb{R}^2 \);
(4) \( \rho_{w,v}(\lambda, \mu) = 0 \) if either \( \lambda \ll -1 \) or \( \mu \ll -1 \); and
(5) \( \rho_{w,v}(\lambda, \mu) \) approaches \( +\infty \) when both \( \lambda \) and \( \mu \) approach \( +\infty \).

**Proof.** (1) The vector field \( A(t, \theta; \lambda, \mu) \) depends on \( \lambda, \mu \) continuously. Then so does the Poincaré map \( \Theta_{\lambda, \mu}(\theta_0) \). The continuity of \( \rho_{w,v}(\lambda, \mu) \) follows from [6, Corollary 2.1, Chap. 2].
(2) The monotonicity of \( \rho_{w,v}(\lambda, \mu) \) follows from Lemma 2.2.
(3) Notice that \( \theta'(t; \theta_0) = 1 \) when \( C_p(\theta(t; \theta_0)) = 0 \). It follows that \( \theta(t; \theta_0) \geq n\pi_p + \pi_p/2 \) for all \( t \geq 0 \) provided \( \theta_0 > n\pi_p + \pi_p/2 \). Then \( \rho_{w,v}(\lambda, \mu) \geq 0 \) follows from the independence of the choice of \( \theta_0 \) in the definition of rotation numbers.
(4) Let \( \lambda_0(w) \in \mathbb{R} \) be the smallest eigenvalue of (1.4)–(1.3). Similarly, \( \lambda_0(v) \) is the smallest \( T \)-periodic eigenvalue of (1.4) with the potential replaced by \( v(t) \). See Zhang [15]. Then, associated with \( \lambda_0(w) \), the following equation
\[
(\phi_p(x'))' + (\lambda_0(w) + w(t))\phi_p(x) = 0
\]
has a nowhere vanishing eigenfunction \( x_0(t) \). Assume that \( x_0(t) > 0 \) for all \( t \in \mathbb{R} \). For \( \lambda = \lambda_0(w) \) and any \( \mu \in \mathbb{R} \), \( x_0(t) \) is also a solution of (1.2). Then the corresponding solution \( \theta(t) \) of (2.4) satisfies \( |\theta(t)| < \pi_p \). Hence \( \rho_{w,v}(\lambda_0(w), \mu) = 0 \) for all \( \mu \in \mathbb{R} \). Analogously, \( \rho_{w,v}(\lambda, \lambda_0(v)) = 0 \) for all \( \lambda \in \mathbb{R} \). Combining the monotonicity of \( \rho_{w,v}(\lambda, \mu) \) proved in (2), we have the result
\[
\rho_{w,v}(\lambda, \mu) = 0, \quad \text{if either } \lambda \leq \lambda_0(w) \text{ or } \mu \leq \lambda_0(v). \quad (2.9)
\]
Obviously, (2.9) is more precise than the statement (4) of the lemma.
(5) Suppose that \( \lambda, \mu > 0 \). For Eq. (1.2), let us introduce new transformations of variables
\[
\hat{x} = x, \quad \hat{y} = -\lambda^{-1/p} \phi_p(x') \quad \text{when } x \geq 0,
\hat{x} = x, \quad \hat{y} = -\mu^{-1/p} \phi_p(x') \quad \text{when } x < 0.
\]
Then Eq. (1.2) is switched to the following new system
\[
\dot{x}' = \begin{cases} -\lambda^{1/p} \phi_p'(\hat{y}), & x \geq 0, \\ -\mu^{1/p} \phi_p'(\hat{y}), & x < 0, \end{cases}
\dot{y}' = \begin{cases} \left(\lambda^{1/p} + \lambda^{-1/p} w(t)\right)\phi_p(\hat{x}), & x \geq 0, \\ \left(\mu^{1/p} + \mu^{-1/p} v(t)\right)\phi_p(\hat{x}), & x < 0. \end{cases} \quad (2.10)
\]
Under the following new \( p \)-polar coordinates
\[
\hat{x} = \hat{r}^{2/p} C_p(\hat{\theta}), \quad \hat{y} = \hat{r}^{2/p} S_p(\hat{\theta}), \quad (2.11)
\]
we know that the equation for $\hat{\theta}$ is
\[
\hat{\theta}' = \begin{cases} 
\lambda^{1/p} + \lambda^{-1/p'} w(t) C_p(\hat{\theta})^p, & C_p(\hat{\theta}) \geq 0, \\
\mu^{1/p} + \mu^{-1/p'} v(t) C_p(\hat{\theta})^p, & C_p(\hat{\theta}) < 0.
\end{cases}
\tag{2.12}
\]

