

Exponential Growth Rates of Periodic Asymmetric Oscillators

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Abstract

In this paper we will study the dynamics of the periodic asymmetric oscillator $x'' + q^+(t)x_+ + q^-(t)x_- = 0$, where $q^+, q^- \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ and $x_+ = \max(x, 0)$, $x_- = \min(x, 0)$ for $x \in \mathbb{R}$. It will be proved that the exponential growth rate

$$\chi(x) := \lim_{t \rightarrow +\infty} \frac{1}{t} \log \sqrt{(x(t))^2 + (x'(t))^2}$$

does exist for each non-zero solution $x(t)$ of the oscillator. The properties of these rates, or the Lyapunov exponents, will be given using the induced circle diffeomorphism of the oscillator. The proof is extensively based on the Denjoy theorem in topological dynamics and the unique ergodicity theorem in ergodic theory.

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1 Introduction

One of the most important notions in dynamical systems is the so-called *Lyapunov exponents*, which are the exponential growth rates of non-zero solutions of linearization equations of nonlinear equations [9, 10, 15].

For a first-order linear system of ODEs

$$x' = A(t)x, \quad x \in \mathbb{R}^n, \quad (1.1)$$

with periodic coefficient $A(t) \in L^1(\mathbb{S}^1, \mathbb{R}^{n \times n})$, $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, the Lyapunov exponents can be simply obtained by applying the Floquet theory. If one uses $x(t; v)$ to denote the solution of (1.1) satisfying the initial condition $x(0) = v \in \mathbb{R}^n$, the Lyapunov exponent of the solution $x(t; v)$

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \|x(t; v)\| = \chi(v) \in \mathbb{R}$$

does exist for each $v \neq 0$. Moreover, the set of Lyapunov exponents $\{\chi(v) : v \neq 0\}$ has at most n different values. In particular, the Hill's equation

$$x'' + q(t)x = 0, \quad x \in \mathbb{R}, \quad (1.2)$$

where $q(t) \in L^1(\mathbb{S}^1) = L^1(\mathbb{S}^1, \mathbb{R})$, has Lyapunov exponents $\pm\chi$, where $\chi = \chi(q) \in [0, \infty)$.

Based on the multiplicative ergodic theorem, the notion can be established for linear systems with some recurrence [9]. See also [1, 11] for extensions to other kinds of dynamical systems, including random dynamical systems.

We remark that the linear structure of systems is fundamental in these works on Lyapunov exponents.

In this paper, we will study the evolution of solutions of the scalar asymmetric oscillator

$$x'' + q^+(t)x_+ + q^-(t)x_- = 0, \quad x \in \mathbb{R}, \quad (1.3)$$

where $q^+, q^- \in L^1(\mathbb{S}^1)$. Here, for $x \in \mathbb{R}$, $x_+ := \max(x, 0)$ and $x_- := \min(x, 0)$. Equation (1.3) is a Lagrangian equation. In case $q^+ \neq q^-$, the internal force of equation (1.3)

$$f(t, x) = q^+(t)x_+ + q^-(t)x_-$$

is nonlinear in $x \in \mathbb{R}$. Moreover, $f(t, x)$ is not differentiable in x at $x = 0$. In recent years, many problems such as the existence of periodic solutions of the forced asymmetric oscillators

$$x'' + q^+(t)x_+ + q^-(t)x_- = e(t), \quad x \in \mathbb{R},$$

where $e(t) \in L^1(\mathbb{S}^1)$, have been studied extensively. This will lead to a typical nonlinear spectrum problem, i.e., the Fučík spectrum of (1.3) which is a generalization of eigenvalues of Hill's equations [16] and the classical Fučík spectrum [6] as well. For the non-autonomous oscillators (1.3), a nice rotation number approach to

a partial characterization of the Fučík spectrum of (1.3) has been given by Zhang [20]. Very recently, Binding and Rynne [2] have revealed some crucial difference between the spectra of (1.3) and that of the Hill's operators. These ideas apply also to another nonlinear spectrum problem — the spectrum of the p -Laplacian with potentials. See [3, 19]. There is also a lot of work on the Lagrangian stability of forced asymmetric oscillators, via the Moser twist theorem. See, for example, [14, 18].

However, the evolution of asymmetric oscillators (1.3) themselves is not very clear due to the non-autonomy and the nonlinearity. The main concern of this paper is on this aspect. As equation (1.3) can be written as a planar system, the evolution of solutions $x(t)$ of (1.3) can be described by that of the argument functions

$$\theta(t; x) = \arg(x(t) - ix'(t)),$$

and the growth functions

$$r(t; x) = \sqrt{(x(t))^2 + (x'(t))^2}.$$

Here $x(t)$ is any non-zero solution of (1.3), while $\theta(t; x)$ is a continuous representation (in time t). It is well-known that the evolution of $\theta(t; x)$ is clear, because

$$\rho = \lim_{t \rightarrow \infty} \frac{\theta(t; x)}{t} \quad (1.4)$$

always exists and is independent of solutions $x(t)$. In fact, $\rho = \rho(q^+, q^-)$ is the rotation number of Poincaré for system (1.3). This notion can be and has been extended to more general systems. For example, Johnson and Moser [8] have applied some extension of the unique ergodic theorem [10, 12] to introduce the rotation number for linear equation (1.2) with an almost periodic coefficient $q(t)$. Feng and Zhang [5] have further extended this to (1.3) where $q^+(t)$, $q^-(t)$ are (the classical) almost periodic functions [7, Appendix]. Moreover, some reasonable upper bounds of $\rho(q^+, q^-)$ are given in [5] as well. The extension to random dynamical systems has recently been given by Li and Lu [13].

One of the main results of this paper is to give a description for the evolution of $r(t; x)$.

Theorem 1.1 *Let $q^\pm \in L^1(\mathbb{S}^1)$. Then, for any non-zero solution $x(t)$ of (1.3), the following exponential growth rate does exist*

$$\chi(x) := \lim_{t \rightarrow +\infty} \frac{1}{t} \log r(t; x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \sqrt{(x(t))^2 + (x'(t))^2}. \quad (1.5)$$

These exponential growth rates $\chi(x)$ can be called the *Lyapunov exponents* of the asymmetric oscillator (1.3). Recall that $f(t, x)$ is not differentiable in x , the exponents (1.5) are not obtained from the usual linearization technique. Theorem 1.1 shows that the exponential growth rates can be defined in a direct way for

certain nonlinear systems such as (1.3). This idea is new for the study of nonlinear systems. Analogous to (1.5), exponents in the negative-time also exist

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log \sqrt{(x(t))^2 + (x'(t))^2} =: \chi_-(x).$$

In the proof of Theorem 1.1, a classical and nice result for dynamics of circle homeomorphisms — the Denjoy theorem — plays an important role. In fact, it will be proved in Section 2 that the induced Poincaré map $\mathcal{H} = \mathcal{H}_{q^+, q^-}$ of (1.3) on the circle is not C^2 , but is C^1 and $\log(\mathcal{H}_{q^+, q^-})'$ is globally Lipschitzian and therefore $\log(\mathcal{H}_{q^+, q^-})'$ has bounded variation. See Proposition 2.1. Hence these homeomorphisms \mathcal{H}_{q^+, q^-} meet with the regularities in Denjoy theorem and the dynamics of \mathcal{H}_{q^+, q^-} can be described completely. A complete proof of Theorem 1.1 is given in Section 3.

In Section 4 we will study some properties of $\chi(x) = \chi(x, q^+, q^-)$ of (1.3). In case $\rho(q^+, q^-)$ is rational, we will give some formulas to compute the Lyapunov exponents $\chi(x)$ of (1.3) using the diffeomorphisms \mathcal{H}_{q^+, q^-} . See Theorem 4.2. Based on the Denjoy theorem and the unique ergodicity theorem in ergodic theory, we find that $\chi(x) = 0$ for all solutions $x(t)$ of (1.3) in case $\rho(q^+, q^-)$ is irrational. See Theorem 4.3.

Due to the switching of two Hill's equations

$$x'' + q^+(t)x = 0, \quad x'' + q^-(t)x = 0,$$

at $x = 0$, the asymmetric oscillator (1.3) may admit more than two different exponents. In Theorem 4.1, we will show that any oscillator (1.3) has at most countably infinitely different Lyapunov exponents.

The results on growth of solutions $x(t)$ of (1.3) in this paper, together with the known results on argument functions $\theta(t; x)$, give a relatively complete description on the evolution of periodic asymmetric oscillators.

2 Regularities and rotation numbers of the Poincaré maps

Let $q^\pm \in L^1(\mathbb{S}^1)$ be given. In order to study the dynamics of asymmetric oscillator (1.3), the following simple idea is useful. In the polar coordinates $x = r \cos \theta$, $x' = -r \sin \theta$, $r > 0$, $\theta \in \mathbb{R}$, equation (1.3) can be written as

$$\theta' = A(t, \theta), \tag{2.1}$$

$$r' = rG(t, \theta), \tag{2.2}$$

where the vector fields $A, G : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given by

$$A(t, \theta) = \begin{cases} q^+(t) \cos^2 \theta + \sin^2 \theta, & \text{when } \cos \theta \geq 0, \\ q^-(t) \cos^2 \theta + \sin^2 \theta, & \text{when } \cos \theta \leq 0, \end{cases} \tag{2.3}$$

$$G(t, \theta) = \begin{cases} (q^+(t) - 1) \cos \theta \sin \theta, & \text{when } \cos \theta \geq 0, \\ (q^-(t) - 1) \cos \theta \sin \theta, & \text{when } \cos \theta \leq 0. \end{cases} \tag{2.4}$$

Due to the positive homogeneity of $f(t, x)$ in x , one sees from equation (2.1) that the evolution of θ is independent of r . Equation (2.1) is 2π -periodic in both t and θ . Hence the rotation number of (2.1) or that of (1.3)

$$\rho = \rho(q^+, q^-) := \lim_{t \rightarrow \infty} \frac{\theta(t) - \theta(0)}{t} (\geq 0)$$

is well-defined. Here $\theta(t)$ is any solution of (2.1). This definition is the same as (1.4).

First, we give some elementary, but yet important, properties on vector fields $A(t, \theta)$ and $G(t, \theta)$ in (2.3) and (2.4).

Lemma 2.1 (i) $A(t, \theta)$ and $G(t, \theta)$ are 2π -periodic in both t and θ .

(ii) $A(t, \theta)$ is continuously differentiable in θ . In fact, one has

$$\frac{\partial A(t, \theta)}{\partial \theta} \equiv -2G(t, \theta). \tag{2.5}$$

(iii) $G(t, \theta)$ is globally Lipschitz continuous in θ . More precisely, let us define

$$r(t) := \max(|q^+(t) - 1|, |q^-(t) - 1|) \in L^1(\mathbb{S}^1). \tag{2.6}$$

Then

$$|G(t, \theta)| \leq r(t)/2, \quad t, \theta \in \mathbb{R}, \tag{2.7}$$

and

$$|G(t, \theta_2) - G(t, \theta_1)| \leq r(t)|\theta_2 - \theta_1|, \quad t, \theta_1, \theta_2 \in \mathbb{R}. \tag{2.8}$$

(iv) Using the notations $y_+ = \max(y, 0)$ and $y_- = \min(y, 0)$ for $y \in \mathbb{R}$, the function $G(t, \theta)$ can be decomposed as

$$G(t, \theta) \equiv q^+(t) \sin \theta (\cos \theta)_+ + q^-(t) \sin \theta (\cos \theta)_- - \cos \theta \sin \theta. \tag{2.9}$$

We remark that, in case $q^+(t) \neq q^-(t)$, $\frac{\partial A(t, \theta)}{\partial \theta}$ is not differentiable in θ at those θ 's such that $\cos \theta = 0$. That is, $A(t, \theta)$ is not C^2 in θ in asymmetric case. This is a crucial difference between the symmetric and asymmetric oscillators.

Now we consider solutions of (2.1)–(2.2). Due to the positive homogeneity of (1.3) in x , we need only to consider solutions of (2.1)–(2.2) with initial points from the unit circle. More precisely, given $\vartheta \in \mathbb{R}$, we use $\theta = \Theta(t, \vartheta)$, $t \in \mathbb{R}$, to denote the solution of (2.1) with the initial value $\theta(0) = \vartheta$. The solution of (2.2) with $r(0) = 1$ is denoted by $R(t, \vartheta)$ and is given by

$$r = R(t, \vartheta) = \exp \left(\int_0^t G(s, \Theta(s, \vartheta)) ds \right), \quad t \in \mathbb{R}. \tag{2.10}$$

For convenience, we always write

$$\hat{G}(t, \vartheta) \equiv G(t, \Theta(t, \vartheta)). \tag{2.11}$$

The equalities (2.12) and (2.13) below can be deduced from the 2π -periodicity of $A(t, \theta)$ in t and θ .

Lemma 2.2 For all $t, \vartheta \in \mathbb{R}$ and $n \in \mathbb{Z}$, there hold

$$\Theta(t, \vartheta + 2n\pi) \equiv \Theta(t, \vartheta) + 2n\pi, \tag{2.12}$$

$$\Theta(t + 2n\pi, \vartheta) \equiv \Theta(t, \Theta(2n\pi, \vartheta)). \tag{2.13}$$

Define the Poincaré map $\mathcal{H} : \mathbb{R} \rightarrow \mathbb{R}$ of (2.1) by

$$\mathcal{H}(\vartheta) = \mathcal{H}_{q^+, q^-}(\vartheta) := \Theta(2\pi, \vartheta), \quad \vartheta \in \mathbb{R}.$$

Then \mathcal{H} satisfies

$$\mathcal{H}(\vartheta + 2n\pi) \equiv \mathcal{H}(\vartheta) + 2n\pi, \quad n \in \mathbb{Z}. \tag{2.14}$$

See (2.12). The map \mathcal{H} induces an orientation-preserving homeomorphism on the circle \mathbb{S}^1 .