Denote by $\hat{\theta}(t; \hat{\theta}_0, \lambda, \mu)$ the solution of (2.12). Then we have the trivial estimates
\[
\min \left\{ \lambda^{1/p}, \mu^{1/p} \right\} t - \min \left\{ \lambda^{1/p'}, \mu^{1/p'} \right\} \int_0^t \max \{|w(s)|, |v(s)|\} \, ds 
\leq \hat{\theta}(t; \hat{\theta}_0, \lambda, \mu) - \hat{\theta}_0 
\leq \max \left\{ \lambda^{1/p}, \mu^{1/p} \right\} t + \max \left\{ \lambda^{1/p'}, \mu^{1/p'} \right\} \int_0^t \max \{|w(s)|, |v(s)|\} \, ds
\]
for all $t \geq 0$ and all $\hat{\theta}_0 \in \mathbb{R}$. Thus
\[
\min \left\{ \lambda^{1/p}, \mu^{1/p} \right\} - \frac{\max \{|w|, |v|\}}{\min \left\{ \lambda^{1/p'}, \mu^{1/p'} \right\}} \leq \lim_{t \to +\infty} \frac{\hat{\theta}(t; \hat{\theta}_0, \lambda, \mu) - \hat{\theta}_0}{t} 
\leq \max \left\{ \lambda^{1/p}, \mu^{1/p} \right\} + \frac{\max \{|w|, |v|\}}{\min \left\{ \lambda^{1/p'}, \mu^{1/p'} \right\}}
\tag{2.13}
\]
for all $\hat{\theta}_0 \in \mathbb{R}$, where $\overline{q}$ is the mean value for $q \in L^1(\mathbb{R}/2\pi p \mathbb{Z})$.

Notice that both (2.2) and (2.10) are equivalent to (1.2). Moreover, one has the relation
\[
x = \ldot{x}, \quad y = \lambda^{1/p'} \ldot{y} \quad \text{when } x \geq 0, \\
x = \ldot{x}, \quad y = \mu^{1/p'} \ldot{y} \quad \text{when } x < 0.
\]

Switching to the corresponding $p$-polar coordinates (2.1) and (2.11), the arguments $\theta$ and $\hat{\theta}$ are related by an orientation-preserving homeomorphism $\mathcal{H} : \mathbb{R} \to \mathbb{R}$. Thus $\mathcal{H}$ preserves solutions
\[
\theta(t; \mathcal{H}(\hat{\theta}_0), \lambda, \mu) \equiv \mathcal{H}(\hat{\theta}(t; \hat{\theta}_0, \lambda, \mu)).
\]

It is not difficult to write out the explicit expression of $\mathcal{H}$. However, let us just mention that $\mathcal{H}$ fixes the points $\{m\pi_p, \pi_p/2 + m\pi_p : m \in \mathbb{Z} \} \subset \mathbb{R}$. In particular,
\[
\lim_{|\hat{\theta}| \to +\infty} \frac{\mathcal{H}(\hat{\theta})}{\hat{\theta}} = 1.
\]

When $\min \{\lambda, \mu\} \gg 1$, it follows from (2.13) that
\[
\lim_{t \to +\infty} \hat{\theta}(t; \hat{\theta}_0, \lambda, \mu) = +\infty
\]
for all \( \hat{\theta}_0 \in \mathbb{R} \). Now the rotation number of (1.5) can be computed in another way

\[
\rho_{w,v}(\lambda, \mu) = \lim_{t \to +\infty} \theta(t; \mathcal{H}(\hat{\theta}_0), \lambda, \mu) \quad \text{where} \quad \hat{\theta}_0 = \lim_{t \to +\infty} \mathcal{H}(\hat{\theta}(t; \hat{\theta}_0, \lambda, \mu)) \quad \text{and} \quad \hat{\theta}(t; \hat{\theta}_0, \lambda, \mu) = \lim_{t \to +\infty} \hat{\theta}(t; \hat{\theta}_0, \lambda, \mu).
\]  

(2.14)

When \( \min\{\lambda, \mu\} \to +\infty \), it is easy to see from (2.14) and (2.13) that \( \rho_{w,v}(\lambda, \mu) \to +\infty \). In fact, (2.13) has given both lower and upper bounds for \( \rho_{w,v}(\lambda, \mu) \) when \( \lambda > 0 \) and \( \mu > 0 \).

3. Periodic Fučik Spectrum

Recall that the periodic Fučik spectrum of (1.2), denoted by \( F_{w,v} \), is the set of all those \( (\lambda, \mu) \in \mathbb{R}^2 \) such that system (1.2) has nontrivial \( T \)-periodic solutions.

We know from Lemma 2.3(4) that the following two straight lines in \((\lambda, \mu)\)-plane are always in \( F_{w,v} \):

\[
L_{w}^{+} : \lambda = \lambda_0(w), \quad L_{v}^{-} : \mu = \lambda_0(v).
\]

Let \((w, v)\) be fixed. The proof of the following lemma is like that for Lemma 3.1 in Zhang [16].

Lemma 3.1. Let \((\lambda, \mu) \in F_{w,v} \). Then there exists \( n \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) such that \( \rho_{w,v}(\lambda, \mu) = n \).

Let \( h : \mathbb{R} \to \mathbb{R} \) be a homeomorphism satisfying

\[
h(\theta_0 + 2\pi_p) = h(\theta_0) + 2\pi_p, \quad \theta_0 \in \mathbb{R}.
\]  

(3.1)

Define the rotation number \( \rho(h) \) of \( h \) by

\[
\rho(h) = \lim_{n \to +\infty} \frac{h^n(\theta_0) - \theta_0}{2m\pi_p}
\]  

(3.2)

(independent of \( \theta_0 \)). Here \( h^n \) is the \( n \)th iteration of the map \( h : \mathbb{R} \to \mathbb{R} \). The following lemma is proved in Gan and Zhang [5].

Lemma 3.2. Let \( h \) be a homeomorphism of \( \mathbb{R} \) satisfying (3.1) and \( n \) an integer.