A crucial fact about equation (2.1) is the following one. We use $\|q\|_{L^1} = \int_0^{2\pi} |q(t)|dt$ to denote the L^1 norm for $q \in L^1(\mathbb{S}^1)$.

Proposition 2.1 Let $q^\pm \in L^1(\mathbb{S}^1)$. Then the Poincaré map $\mathcal{H} : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 diffeomorphism of \mathbb{R} . Moreover, the function $\log \mathcal{H}'(\vartheta)$ is globally Lipschitz continuous in $\vartheta \in \mathbb{R}$. In particular, the function $\log \mathcal{H}'(\vartheta)$, $\vartheta \in [0, 2\pi]$, has bounded variation.

Proof. As noticed before, the vector field $A(t, \theta)$ is continuously differentiable in θ . Hence, from the continuously differentiable dependence of solutions on initial values, we know that the solutions $\Theta(t, \vartheta)$ is C^1 in ϑ . In fact, by differentiating equation (2.1) with respect to the initial value ϑ , we know that $V(t) := \frac{\partial \Theta(t, \vartheta)}{\partial \theta}$ satisfies the variational equation

$$V'(t) = \frac{\partial A(t, \Theta(t, \vartheta))}{\partial \theta} V(t) \equiv -2\hat{G}(t, \vartheta)V(t), \quad V(0) = 1,$$

where equality (2.5) and definition (2.11) are used. Integrating this equation, we get

$$\frac{\partial \Theta(t, \vartheta)}{\partial \theta} = \exp\left(-2 \int_0^t \hat{G}(s, \vartheta) ds\right). \tag{2.15}$$

In particular,

$$\log \mathcal{H}'(\vartheta) = -2 \int_0^{2\pi} \hat{G}(s, \vartheta) ds. \tag{2.16}$$

Using the function $r(t)$ in (2.6), we get from (2.7) and (2.15) the following estimates

$$\exp(-\|r\|_{L^1}) \leq \frac{\partial \Theta(t, \vartheta)}{\partial \theta} \leq \exp(\|r\|_{L^1}), \quad t \in [0, 2\pi], \vartheta \in \mathbb{R}.$$

In particular we have

$$|\Theta(t, \vartheta_2) - \Theta(t, \vartheta_1)| \leq \exp(\|r\|_{L^1})|\vartheta_2 - \vartheta_1|, \quad t \in [0, 2\pi], \vartheta_1, \vartheta_2 \in \mathbb{R}. \tag{2.17}$$

Now it follows (2.8), (2.11) and (2.17) that

$$|\hat{G}(t, \vartheta_2) - \hat{G}(t, \vartheta_1)| \leq r(t) \exp(\|r\|_{L^1}) |\vartheta_2 - \vartheta_1|, \quad t \in [0, 2\pi], \vartheta_1, \vartheta_2 \in \mathbb{R}.$$

By (2.16) we have

$$|\log \mathcal{H}'(\vartheta_2) - \log \mathcal{H}'(\vartheta_1)| \leq 2 \int_0^{2\pi} r(t) e^{\|r\|_{L^1}} |\vartheta_2 - \vartheta_1| dt = 2\|r\|_{L^1} e^{\|r\|_{L^1}} |\vartheta_2 - \vartheta_1|$$

for all $\vartheta_i \in \mathbb{R}$. This proves that $\log \mathcal{H}'(\vartheta)$ is globally Lipschitz continuous in \mathbb{R} . ■

Remark 2.1 (i) In case $q^+ = q^- = q \in L^1(\mathbb{S}^1)$, equation (1.3) is the Hill's equation (1.2). The Poincaré matrix M of (1.2) induces a linear map from \mathbb{R}^2 to itself which is analytical. Now the corresponding Poincaré map $\mathcal{H} : \mathbb{R} \rightarrow \mathbb{R}$ is a lifting of the circle diffeomorphism

$$\mathbb{S}^1 \subset \mathbb{R}^2 \mapsto \mathbb{S}^1, \quad u \mapsto Mu / \|Mu\|.$$

Then \mathcal{H} is an analytical diffeomorphism in this case. For the asymmetric case, Proposition 2.1 has provided the necessary regularity on \mathcal{H} which is sufficient to apply Denjoy theorem.

(ii) Equalities (2.10) and (2.15) show that the solutions $\Theta(t, \vartheta)$ and $R(t, \vartheta)$ of (2.1)–(2.2) satisfy

$$\frac{\partial \Theta(t, \vartheta)}{\partial \theta} \equiv \frac{1}{R^2(t, \vartheta)}. \tag{2.18}$$

Equality (2.18) is the same as the area-preserving property of the solution mapping of (1.3) in phase plane. This is useful in studying eigenvalues and Fučík spectrum associated with (1.3). See [19, 20].

Note that the rotation number $\rho = \rho(q^+, q^-) \geq 0$ is not defined modulo 1. This will be convenient in studying differential equations, as seen from [21].

As a consequence of the Denjoy theorem [10, Theorem 12.1.1], we have the following result. Suppose that the rotation number $\rho = \rho(q^+, q^-)$ of (1.3) is irrational. Then there exists an increasing homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h(\vartheta + 2n\pi) = h(\vartheta) + 2n\pi, \quad \vartheta \in \mathbb{R}, n \in \mathbb{Z}, \tag{2.19}$$

$$\mathcal{H}(\vartheta) = h^{-1}(h(\vartheta) + 2\pi\rho), \quad \vartheta \in \mathbb{R}. \tag{2.20}$$

Equalities (2.19) and (2.20) mean that \mathcal{H} is topologically conjugate to the translation $R_{2\pi\rho} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\theta \mapsto \theta + 2\pi\rho$.

3 Existence of exponential growth rates

In this section, we will give the complete proof of Theorem 1.1. In doing so, some properties on $\chi(\vartheta) := \chi(\cos \vartheta, -\sin \vartheta)$, the exponential growth rate of the solution with initial direction $(\cos \vartheta, -\sin \vartheta)$, will be found. We give the proof of Theorem 1.1 by considering two cases separately.