Then

1. \( \rho(h) \geq n \) if and only if \( \max_{\theta_0 \in \mathbb{R}} (h(\theta_0) - \theta_0) \geq 2n\pi_p \).
2. \( \rho(h) \leq n \) if and only if \( \min_{\theta_0 \in \mathbb{R}} (h(\theta_0) - \theta_0) \leq 2n\pi_p \).
Recall that the Poincaré map $\Theta_{\lambda, \mu} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\Theta_{\lambda, \mu}(\theta_0) := \theta(2\pi p; \theta_0, \lambda, \mu).$$

By the periodicity equalities (2.6) and (2.7) for solutions $\theta(t; \theta_0, \lambda, \mu)$, we know that $\Theta_{\lambda, \mu}$ satisfies (3.1), and $\rho_{w,v}(\lambda, \mu)$ defined in (2.8) is the same as $\rho(\Theta_{\lambda, \mu})$ defined by (3.2). Introduce the following two functions

$$M(\lambda, \mu) = \max_{\theta_0 \in \mathbb{R}} (\Theta_{\lambda, \mu}(\theta_0) - \theta_0),$$

$$N(\lambda, \mu) = \min_{\theta_0 \in \mathbb{R}} (\Theta_{\lambda, \mu}(\theta_0) - \theta_0).$$

By Lemma 2.3, it is not difficult to prove that for any given $M$, both of $\rho_{w,v}(\lambda, \mu)$ and Lemma 3.2, it is easy to prove the following result.

**Lemma 3.3.** Fix $(w, v)$. For any $n \in \mathbb{N}$, the boundary of $\rho_{w,v}^{-1}(n)$ is given by

$$\partial \rho_{w,v}^{-1}(n) = M_n \cup N_n.$$

For any $(x_0, y_0) \in \mathbb{R}^2$, the unique solution $(x(t; x_0, y_0, \lambda, \mu), y(t; x_0, y_0, \lambda, \mu))$ of (2.2) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$ is well defined for all $t \in \mathbb{R}$. The Poincaré map $P_{\lambda, \mu} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of (2.2) is defined by

$$P_{\lambda, \mu}(x_0, y_0) = (x(2\pi p; x_0, y_0, \lambda, \mu), y(2\pi p; x_0, y_0, \lambda, \mu)).$$

Notice that system (2.2) is a Hamiltonian system. As a consequence, $P_{\lambda, \mu} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an area-preserving homeomorphism.

Denote by $r(t; \theta_0, \lambda, \mu)$ the unique solution of (2.3) satisfying the initial condition $r(0; \theta_0, \lambda, \mu) = 1$. Define

$$R_{\lambda, \mu}(\theta_0) := r(2\pi p; \theta_0, \lambda, \mu).$$

The following equality (3.5) is the same as the area-preserving property of $P_{\lambda, \mu}$.

**Lemma 3.4.** There holds the following equality

$$\frac{d\Theta_{\lambda, \mu}(\theta_0)}{d\theta_0} = \frac{1}{R_{\lambda, \mu}^2(\theta_0)}.$$  

**Proof.** The following proof is similar to that in [17] in studying Lyapunov exponents of asymmetric oscillators. Since the vector-filed $A(t, \theta; \lambda, \mu, w, v)$ for $\theta$
(see (2.4)) is continuously differentiable in $\theta \in \mathbb{R}$, see (2.5), the solution $\theta(t; \theta_0, \lambda, \mu) := \theta(t; \theta_0, \lambda, \mu, w, v)$ is continuously differentiable in the initial value $\theta_0 \in \mathbb{R}$. Moreover, by Eq. (2.4), the differential

$$\tilde{\theta}(t; \theta_0, \lambda, \mu) := \frac{\partial \theta(t; \theta_0, \lambda, \mu)}{\partial \theta_0}$$

satisfies the following variational equation

$$\frac{d}{dt} \tilde{\theta}(t; \theta_0, \lambda, \mu) = \left. \frac{\partial A(t, \theta; \lambda, \mu, w, v)}{\partial \theta} \right|_{\theta = \theta(t; \theta_0, \lambda, \mu)} \cdot \tilde{\theta}(t; \theta_0, \lambda, \mu)$$

with the initial value $\tilde{\theta}(0; \theta_0, \lambda, \mu) = 1$. Thus

$$\frac{d}{dt} \log \tilde{\theta}(t; \theta_0, \lambda, \mu) = \left. \frac{\partial A(t, \theta; \lambda, \mu, w, v)}{\partial \theta} \right|_{\theta = \theta(t; \theta_0, \lambda, \mu)} \cdot \tilde{\theta}(t; \theta_0, \lambda, \mu) = -2G(t, \theta(t; \theta_0, \lambda, \mu); \lambda, \mu, w, v) \quad \text{(by (2.5))}$$

$$= -2 \frac{d}{dt} \log r(t; \theta_0, \lambda, \mu) \quad \text{(by (2.3))}.$$ 

Since $\tilde{\theta}(0; \theta_0, \lambda, \mu) = 1$ and $r(0; \theta_0, \lambda, \mu) = 1$, we conclude that

$$(r(t; \theta_0, \lambda, \mu))^2 \cdot \frac{\partial \theta(t; \theta_0, \lambda, \mu)}{\partial \theta_0} \equiv 1.$$ 

Let $t = 2\pi_p$. We get (3.5). \hfill \Box

Now we can prove that all curves $\mathcal{M}_n$ and $\mathcal{N}_n$ are actually the Fučík curves of problem (1.2)–(1.3).

**Theorem 3.1.** All curves $\mathcal{M}_n$ and $\mathcal{N}_n$ are in $\mathcal{F}_{w,v}$, i.e.,

$$\mathcal{L}_w^+ \cup \mathcal{L}_w^- \cup \left( \bigcup_{n \in \mathbb{N}} (\mathcal{M}_n \cup \mathcal{N}_n) \right) \subset \mathcal{F}_{w,v}. \quad (3.6)$$

Moreover, for any $(\lambda, \mu) \in \mathcal{M}_n \cup \mathcal{N}_n$, each eigen-function $E_n(t)$ associated with $(\lambda, \mu)$ has precisely $2n$ zeros in the interval $[0, 2\pi_p]$.

**Proof.** Let $(\lambda, \mu) \in \mathcal{M}_n \cup \mathcal{N}_n$. By (3.3) and (3.4), there must be some $\tilde{\theta}_0 \in \mathbb{R}$ such that

$$\theta(2\pi_p; \tilde{\theta}_0, \lambda, \mu) = \Theta_{\lambda,\mu}(\tilde{\theta}_0) = \tilde{\theta}_0 + 2n\pi_p, \quad (3.7)$$

and

$$\left. \frac{d}{d\theta_0} \Theta_{\lambda,\mu}(\theta_0) \right|_{\theta_0 = \tilde{\theta}_0} = 1. \quad (3.8)$$

By (3.5), equality (3.8) is the same as

$$r(2\pi_p; \tilde{\theta}_0, \lambda, \mu) = R_{\lambda,\mu}(\tilde{\theta}_0) = 1. \quad (3.9)$$
Geometrically, equalities (3.7) and (3.9) imply that, in the p-polar coordinates, the solution of (1.2) starting at the point \((r, \theta) = (1, \dot{\theta}_0)\) will return to itself after one period. That is, \((\lambda, \mu)\) must be in \(\mathcal{F}_{w,v}\).

Let now \((\lambda, \mu) \in \mathcal{M}_n \cup \mathcal{N}_n\). We have \(\rho(\lambda + w, \mu + v) = n\). By an eigen-function \(E_n(t)\) associated with \((\lambda, \mu)\), we mean that \(E_n(t) \neq 0\) is \(2\pi p\)-periodic and satisfies Eq. (1.2). In terminology of the p-polar coordinates, we have

\[
E_n(t) = c_0 \cdot (r(t; \theta_0, \lambda, \mu))^{2/p} C_p(\theta(t; \theta_0, \lambda, \mu))
\]

for some \(\theta_0 \in \mathbb{R}\) and some non-zero constant \(c_0\). Let \(N\) be the number of the nodes of \(E_n(t)\) in the interval \([0, 2\pi_p]\), i.e.

\[
N := \#\{t \in [0, 2\pi_p] : E_n(t) = 0\}. \tag{3.10}
\]

As \(E_n(t)\) is \(2\pi p\)-periodic, we know that

\[
\#\{t \in [0, 2m\pi_p] : E_n(t) = 0\} =Nm, \quad m \in \mathbb{N}. \tag{3.11}
\]

It is well known that the rotation numbers can also be defined using the asymptotical growth rate of nodes [4]. Precisely, in the present case, we have

\[
n = \rho(\lambda + w, \mu + v) = \lim_{\tau \to +\infty} \frac{\pi_p \cdot \#\{t \in [0, \tau] : E_n(t) = 0\}}{\tau} = \lim_{m \to +\infty} \frac{\pi_p \cdot \#\{t \in [0, 2m\pi_p] : E_n(t) = 0\}}{2m\pi_p} = \frac{N}{2},
\]

following from (3.11). Thus we have necessarily \(N = 2n\). By the definition (3.10) of \(N\), we know that \(E_n(t)\) has precisely \(2n\) zeros in the interval \([0, 2\pi_p]\).

A typical example for the rotation number approach of the periodic Fučík spectrum is plotted in Fig. 1, where \(p = 4\), \(T = 2\pi_4\), and \(w(t) = \cos(\pi t/\pi_4)\), \(v(t) = 2\cos(\pi t/\pi_4)\).

On the surface \(\rho = \rho_{w,v}(\lambda, \mu)\), for each \(n \in \mathbb{Z}^+\), one sees that there is a platform of the height \(n\) with its boundary curves \(\mathcal{M}_n\) and \(\mathcal{N}_n\) being the Fučík curves of Theorem 3.1. It seems that for \(n \in \mathbb{N}\), one has also some platform of the height of \(n - 1/2\). In this case, the corresponding boundary curves are contained in the anti-periodic Fučík spectrum of (1.2), which is the set of those \((\lambda, \mu)\) such that Eq. (1.2) has non-trivial solutions \(x(t)\) satisfying

\[
x(2\pi_p) = -x(0), \quad x'(2\pi_p) = -x'(0).
\]

See the rotation number approach to anti-periodic eigenvalues of the p-Laplacian with potentials in Zhang [15].