3.1 Rational rotation numbers

Suppose that $\rho = \rho(q^+, q^-)$ is rational. Denote $\rho = k/\ell$, $k, \ell \in \mathbb{Z}$, $k \geq 0$, $\ell \geq 1$, and k, ℓ are co-prime. We introduce the map

$$\check{\mathcal{H}}(\vartheta) = \mathcal{H}^\ell(\vartheta) = \Theta(2\ell\pi, \vartheta), \quad \vartheta \in \mathbb{R}.$$

Then $\check{\mathcal{H}}$ is the corresponding $2\ell\pi$ -Poincaré map of (2.1) by considering q^\pm as in $L^1(\mathbb{R}/2\ell\pi\mathbb{Z})$ of $2\ell\pi$ -periodic functions. It is evident that $\check{\mathcal{H}}$ also satisfies the periodicity equality (2.14). The rationality of $\rho = k/\ell$ implies that there is at least one $\vartheta \in [0, 2\pi)$ such that

$$\check{\mathcal{H}}(\vartheta) = \vartheta + 2k\pi. \tag{3.1}$$

Let α_0 be the minimal $\vartheta \in [0, 2\pi)$ satisfying (3.1) and $J := [\alpha_0, \alpha_0 + 2\pi]$. Denote

$$\Omega = \{\vartheta \in J : \vartheta \text{ satisfies (3.1)}\}.$$

Then $\Omega \subset J$ is a non-empty closed set and has the following properties.

- Case 1. All $\vartheta \in \Omega$ satisfy also (3.1), or
- Case 2. On the complement $J \setminus \Omega = \cup_i (\alpha_i, \beta_i)$, a union of (at most countably many) open intervals, labeling using the natural ordering in \mathbb{R} , one has alternatively

$$\check{\mathcal{H}}(\vartheta) < \vartheta + 2k\pi \quad \forall \vartheta \in (\alpha_i, \beta_i), \tag{3.2}$$

or

$$\check{\mathcal{H}}(\vartheta) > \vartheta + 2k\pi \quad \forall \vartheta \in (\alpha_i, \beta_i). \tag{3.3}$$

Let us first consider Case 1. For $\vartheta_0 \in \Omega$, equality (3.1) is

$$\Theta(2\ell\pi, \vartheta_0) = \vartheta_0 + 2k\pi,$$

which means, by (2.12) and (2.13), that $\Theta(t, \vartheta_0) - kt/\ell =: \varphi(t, \vartheta_0)$ is $2\ell\pi$ -periodic in t . Hence $\hat{G}(t, \vartheta_0) = G(t, \Theta(t, \vartheta_0))$ is also $2\ell\pi$ -periodic in t . Thus the exponential growth rate

$$\chi(\vartheta_0) := \lim_{t \rightarrow \infty} \frac{1}{t} \log R(t, \vartheta_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \hat{G}(s, \vartheta_0) ds \equiv \overline{\hat{G}(\cdot, \vartheta_0)} \tag{3.4}$$

does exist and is just the mean value of $2\ell\pi$ -periodic function $\hat{G}(\cdot, \vartheta_0)$.

Now we consider Case 2. Let $\vartheta_0 \in (\alpha, \beta)$, where α and β are as in Case 1 and either (3.2) or (3.3) is true. For definiteness, let us assume that (3.3) is true. By introducing

$$\hat{\mathcal{H}}(\vartheta) := \check{\mathcal{H}}(\vartheta) - 2k\pi, \quad \vartheta \in \mathbb{R},$$

we have

$$\hat{\mathcal{H}}(\alpha) = \alpha, \quad \hat{\mathcal{H}}(\beta) = \beta, \quad \hat{\mathcal{H}}(\vartheta) > \vartheta \quad \text{for all } \vartheta \in (\alpha, \beta). \tag{3.5}$$

Hence $\hat{\mathcal{H}}$ is an increasing self-diffeomorphism of the interval $[\alpha, \beta]$ with the end-points being fixed points of $\hat{\mathcal{H}}$. From the dynamics of interval homeomorphisms, we have from (3.5)

$$\lim_{n \rightarrow +\infty} \hat{\mathcal{H}}^n(\vartheta_0) = \beta. \tag{3.6}$$

By (2.12) and (2.13), we have

$$\Theta(2n\ell\pi, \vartheta_0) = \tilde{\mathcal{H}}^n(\vartheta_0) = \hat{\mathcal{H}}^n(\vartheta_0) + 2nk\pi, \quad \Theta(2n\ell\pi, \beta) = \beta + 2nk\pi, \quad n \in \mathbb{Z}.$$

Denote

$$\varepsilon_n := \Theta(2n\ell\pi, \vartheta_0) - \Theta(2n\ell\pi, \beta), \quad n \in \mathbb{Z}.$$

Then (3.6) means that

$$\lim_{n \rightarrow +\infty} \varepsilon_n = 0. \tag{3.7}$$

Now, for any $t \in [2n\ell\pi, 2(n+1)\ell\pi]$, we have from (2.13) the following estimate

$$\begin{aligned} |\Theta(t, \vartheta_0) - \Theta(t, \beta)| &= |\Theta(t - 2n\ell\pi, \Theta(2n\ell\pi, \vartheta_0)) - \Theta(t - 2n\ell\pi, \Theta(2n\ell\pi, \beta))| \\ &\leq \exp(\ell\|r\|_{L^1})|\Theta(2n\ell\pi, \vartheta_0) - \Theta(2n\ell\pi, \beta)| = \exp(\ell\|r\|_{L^1})\varepsilon_n, \end{aligned}$$

where inequality (2.17) is used. By (2.8) we get

$$|\hat{G}(t, \vartheta_0) - \hat{G}(t, \beta)| \leq \exp(\ell\|r\|_{L^1})\varepsilon_n r(t), \quad t \in [2n\ell\pi, 2(n+1)\ell\pi], \quad n \in \mathbb{Z}^+.$$

Roughly speaking, as $t \rightarrow +\infty$, the function $\hat{G}(t, \vartheta_0)$ is uniformly asymptotic to the $2\ell\pi$ -periodic function $\hat{G}(t, \beta)$. Thus, for $t \in [2n\ell\pi, 2(n+1)\ell\pi]$,

$$\begin{aligned} &\left| \frac{1}{t} \int_0^t (\hat{G}(s, \vartheta_0) - \hat{G}(s, \beta)) ds \right| \\ &\leq \frac{1}{2n\ell\pi} \sum_{k=0}^n \int_{2k\ell\pi}^{2(k+1)\ell\pi} |\hat{G}(s, \vartheta_0) - \hat{G}(s, \beta)| ds \\ &\leq \frac{1}{2n\ell\pi} \sum_{k=0}^n \int_{2k\ell\pi}^{2(k+1)\ell\pi} \exp(\ell\|r\|_{L^1})\varepsilon_k r(s) ds \\ &= \frac{\|r\|_{L^1} \exp(\ell\|r\|_{L^1})}{2\pi} \frac{1}{n} \sum_{k=0}^n \varepsilon_k. \end{aligned} \tag{3.8}$$

Now (3.7) implies that the quantity in (3.8) will go to 0 as $n \rightarrow +\infty$. Thus (3.8) shows that

$$\chi(\vartheta_0) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \hat{G}(s, \vartheta_0) ds = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \hat{G}(s, \beta) ds = \chi(\beta)$$

exists as well and is equal to $\chi(\beta)$, where formula (3.4) in Case 1 is used.

3.2 Irrational rotation numbers

Suppose that $\rho = \rho(q^+, q^-)$ is irrational. By the definition of \mathcal{H} , (2.20) is

$$\Theta(2\pi, \vartheta) = h^{-1}(h(\vartheta) + 2\pi\rho). \tag{3.9}$$

From a fundamental result due to Bohr, see, for example, [7, Theorem 2.6], we know that (3.9) implies that there exists a continuous function $\omega(u, v) : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

- $\omega(u, v)$ is 2π -periodic in both u and v , and
- for any $\vartheta_0 \in \mathbb{R}$, the solution $\Theta(t, \vartheta_0)$ can be rewritten as

$$\Theta(t, \vartheta_0) \equiv h(\vartheta_0) + \rho t + \omega(t, h(\vartheta_0) + \rho t). \tag{3.10}$$

By (3.10) we know that $\hat{G}(t, \vartheta_0)$ can be rewritten as

$$\hat{G}(t, \vartheta_0) \equiv G(t, h(\vartheta_0) + \rho t + \omega(t, h(\vartheta_0) + \rho t)).$$

Note that $G(t, \theta)$ is 2π -periodic in both t and θ . If we introduce a function $\check{G}_{\vartheta_0} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\check{G}_{\vartheta_0}(u, v) = G(u, h(\vartheta_0) + v + \omega(u, h(\vartheta_0) + v)),$$

then $\check{G}_{\vartheta_0}(u, v)$ is 2π -periodic in both u and v . Moreover, the function $\hat{G}(t, \vartheta_0)$ can be rewritten as $\hat{G}(t, \vartheta_0) \equiv \check{G}_{\vartheta_0}(t, \rho t)$. Now the exponential rate is transformed to the following limit

$$\chi(\vartheta_0) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \hat{G}(s, \vartheta_0) ds. \tag{3.11}$$

Recall that $G(t, \theta)$ is related to (q^+, q^-) by (2.4). In case that $q^\pm \in L^1(\mathbb{S}^1)$, $\hat{G}(t, \vartheta_0)$ is not continuous in t and is not an almost periodic function in the classical sense. To guarantee the existence of the mean value (3.11) for general $q^\pm \in L^1(\mathbb{S}^1)$, one may exploit the following result.

Lemma 3.1 *Given a continuous almost periodic function $b(t)$, for any $q(t) \in L^1(\mathbb{S}^1)$, the following mean value*

$$M_b(q) := \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t q(s)b(s) ds \tag{3.12}$$

is well-defined. Moreover, as a linear functional,

$$(L^1(\mathbb{S}^1), \|\cdot\|_{L^1}) \rightarrow \mathbb{R}, \quad q \mapsto M_b(q)$$

is bounded. More precisely,

$$|M_b(q)| \leq \|b\|_{C(\mathbb{R})} \|q\|_{L^1}. \tag{3.13}$$

Proof. At first, let $q \in C(\mathbb{S}^1)$. Then the function $q(t)b(t)$ is a classical almost periodic function. Hence the mean value $M_b(q)$ is well-defined. Obviously, (3.13) is satisfied for $q(t) \in C(\mathbb{S}^1)$.

Next let $q(t) \in L^1(\mathbb{S}^1)$ be fixed. We can take any sequence $q_n(t) \in C(\mathbb{S}^1)$ such that

$$\lim_{n \rightarrow \infty} \|q_n - q\|_{L^1} = \lim_{n \rightarrow \infty} \|q_n - q\|_{L^1(\mathbb{S}^1)} = 0. \tag{3.14}$$

As $q_n \in C(\mathbb{S}^1)$, we know that $M_b(q_n)$ is well-defined for all n , and from (3.13),

$$|M_b(q_n) - M_b(q_m)| \leq \|b\|_{C(\mathbb{R})} \|q_n - q_m\|_{L^1} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

Hence $\{M_b(q_n)\}$ is a Cauchy sequence of real numbers. Thus the following limit exists

$$M_* = \lim_{n \rightarrow \infty} M_b(q_n). \tag{3.15}$$

In the following, we prove that (3.12) is convergent to M_* as $t \rightarrow +\infty$. The proof for the case $t \rightarrow -\infty$ is similar. As $q(t) \in L^1(\mathbb{S}^1)$, $q_n(t) \in C(\mathbb{S}^1)$, we have, for any $t \in [2m\pi, 2(m+1)\pi]$, $m \in \mathbb{N}$, the following trivial estimate

$$\begin{aligned} & \left| \frac{1}{t} \int_0^t q(s)b(s)ds - \frac{1}{t} \int_0^t q_n(s)b(s)ds \right| \\ & \leq \frac{\|b\|_{C(\mathbb{R})}}{2\pi} \left(1 + \frac{1}{m} \right) \|q - q_n\|_{L^1} \leq \frac{\|b\|_{C(\mathbb{R})}}{\pi} \|q - q_n\|_{L^1} < \varepsilon, \end{aligned}$$

by fixing an $n = N = N_\varepsilon$, where N is so large that

$$\|q_N - q\|_{L^1} < \pi\varepsilon/\|b\|_{C(\mathbb{R})}, \quad |M_b(q_N) - M_*| < \varepsilon.$$

See (3.14) and (3.15). Now we have

$$\begin{aligned} & \left| \frac{1}{t} \int_0^t q(s)b(s)ds - M_* \right| \\ & \leq \left| \frac{1}{t} \int_0^t q(s)b(s)ds - \frac{1}{t} \int_0^t q_N(s)b(s)ds \right| \\ & \quad + \left| \frac{1}{t} \int_0^t q_N(s)b(s)ds - M_b(q_N) \right| + |M_b(q_N) - M_*| \\ & < 2\varepsilon + \left| \frac{1}{t} \int_0^t q_N(s)b(s)ds - M_b(q_N) \right|. \end{aligned}$$