From Theorem 3.1, \(\mathcal{M}_n\) and \(\mathcal{N}_n\) are called the \(nth\) Fučík curves of (1.2), while \(\mathcal{L}_w, \mathcal{L}_v\) are called the zeroth Fučík curves of (1.2). As in the case \(p = 2\), one can expect that for non-constant potentials \(w(t), \ v(t)\), the rotation number
approach (3.6) can yield only part of the Fučík curves of (1.2)–(1.3). However, these curves in (3.6) are most convenient in applications, because when \( w = v = q \in L^1(\mathbb{R}/2\pi p \mathbb{Z}) \), \( M_n \) and \( N_n \) are just the Fučík curves emanating from \( (\lambda_n(q), \lambda_n(q)) \) and \( (\lambda_n(q), \lambda_n(q)) \) respectively. Here \( \lambda_n(q) \leq \lambda_n(q) \) are just the \( n \)th variational \( T \)-periodic eigenvalues of

\[
(\phi_p(x'))' + (\lambda + q(t))\phi_p(x) = 0.
\]

See Zhang [15].

4. Multiple Periodic Solutions

In this section, we study multiple periodic solutions of nonlinear equations (1.7), where \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is assumed to be \( 2\pi p \)-periodic in the first variable, satisfies the Carathéodory condition, and possesses some asymptotic properties at the origin and near infinity. That is, there exist \( w_0, v_0, w_\infty, v_\infty \in L^1(\mathbb{R}/2\pi p \mathbb{Z}) \) such that the following limits

\[
\lim_{x \to 0^+} \frac{f(t, x)}{\phi_p(x)} = w_0(t), \quad \lim_{x \to 0^-} \frac{f(t, x)}{\phi_p(x)} = v_0(t)
\]

and

\[
\lim_{x \to +\infty} \frac{f(t, x)}{\phi_p(x)} = w_\infty(t), \quad \lim_{x \to -\infty} \frac{f(t, x)}{\phi_p(x)} = v_\infty(t)
\]

exist and are uniform in \( t \).
Let $\phi_p(x') = -y$. Equation (1.7) can be rewritten as a planar system

$$
\begin{cases}
x' = -\phi_p(y), \\
y' = f(t, x).
\end{cases}
$$

(4.3)

For each initial value $z_0 = (r_0, \theta_0)$ in the $p$-polar coordinates, with $r_0 > 0$ and $\theta_0 \in \mathbb{R}$, we assume that (4.3) has the unique solution $(x(t; z_0), y(t; z_0)) \in \mathbb{R}$ defined on $t \in [0, 2\pi_p]$ satisfying the initial conditions

$$
x(0; z_0) = r_0^{2/p} C_p(\theta_0), \quad y(0; z_0) = r_0^{2/p} S_p(\theta_0).
$$

The time $2\pi_p$-rotation number of the solution $(x(t; z_0), y(t; z_0))$ is defined as

$$
\text{Rot}_f(z_0) = \frac{\theta(2\pi_p; z_0) - \theta(0; z_0)}{2\pi_p} \in \mathbb{R},
$$

where $\theta(t; z_0)$, with $\theta(0; z_0) = \theta_0$, is the argument of the solution $(x(t; z_0), y(t; z_0))$ switched to the $p$-polar coordinates (2.1). In general, Rot$_f(z_0)$ depends upon the initial point $z_0$.

By conditions (4.1) and (4.2), when $x$ is near the origin or near infinity, equation (1.7) can be approximated by

$$
\begin{align*}
(\phi_p(x'))' + w_0(t)\phi_p(x_+) + v_0(t)\phi_p(x_-) &= 0, \\
(\phi_p(x'))' + w_\infty(t)\phi_p(x_+) + v_\infty(t)\phi_p(x_-) &= 0,
\end{align*}
$$

(4.4)

(4.5)

respectively. The time $2\pi_p$-rotation numbers of systems (4.4) and (4.5) are denoted by Rot$_0(z_0)$ and Rot$_\infty(z_0)$ respectively, which are determined by potentials $(w_0, v_0)$ and $(w_\infty, v_\infty)$ respectively, and by the initial point $z_0$ as well. However, due to the positive homogeneity of equations (4.4) and (4.5) in $x$, we have, in the $p$-polar coordinates,

$$
\text{Rot}_0(r_0, \theta_0) = \text{Rot}_0(1, \theta_0), \quad \text{Rot}_\infty(r_0, \theta_0) = \text{Rot}_\infty(1, \theta_0)
$$

(4.6)

for any $\theta_0 \in \mathbb{R}$ and $r_0 > 0$.

By (4.6), let us simply take $\Gamma_0$ as

$$
\Gamma_0 = \{ z_0 = (r, \theta) : r = 1, \; \theta \in \mathbb{R}, \}
$$

the unit circle in the $p$-polar coordinates. In the following we establish some fundamental relations between $\rho(w_0, v_0)$ and Rot$_0(z_0)$, and between $\rho(w_\infty, v_\infty)$ and Rot$_\infty(z_0)$.