By the definition of $M_b(q_N)$, there exists $T = T_\varepsilon \gg 1$ such that for all $t > T$, one has

$$\left| \frac{1}{t} \int_0^t q(s)b(s)ds - M_* \right| < 3\varepsilon.$$

Thus

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t q(s)b(s)ds = M_*.$$

The proof here shows also that M_* in (3.15) is independent of the choice of q_n satisfying (3.14). The inequality (3.13) for general $q(t) \in L^1(\mathbb{S}^1)$ follows simply the corresponding result for $q(t) \in C(\mathbb{S}^1)$ and result (3.15) because we have

$$|M_*| = \lim_{n \rightarrow \infty} |M_b(q_n)| \leq \lim_{n \rightarrow \infty} \|b\|_{C(\mathbb{R})} \|q_n\|_{L^1} = \|b\|_{C(\mathbb{R})} \|q\|_{L^1}$$

by the choice (3.14) of the sequence $\{q_n\}$. ■

Recall from (2.9) that the function $\hat{G}(t, \vartheta_0)$ can be decomposed into

$$\hat{G}(t, \vartheta_0) = q^+(t)b_1(t) + q^-(t)b_2(t) - b_3(t), \tag{3.16}$$

$$b_1(t) = \sin(\Theta(t, \vartheta_0))(\cos(\Theta(t, \vartheta_0)))_+, \tag{3.17}$$

$$b_2(t) = \sin(\Theta(t, \vartheta_0))(\cos(\Theta(t, \vartheta_0)))_-, \tag{3.18}$$

$$b_3(t) = \sin(\Theta(t, \vartheta_0)) \cos(\Theta(t, \vartheta_0)). \tag{3.19}$$

Due to (3.10), the functions $b_i(t)$ are continuous quasi-periodic functions of frequencies 1 and ρ [7, Appendix]. Applying Lemma 3.1 to $b = b_1, b_2, b_3$ in (3.17)–(3.19) and $q = q^+, q^-, -1$ respectively, we know from (3.16) that the limit (3.11) does exist for general $q^\pm \in L^1(\mathbb{S}^1)$. Now the proof of Theorem 1.1 is complete. ■

Remark 3.1 When $\rho(q^+, q^-)$ is rational, the existence of $\chi(\vartheta_0), \vartheta_0 \in \mathbb{R}$, can also be proved using the decompositions (3.16). In this case, the functions $b_i(t)$ in (3.17)–(3.19) are either periodic or asymptotic to periodic functions.

4 Properties of exponential growth rates

For the Hill’s equation (1.2), where $q(t) \in L^1(\mathbb{S}^1)$, from the Floquet theory of (1.2) we have the following properties on Lyapunov exponents of (1.2).

- When equation (1.2) is elliptic or parabolic, then $\chi(\vartheta_0) = 0$ for all $\vartheta_0 \in \mathbb{R}$.
- When equation (1.2) is hyperbolic, then there exists $\chi(q) > 0$ such that either $\chi(\vartheta_0) = \chi(q)$ or $\chi(\vartheta_0) = -\chi(q)$ for any $\vartheta_0 \in \mathbb{R}$. Hence the maximal Lyapunov exponent $\chi(q) \geq 0$ is well-defined for any case.

The maximal Lyapunov exponent $\chi(q) : q \in (L^1(\mathbb{S}^1), \|\cdot\|_{L^1}) \rightarrow \mathbb{R}$ is continuous in the usual L^1 topology. Furthermore, Zhang [21] has recently obtained a very strong continuity result of $\chi(q)$ in q . That is, the maximal Lyapunov exponent $\chi(q) : q \in (L^1(\mathbb{S}^1), w) \rightarrow \mathbb{R}$ is continuous, where w indicates the topology of the weak convergence in the space $L^1(\mathbb{S}^1)$.

In this section, we will first reveal some difference for Lyapunov exponents of asymmetric oscillators (1.3) and the Hill’s equations (1.2).

Example 4.1 It is well-known that the constant Hill’s equation

$$x'' + ax = 0$$

has the exponents $\pm\chi(a)$, where $\chi(a) := (\max(-a, 0))^{1/2}$. For the asymmetric oscillator

$$x'' + ax_+ + bx_- = 0, \quad a, b \in \mathbb{R}, \tag{4.1}$$

all solutions of (4.1) can be found explicitly. It is easy to see that all exponents of (4.1) are $\pm\chi(a)$ and $\pm\chi(b)$. Hence, in case $a, b < 0$ and $a \neq b$, system (4.1) has exactly four different exponents.

This simple example shows that the structure of Lyapunov exponents for asymmetric oscillators is different from that for Hill’s equations. In general, we have the following result on the number of Lyapunov exponents of (1.3).

Theorem 4.1 *Let $q^\pm \in L^1(\mathbb{S}^1)$. The periodic asymmetric oscillator (1.3) has at most countably infinitely many different Lyapunov exponents $\chi(\vartheta_0)$. That is, the set $\{\chi(\vartheta_0) : \vartheta_0 \in \mathbb{R}\}$ is at most countably infinite.*

We give the proof of Theorem 4.1 in two cases.

At first we assume that $\rho(q^+, q^-) = k/\ell$ is rational and we will keep the notations in Section 3.1. Recall that $\mathcal{H}(\vartheta) = \mathcal{H}^\ell(\vartheta) = \Theta(2\ell\pi, \vartheta)$. Define

$$\check{\mathcal{R}}(\vartheta) := R(2\ell\pi, \vartheta),$$

where $R(t, \vartheta)$ are solutions of (2.2). By the area-preserving property (2.18), we have

$$\check{\mathcal{H}}'(\vartheta) = \frac{\partial \check{\mathcal{H}}(\vartheta)}{\partial \vartheta} \equiv \frac{1}{(\check{\mathcal{R}}(\vartheta))^2}. \tag{4.2}$$

From the proof in Section 3.1, all different Lyapunov exponents $\chi(\vartheta_0)$ are given by those exponents $\chi(\vartheta)$ at those $\vartheta \in \Omega := \{\vartheta \in [0, 2\pi) : \check{\mathcal{H}}(\vartheta) = \vartheta + 2k\pi\}$.

Theorem 4.2 *Suppose that $\rho(q^+, q^-) = k/\ell$ is rational. Then, for any $\vartheta \in \Omega$,*

$$\chi(\vartheta) = -\frac{1}{4\ell\pi} \log \check{\mathcal{H}}'(\vartheta). \tag{4.3}$$

Proof. Let $\vartheta \in \Omega$. Then the function

$$\hat{G}(t, \vartheta) = G(t, \Theta(t, \vartheta))$$

is $2\ell\pi$ -periodic in t . By (3.4),

$$\chi(\vartheta) = \frac{1}{2\ell\pi} \int_0^{2\ell\pi} \hat{G}(t, \vartheta) dt = \frac{\log \check{\mathcal{R}}(\vartheta)}{2\ell\pi}.$$

Now (4.3) simply follows (4.2). ■

Since $\check{\mathcal{H}} = \mathcal{H}^\ell$, formula (4.3) can be rewritten as

$$\chi(\vartheta) = -\frac{1}{4\pi} \frac{1}{\ell} \sum_{i=0}^{\ell-1} \log \mathcal{H}'(\mathcal{H}^i(\vartheta)), \quad \vartheta \in \Omega. \tag{4.4}$$

Formula (4.4) shows that $\chi(\vartheta)$ can be computed using the average of the function $-\frac{1}{4\pi} \log \mathcal{H}'$ over the periodic orbit $\{\mathcal{H}^i(\vartheta) : 0 \leq i \leq \ell - 1\}$ of the diffeomorphism \mathcal{H} .

In case that $\rho(q^+, q^-)$ is irrational, we can prove that the evolution of $r(t; x)$ is trivial in the following sense.

Theorem 4.3 *Let $q^\pm \in L^1(\mathbb{S}^1)$. In case that the rotation number $\rho(q^+, q^-)$ is irrational, we have $\chi(\vartheta_0) = 0$ for all $\vartheta_0 \in \mathbb{R}$.*

The proof of Theorem 4.3 is based on the unique ergodicity theorem [10] and the area-preserving property of systems (1.3). We will complete it in several steps.

Recall from the proof in Section 3.2 that for each ϑ_0 , the exponent $\chi(\vartheta_0)$ is given by the limit (3.11). In particular, we have

$$\chi(\vartheta_0) = \lim_{n \rightarrow +\infty} \frac{1}{2n\pi} \int_0^{2n\pi} G(t, \Theta(t, \vartheta)) dt, \quad \vartheta_0 \in \mathbb{R}. \tag{4.5}$$

One crucial observation is

$$\begin{aligned} \int_0^{2n\pi} G(t, \Theta(t, \vartheta_0)) dt &= \sum_{k=0}^{n-1} \int_0^{2\pi} G(t + 2k\pi, \Theta(t + 2k\pi, \vartheta_0)) dt \\ &= \sum_{k=0}^{n-1} \int_0^{2\pi} G(t + 2k\pi, \Theta(t, \Theta(2k\pi, \vartheta_0))) dt \quad (\text{by (2.13)}) \\ &= \sum_{k=0}^{n-1} \int_0^{2\pi} G(t, \Theta(t, \Theta(2k\pi, \vartheta_0))) dt \\ &= \sum_{k=0}^{n-1} \check{G}(\Theta(2k\pi, \vartheta_0)) = \sum_{k=0}^{n-1} \check{G}(\mathcal{H}^k(\vartheta_0)), \end{aligned} \tag{4.6}$$

where

$$\check{G}(\vartheta) := \int_0^{2\pi} G(t, \Theta(t, \vartheta)) dt, \quad \vartheta \in \mathbb{R}. \tag{4.7}$$

Lemma 4.1 *The function $\check{G}(\vartheta)$ defined by (4.7) is 2π -periodic and continuous.*

Proof. As $G(t, \theta)$ is 2π -periodic in both t and θ , the 2π -periodicity of $\check{G}(\vartheta)$ in ϑ follows (2.12).

By formula (2.16), the function $\check{G}(\vartheta)$ of (4.7) can be rewritten as $-\frac{1}{2} \log \mathcal{H}'(\vartheta)$. Now Proposition 2.1 shows that $\check{G}(\vartheta)$ is actually Lipschitz continuous. ■

Lemma 4.1 shows that $\check{G}(\vartheta)$ can be considered as a continuous function on the circle \mathbb{S}^1 . The diffeomorphism $\mathcal{H} = \mathcal{H}_{q^+, q^-} : \mathbb{R} \rightarrow \mathbb{R}$ in Section 2 associated with (1.3) can be considered as a self-diffeomorphism of \mathbb{S}^1 , still denoted by $\mathcal{H} = \mathcal{H}_{q^+, q^-}$.

Lemma 4.2 *Suppose that $\rho(q^+, q^-)$ is irrational. Then $\mathcal{H} = \mathcal{H}_{q^+, q^-} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is uniquely ergodic.*

Proof. By (2.19), the map h can be considered as an orientation-preserving diffeomorphism of \mathbb{S}^1 . Now the conjugacy (2.20) can be understood a conjugacy between $\mathcal{H} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and the rigid rotation $R_{2\pi\rho} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. As ρ is irrational, it is well-known that $R_{2\pi\rho}$ has the unique ergodic measure which is the Lebesgue measure ν_0 of \mathbb{S}^1 with $\nu_0(\mathbb{S}^1) = 1$. By (2.20), $\mathcal{H} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ has also the unique ergodic measure ν which is

$$\nu(B) = \nu_0(h(B)) \quad \text{for all Borel measurable subsets } B \text{ of } \mathbb{S}^1. \tag{4.8}$$

This proves the unique ergodicity of \mathcal{H} . ■

Now we recall the unique ergodicity theorem [10, Proposition 4.1.13].

Lemma 4.3 *Let X be a compact metric space and $\phi : X \rightarrow X$ a continuous map. Suppose that $\{\phi^n\}_{n \in \mathbb{Z}^+}$ has a unique ergodic invariant Borel probability measure λ . Then for any continuous function $f : X \rightarrow \mathbb{R}$, the following convergence*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\phi^k(x)) = \int_X f d\lambda \tag{4.9}$$

is uniform in $x \in X$.

Now we complete the proof of Theorem 4.3. Applying Lemma 4.3 to (4.5) and (4.6), we know that the convergence (4.5) is uniform in $\vartheta_0 \in \mathbb{S}^1$. Moreover, by (4.9), the exponents $\chi(\vartheta_0)$ are independent of ϑ_0 and are always equal to

$$\gamma := \frac{1}{2\pi} \int_{\mathbb{S}^1} \check{G}(\theta) d\nu(\vartheta),$$

where ν is the unique ergodic measure of \mathcal{H} which is given by (4.8). That is, we have the following uniform convergence

$$\lim_{n \rightarrow +\infty} \frac{1}{2n\pi} \int_0^{2n\pi} G(t, \Theta(t, \vartheta_0)) dt = \gamma \quad \text{uniformly in } \vartheta_0 \in \mathbb{S}^1. \tag{4.10}$$

The proof of Theorem 4.3 will be complete by proving that $\gamma = 0$. To this end, we will exploit the Lagrangian structure of the asymmetric oscillators (1.3). Recall from (2.10) that

$$\int_0^{2n\pi} G(t, \Theta(t, \vartheta_0)) dt \equiv \log R(2n\pi, \vartheta_0).$$

Assume that $\gamma > 0$. By (4.10), we know that, as $n \rightarrow +\infty$,

$$\log R(2n\pi, \vartheta_0) = 2n\pi(\gamma + o(1))$$

uniformly in ϑ_0 . In particular, there exists $N_0 > 0$ such that for any $n > N_0$ one has

$$R(2n\pi, \vartheta_0) > \exp(\gamma n\pi) \quad \text{for all } \vartheta_0 \in \mathbb{R}.$$