Lemma 4.1. Given $n \in \mathbb{N}$. Then

$$
\rho(w_0, v_0) < n \iff \max_{z_0 \in \Gamma_0} \text{Rot}_0(z_0) < n,
$$

(4.7)

$$
\rho(w_0, v_0) > n \iff \min_{z_0 \in \Gamma_0} \text{Rot}_0(z_0) > n.
$$

(4.8)

**Proof.** Let

$$
h(\theta) = \theta(2\pi_p; \theta_0, 0, 0, w_0, v_0).
$$
Then \( h \) is a homeomorphism of \( \mathbb{R} \) satisfying (3.1). By Lemma 3.2 and (4.6), we know that
\[
\rho(h) < n \iff \max_{\theta_0 \in \mathbb{R}} (h(\theta_0) - \theta_0) < 2n\pi_p
\]
\[
\iff \max_{\theta_0 \in \mathbb{R}} \frac{\theta(2\pi_p; \theta_0, w_0, v_0) - \theta_0}{2\pi_p} < n
\]
\[
\iff \max_{\theta_0 \in \mathbb{R}} \text{Rot}_0(1, \theta_0) < n
\]
\[
\iff \max_{z_0 \in \Gamma_0} \text{Rot}_0(z_0) < n.
\]
Thus we obtain (4.7). Result (4.8) can be proved in a similar way.

Notice that (4.7) and (4.8) are also true for \( \rho(w_\infty, v_\infty) \) and \( \text{Rot}_\infty(z_0) \).

Recall that Eqs. (4.4) and (4.5) are the approximating equations of (1.7) near the origin and the infinity. The following lemma shows that \( \text{Rot}_f \) can be approximated by \( \text{Rot}_0 \) near the origin, and by \( \text{Rot}_\infty \) near infinity, respectively.

**Lemma 4.2.** For any \( \varepsilon > 0 \), there exist \( \delta > 0 \) sufficiently small and \( R > 0 \) sufficiently large such that
\[
|\text{Rot}_f(z_0) - \text{Rot}_0(z_0)| \leq \varepsilon, \quad \forall z_0 = (r_0, \theta_0), \quad 0 < r_0 \leq \delta,
\]
\[
|\text{Rot}_f(z_0) - \text{Rot}_\infty(z_0)| \leq \varepsilon, \quad \forall z_0 = (r_0, \theta_0), \quad r_0 \geq R.
\]

**Proof.** By the asymptotic properties (4.1) and (4.2), we can write \( f(t, x) \) as
\[
f(t, x) = w_0(t) \phi_p(x_+) + v_0(t) \phi_p(x_+) + \gamma_0(t, x)
\]
\[
= w_\infty(t) \phi_p(x_+) + v_\infty(t) \phi_p(x_+) + \gamma_\infty(t, x),
\]
where
\[
\lim_{|x| \to 0} \frac{\gamma_0(t, x)}{\phi_p(x)} = 0 \quad \text{and} \quad \lim_{|x| \to +\infty} \frac{\gamma_\infty(t, x)}{\phi_p(x)} = 0 \quad \text{uniformly in } t.
\]
In particular, we know from (4.11)–(4.13) that there exists some \( l(t) \in L^1(\mathbb{R}/2\pi_p \mathbb{Z}, \mathbb{R}) \) such that \( l(t) \geq 0 \) and
\[
|f(t, x)| \leq l(t)|x|^{p-1} \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}.
\]
Let \( y = -\phi_p(x') \). In the \( p \)-polar coordinates (2.1), Eq. (1.7) can be rewritten as
\[
(\log r)' = G_f(t, r, \theta),
\]
\[
\theta' = A_f(t, r, \theta),
\]
where \( G_f \) and \( A_f \) can be constructed explicitly. Condition (4.14) means that there exists some \( \tilde{l}(t) \in L^1(\mathbb{R}/2\pi_p \mathbb{Z}, \mathbb{R}) \) such that \( \tilde{l}(t) \geq 0 \) and
\[
|G_f(t, r, \theta)| \leq \tilde{l}(t) \quad \forall t, \quad r, \quad \theta.
\]
For any \( r_0 > 0 \) and \( \theta_0 \in \mathbb{R} \), let \( (r(t; r_0, \theta_0), \theta(t; r_0, \theta_0)) \) be the solution of (4.15)+(4.16) satisfying \( (r(0; r_0, \theta_0), \theta(0; r_0, \theta_0)) = (r_0, \theta_0) \). Now (4.17) yields
\[
 r_0 \exp \left( -\int_0^{2\pi_p} \hat{l}(t) dt \right) \leq r(t; r_0, \theta_0) \leq r_0 \exp \left( \int_0^{2\pi_p} \hat{l}(t) dt \right)
\]
for all \( t \in [0, 2\pi_p] \), \( r_0 > 0 \) and \( \theta_0 \in \mathbb{R} \). In particular, for any \( \delta_0 > 0 \) small and any \( R_0 > 0 \) large, there exist \( \delta \in (0, \delta_0) \) and \( R \in (R_0, \infty) \) such that
\[
 r_0 \in (0, \delta) \implies r(t; r_0, \theta_0) < \delta_0, \tag{4.18}
\]
\[
 r_0 \in (R, \infty) \implies r(t; r_0, \theta_0) > R_0, \tag{4.19}
\]
for all \( t \in [0, 2\pi_p] \) and all \( \theta_0 \in \mathbb{R} \). Now we consider (4.15)+(4.16) when \( r \) is small. Due to (4.11), system (4.15)+(4.16) can be rewritten as
\[
 (\log r)' = G_0(t, \theta) + \tilde{G}_0(t, r, \theta),
\]
\[
 \theta' = A_0(t, \theta) + \tilde{A}_0(t, r, \theta), \tag{4.20}
\]
where
\[
 G_0(t, \theta) = G(t, \theta; 0, 0, w_0, v_0),
\]
\[
 A_0(t, \theta) = A(t, \theta; 0, 0, w_0, v_0)
\]
are as in the right-hand sides of (2.3) and (2.4), and
\[
 \tilde{G}_0(t, r, \theta) = \frac{\rho \gamma_0(t, r^{2/p}C_p(\theta)) \phi_p(S_p(\theta))}{2r^{2/p}},
\]
\[
 \tilde{A}_0(t, r, \theta) = \frac{\gamma_0(t, r^{2/p}C_p(\theta)) C_p(\theta)}{r^{2/p}}.
\]
On the other hand, Eq. (4.4) is equivalent to the following system
\[
 (\log r)' = G_0(t, \theta), \tag{4.21}
\]
\[
 \theta' = A_0(t, \theta). \tag{4.22}
\]
We use \( (\hat{r}(t; r_0, \theta_0), \hat{\theta}(t; r_0, \theta_0)) \) to denote the solution of (4.21)+(4.22) satisfying \( (\hat{r}(0), \hat{\theta}(0)) = (r_0, \theta_0) \).

The first condition of (4.13) shows that for any \( \varepsilon > 0 \), there exists some \( \delta_0 > 0 \) small such that
\[
 |\hat{A}_0(t, r, \theta)| < \varepsilon \quad \forall \ t \in \mathbb{R}, \ \forall 0 < r < \delta_0, \ \forall \theta \in \mathbb{R}. \tag{4.23}
\]
Let \( \delta > 0 \) be such that (4.18) is true. Thus
\[
 r_0 \in (0, \delta) \implies 0 < r(t; r_0, \theta_0) < \delta_0, \ \forall \ t \in [0, 2\pi_p]. \tag{4.24}
\]
Define \( \hat{A}(t; r_0, \theta_0) := \hat{A}_0(t, r(t; r_0, \theta_0), \theta(t; r_0, \theta_0)) \). Then (4.23) and (4.24) imply
\[
 r_0 \in (0, \delta) \implies |\hat{A}(t; r_0, \theta_0)| < \varepsilon \quad \forall \ t \in [0, 2\pi_p]. \tag{4.25}
\]
\[ (4.19) \] at infinity.

curves \( \Gamma \)

\[ \begin{align*}
\text{where, for each } t, \text{ there exists } \tau = \tau_t \in (0, 1) \text{ such that } \\
A(t) = \partial A_0 \bigg|_{\theta = \hat{\theta}(t; r_0, \theta_0) + \tau \hat{\theta}(t)} = -2G(t, \hat{\theta}(t; r_0, \theta_0) + \tau \hat{\theta}(t), 0, 0, \omega_0, \nu_0). 
\end{align*} \]

See (2.5). From formula (2.3), it is easy to see that there exists \( \bar{l}(t) \in L^1(\mathbb{R}/2\pi_p \mathbb{Z}, \mathbb{R}) \), independent of \( \hat{\theta}(t) \), such that \( \bar{l}(t) \geq 0 \) and \( |a(t)| \leq \bar{l}(t) \). Equality (4.26) is a linear equation for \( \hat{\theta}(t) \). Using the condition \( \hat{\theta}(0) = 0 \), we have

\[ \hat{\theta}(2\pi_p) = \int_0^{2\pi_p} \bar{\Lambda}(s; r_0, \theta_0) \exp \left( \int_s^{2\pi_p} a(u) \, du \right) ds. \quad (4.27) \]

By the definition of \( 2\pi_p \)-rotation numbers, we have then

\[ |\text{Rot}_{\infty}(\bar{z}_0) - \text{Rot}_{\bar{f}}(\bar{z}_0)| = \left| \hat{\theta}(2\pi_p) \right| \]

\[ \leq \frac{1}{2\pi_p} \exp \left( \int_0^{2\pi_p} \bar{l}(u) \, du \right) \int_0^{2\pi_p} |\bar{\Lambda}(s; r_0, \theta_0)| \, ds \quad (\text{by } (4.27)) \]

\[ < \exp \left( \int_0^{2\pi_p} \bar{l}(u) \, du \right) \cdot \varepsilon \quad (\text{by } (4.25)). \]

This proves (4.9). Result (4.10) can be obtained similarly by using the estimate (4.19) at infinity.

Let \( \mathcal{A} \subseteq \mathbb{R}^2 \setminus \{0\} \) be a closed topological annulus between two disjoint Jordan curves \( \Gamma_i, \Gamma_e \), with \( \Gamma_e \) star-shaped around the origin and internal to \( \Gamma_e \). The following theorem can be derived immediately from the generalized Poincaré–Birkhoff fixed point theorem of Ding [3].