Consider the unit disc

$$D = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.$$

Under the evolution of (1.3), D is transformed to a twist disc

$$D_{2n\pi} = \{(r, \theta) : 0 \leq r \leq R(2n\pi, \Theta(-2n\pi, \theta)), \\ \Theta(2n\pi, 0) \leq \theta \leq \Theta(2n\pi, 2\pi) = \Theta(2n\pi, 0) + 2\pi\}$$

at time $t = 2n\pi$, where (2.12) is used. When $n > N_0$, the area of $D_{2n\pi}$ is

$$\begin{aligned} & \frac{1}{2} \int_{\Theta(2n\pi,0)}^{\Theta(2n\pi,0)+2\pi} R^2(2n\pi, \Theta(-2n\pi, \theta)) d\theta > \frac{1}{2} \int_{\Theta(2n\pi,0)}^{\Theta(2n\pi,0)+2\pi} \exp(\gamma n\pi) d\theta \\ & = \pi \exp(\gamma n\pi) \rightarrow +\infty. \end{aligned}$$

It is a contradiction to the area-preserving property of (1.3). Similarly, it is impossible that $\gamma < 0$. Hence $\gamma = 0$ and the proof of Theorem 4.3 is complete. ■

Now we complete the proof of Theorem 4.1. In case that $\rho(q^+, q^-)$ is irrational, we know from Theorem 4.3 that the set $\{\chi(\vartheta_0) : \vartheta_0 \in \mathbb{R}\} = \{0\}$.

Next we assume that $\rho(q^+, q^-)$ is rational. In this case, $\chi(\vartheta)$ are given by (4.3) using $\log \mathcal{H}'(\vartheta)$ at fixed points ϑ of \mathcal{H} . For all degenerate fixed points ϑ of \mathcal{H} , i.e., $\mathcal{H}'(\vartheta) = 1$, the corresponding exponents are always 0. For any non-degenerate fixed point ϑ of \mathcal{H} , i.e., $\mathcal{H}'(\vartheta) \neq 1$, one has a maximal open interval $(\alpha, \beta) \ni \vartheta$ such that \mathcal{H} has only ϑ as the unique fixed point of \mathcal{H} in (α, β) . Since \mathbb{S}^1 is compact, there are at most countably infinitely many intervals (α, β) . By formula (4.3), one has at most countably infinitely many exponents $\chi(\vartheta)$. ■

Theorem 4.3 can be also proved using the skew-product flow [17] induced by the differential system on 2-torus \mathbb{T}^2

$$\frac{ds}{dt} = 1, \quad \frac{d\theta}{dt} = A(s, \theta).$$

Some ideas in a recent paper [4] may be useful in understanding Lyapunov exponents of asymmetric oscillators (1.3).

Due to a classical example by Millionschikov and Vinograd, the restriction on periodic coefficients $q^\pm(t)$ of (1.3) is necessary in some sense. That is, the results of this paper cannot be completely extended to almost periodic coefficients. However, partial generalizations of the results of this paper to the so-called asymmetric p -Laplacian oscillators with periodic coefficients can be given [22].

Example 4.1 and Theorems 4.1 and 4.2 show that in case $\rho(q^+, q^-)$ is rational, the structure of $\{\chi(\theta)\}$ is delicate. It is then an interesting problem if the periodic asymmetric oscillator (1.3) admits only finitely many different Lyapunov exponents.

References

- [1] L. Arnold, *Random Dynamical Systems*, Springer Verlag, New York, 1998.
- [2] P. L. Binding and B. P. Rynne, *Half-eigenvalues of periodic Sturm-Liouville problems*, J. Differential Equations **206** (2004), 280–306.
- [3] P. L. Binding and B. P. Rynne, *The spectrum of the periodic p -Laplacian*, J. Differential Equations **235** (2007), 199–218.
- [4] F. Colonius, R. Fabbri and R. Johnson, *Chain recurrence, growth rates and ergodic limits*, Ergod. Th. Dynam. Syst. **27** (2007), 1509–1524.
- [5] H. Feng and M. Zhang, *Optimal estimates on rotation number of almost periodic systems*, Z. Angew. Math. Phys. **57** (2006), 183–204.

- [6] S. Fučík, *Boundary value problems with jumping nonlinearities*, Časopis Pěst. Mat. **101** (1976), 69–87.
- [7] J. K. Hale, *Ordinary Differential Equations*, 2nd ed., Wiley, New York, 1969.
- [8] R. Johnson and J. Moser, *The rotation number for almost periodic potentials*, Commun. Math. Phys. **84** (1982), 403–438; Erratum, Commun. Math. Phys. **90** (1983), 317–318.
- [9] R. A. Johnson, K. J. Palmer and G. R. Sell, *Ergodic properties of linear dynamical systems*, SIAM J. Math. Anal. **18** (1987), 1–33.
- [10] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge, 1995.
- [11] M. Kunze, *On Lyapunov exponents for non-smooth dynamical systems with an application to a pendulum with dry friction*, J. Dynam. Differential Equations **12** (2000), 31–116.
- [12] J. Kwapisz, *Poincaré rotation number for maps of the real line with almost periodic displacement*, Nonlinearity **13** (2000), 1841–1854.
- [13] W. Li and K. Lu, *Rotation numbers for random dynamical systems on the circle*, Trans. Amer. Math. Soc. **360** (2008), 5509–5528.
- [14] X. Li and Q. Ma, *Boundedness of solutions for second order differential equations with asymmetric nonlinearity*, J. Math. Anal. Appl. **314** (2006), 233–253.
- [15] S. T. Liao, *On characteristic exponents construction of a new Borel set for the multiplicative ergodic theorem for vector fields*, Acta Sci. Natur. Univ. Pekinensis **29** (1993), 277–302.
- [16] W. Magnus and S. Winkler, *Hill's Equations*, corrected reprint of 1966 edition, Dover, New York, 1979.
- [17] W. Shen and Y. Yi, *Almost automorphic and almost periodic dynamics in skew-product semiflows*, Mem. Amer. Math. Soc. **136** (1998), no. 647, 93 pp.
- [18] X. P. Wang, *Invariant tori and boundedness in asymmetric oscillations*, Acta Math. Sinica Engl. Ser. **19** (2003), 765–782.
- [19] M. Zhang, *The rotation number approach to eigenvalues of the one-dimensional p -Laplacian with periodic potentials*, J. London Math. Soc. (2) **64** (2001), 125–143.
- [20] M. Zhang, *The rotation number approach to the periodic Fučík spectrum*, J. Differential Equations **185** (2002), 74–96.
- [21] M. Zhang, *Continuity in weak topology: higher order linear systems of ODE*, Sci. China Ser. A **51** (2008), 1036–1058.
- [22] M. Zhang, Z. Zhou and M. Li, *Lyapunov exponents and switching times of asymmetric p -Laplacian oscillators*, Preprint, 2008. <http://faculty.math.tsinghua.edu.cn/~mzhang/publs/publ.htm>