**Lemma 4.3.** Assume that \( \mathcal{A} \) is as above and there is \( j \in \mathbb{Z} \) such that, either

\[ \text{Rot}_{\bar{f}}(\bar{z}_0) > j \quad \forall \bar{z}_0 \in \Gamma_i, \quad \text{and} \quad \text{Rot}_{\bar{f}}(\bar{z}_0) < j \quad \forall \bar{z}_0 \in \Gamma_e, \]

or,

\[ \text{Rot}_{\bar{f}}(\bar{z}_0) < j \quad \forall \bar{z}_0 \in \Gamma_i, \quad \text{and} \quad \text{Rot}_{\bar{f}}(\bar{z}_0) > j \quad \forall \bar{z}_0 \in \Gamma_e. \]

Then system (4.3) has at least two distinct \( 2\pi_p \)-periodic solutions \( z_1(t) \) and \( z_2(t) \) in \( \mathcal{A} \). The corresponding solutions \( x_1(t) \) and \( x_2(t) \) of Eq. (1.7) have exactly \( 2j \) zeros in the interval \([0, 2\pi_p]\).
Rotation Number, Periodic Fučík Spectrum and Multiple Periodic Solutions

Notice that condition (4.1) asserts that \( x(t) \equiv 0 \) is always a trivial \( 2\pi p \)-periodic solution of (1.7). Due to the node properties, we know that these periodic solutions in Lemma 4.3 are always non-trivial.

In order to apply Lemma 4.3 to Eq. (1.7), let us take the following Jordan curves, written as in the \( p \)-polar coordinates \((r, \theta)\) of (2.1),

\[
\Gamma_i = \{ z_0 = (r_0, \theta_0) \in \mathbb{R}^2 : r_0 = \delta \}, \quad \Gamma_e = \{ z_0 = (r_0, \theta_0) \in \mathbb{R}^2 : r_0 = R \},
\]

where \( \delta \) is small and \( R \) is large. By the approximation results for \( 2\pi p \)-rotation numbers in Lemma 4.2 and the equivalent statements of the conditions of Lemma 4.3, which is stated in Lemma 4.1, now we can use rotation numbers \( \rho(w_0, v_0) \) and \( \rho(w_\infty, v_\infty) \) of the asymptotical equations (4.4) and (4.5) to detect periodic solutions of nonlinear equation (1.7) in a very convenient way.

**Theorem 4.1.** Assume that (4.1) and (4.2) hold. Suppose that there exist \( m, n \in \mathbb{N} \) with \( m \leq n \) such that one of the following conditions for rotation numbers is fulfilled

\[
\rho(w_0, v_0) < m \leq n < \rho(w_\infty, v_\infty), \tag{4.28}
\]

\[
\rho(w_\infty, v_\infty) < m \leq n < \rho(w_0, v_0). \tag{4.29}
\]

Then, for each \( j \in \mathbb{N} \) with \( m \leq j \leq n \), Eq. (1.7) has at least two distinct \( 2\pi p \)-periodic solutions with exactly \( 2j \) zeroes on the interval \([0, 2\pi p]\). In particular, Eq. (1.7) has at least \( 2(n - m + 1) \) distinct, non-trivial \( 2\pi p \)-periodic solutions.

**Proof.** Suppose that, for example, condition (4.28) holds. Let \( j \in \mathbb{N} \) be such that \( m \leq j \leq n \). By Lemma 4.1 and (4.6), we know that

\[
\text{Rot}_0(r, \theta_0) < j, \quad \text{Rot}_\infty(r, \theta_0) > j, \quad \forall \theta_0 \in \mathbb{R}, \quad r > 0.
\]

By Lemma 4.2, we have \( \text{Rot}_f(\delta, \theta_0) < j \) for \( \delta \) sufficiently small, and \( \text{Rot}_f(R, \theta_0) > j \) for \( R \) sufficiently large. Thus the first set of conditions in Lemma 4.3 is fulfilled with such a choice of \( j \). Hence Eq. (1.7) has at least two distinct \( 2\pi p \)-periodic solutions with precisely \( 2j \) zeros in \([0, 2\pi p]\). Similarly, (4.29) can yield two solutions of the same node structure.

For different \( j \), \( m \leq j \leq n \), the structures of nodes of solutions are also different. Hence the number of non-trivial \( 2\pi p \)-periodic solutions of (1.7) is at least \( 2(n - m + 1) \).

In Theorem 4.1, we have used rotation numbers of the \( p \)-Laplacian with potentials to give some nice results of multiple periodic solutions. See conditions (4.28) and (4.29). Notice that \( \rho(w, v) \) is defined in some implicit way using potentials \( w(t) \) and \( v(t) \). In order to apply Theorem 4.1 to Eq. (1.7), some explicit estimates for both lower and upper bounds of \( \rho(w, v) \) are important. For this, let us mention that Feng and Zhang [4] have recently applied some basic Sobolev inequalities to construct some very deep upper bounds for rotation numbers \( \rho(w, v) \) using only the \( L^p \)
norms of both $w(t)$ and $v(t)$, and using some mixing type $L^\alpha$ norms of $(w(t), v(t))$, $1 \leq \alpha \leq \infty$. By applying the estimates in [4, 13], one can give many examples of (1.7) for which the other usual topological techniques cannot yield deeper existence results.

Finally, we remark that for the case $p = 2$, Zanini [14] has obtained similar results for multiple periodic solutions of (1.7) by assuming that $f(t, x)$ is asymptotically linear in $x$ at the origin and near infinity. In this case, Margheri, Rebelo and Zanolin [11] have applied the Maslov index to give more precise existence results of multiple periodic solutions. However, for the $p$-Laplacian case, the asymptotical condition (4.1) for $f(t, x)$ near the origin makes the system (4.3) non-differentiable at the origin. It is then an interesting problem that if one can develop the Maslov index for equations (1.7) so that Theorem 4.1 can be improved accordingly.

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