Non-degeneracy and uniqueness of periodic solutions for some superlinear beam equations

Wei Li, Meirong Zhang *
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

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ABSTRACT

In this work we will use some Sobolev constants to explicitly characterize a class of potentials \( q(t) \in L^p(\mathbb{R}/T\mathbb{Z}) \) for which the periodic linear beam equation \( u^{(4)} = q(t)u \) is non-degenerate. As an application, we will obtain the uniqueness of periodic solutions of a certain class of superlinear beam equations.

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1. Introduction

For various kinds of boundary value problems for nonlinear differential equations, the existence of solutions can be studied by many methods. In this work, we are dealing with the uniqueness. Generally speaking, the uniqueness of a solution can be obtained when the nonlinearities are Lipschitz continuous and are not resonant. However, these exclude superlinear nonlinear equations.

Recently, on the basis of some results on eigenvalues of Hill’s equations [14], Ortega and Zhang [6] have introduced some classes of nonlinear second-order ODEs, including some superlinear ones, for which the uniqueness of the periodic solution can be derived. Kunze and Ortega [3] have generalized this approach to some superlinear PDEs. In this work, we take the nonlinear beam equations [9] as models to give an extension to some higher-order superlinear ODEs. The key idea is the following concept.

Definition 1. Given \( q(t) \in L^p(\mathbb{S}_T), \mathbb{S}_T = \mathbb{R}/T\mathbb{Z}, 1 \leq p \leq \infty \), we say that the linear beam equation

\[
u^{(4)} = q(t)u, \quad t \in \mathbb{R}, \quad u \in \mathbb{R},\]

is non-degenerate with respect to the \( T \)-periodic boundary condition

\[
u^{(i)}(0) = \nu^{(i)}(T), \quad 0 \leq i \leq 3,\]

if problem (1) and (2) has only the trivial solution \( u(t) = 0 \). In this case, we also say that \( q(t) \) is a non-degenerate potential of problem (1) and (2).
Eq. (1) is also called a periodic Euler–Bernoulli equation [8]. The non-degeneracy of (1) can be rephrased using periodic eigenvalues of \( u^{(4)} = \lambda q(t)u \). However, many problems concerning with periodic linear beam equations remain open [7,8], especially compared with the theory for Hill’s equations [4]. Hence we will try not to use the eigenvalue theory of (1) and will use some Sobolev inequalities to give a direct characterization of some non-degenerate potentials. See Theorem 2.

The classes \( C(\sigma; A, B) \) of nonlinearities to be considered are given in Definition 3. These nonlinearities \( f(x) \) could grow superlinearly as \( x \to \infty \). Besides the existence for equations of Landesman–Lazer type [12] where the nonlinearities are monotone and one-sided bounded, we will show in Theorem 5 that, for those classes of nonlinear equations, the uniqueness of the periodic solution can be obtained.

2. Sobolev inequalities and non-degenerate potentials

We need some Sobolev inequalities and their optimal constants. For a function \( h(t) \) in the Lebesgue space \( L^1(\mathbb{R}) \) of \( T \)-periodic functions, \( S_T = \mathbb{R}/T\mathbb{Z} \), the mean value of \( h(t) \) is \( \bar{h} = \frac{1}{T} \int_{S_T} h(t) \, dt \). Then \( L^1(S_T) \) can be decomposed as \( L^1(S_T) = \mathbb{R} \oplus \tilde{L}^1(S_T) \), where \( \tilde{L}^1(S_T) = \{ h \in L^1(S_T) : \bar{h} = 0 \} \) and \( \mathbb{R} \) is identified as the set of constant functions of \( L^1(S_T) \). Analogously, the Sobolev space \( H^1(S_T) \) can be decomposed as \( H^1(S_T) = \mathbb{R} \oplus H^1(S_T) \), where \( H^1(S_T) = H^1(S_T) \cap \tilde{L}^1(S_T) \).

Given an exponent \( \gamma \) \( \in [1, \infty] \). There must be some constant \( C > 0 \) such that

\[
C \| \phi \|_{L^2(S_T)}^2 \leq \| \phi'' \|_{L^2(S_T)}^2 \quad \text{for all } \phi \in H^2(S_T).
\]

To simplify the notation, we write the \( L^\gamma \) norm \( \| \cdot \|_{L^\gamma(S_T)} \) as \( \| \cdot \|_{\gamma} \) when the period \( T \) is clear. The best constant in (3) will be denoted by \( M(\gamma, T) \). That is,

\[
M(\gamma, T) = \inf_{\phi \in H^{2}(S_T), \phi \neq 0} \frac{\| \phi'' \|_2^2}{\| \phi \|_{\gamma}^2}.
\]

The scaling of the these constants (in \( T \)) is as follows. For \( \psi(t) \in H^2(S_T) \), we know that \( \psi(t) := \psi(Tt) \in H^2(S_T) \). Since

\[
\| \psi \|_{L^2(S_T)} = T^{-2/\gamma} \| \psi \|_{L^2(S_T)}, \quad \| \psi'' \|_{L^2(S_T)} = T^{3/2} \| \psi'' \|_{L^2(S_T)},
\]

we have

\[
\frac{\| \psi'' \|^2_{L^2(S_T)}}{\| \psi \|^2_{L^2(S_T)}} = \frac{1}{T^{3+2/\gamma}} \frac{\| \psi'' \|^2_{L^2(S_T)}}{\| \psi \|^2_{L^2(S_T)}}
\]

and therefore the following scaling equality:

\[
M(\gamma, T) = M(\gamma) / T^{3+2/\gamma}, \quad \text{where } M(\gamma) := M(\gamma, 1).
\]

The explicit formula for these Sobolev constants is unknown for general exponent \( \gamma \in [1, \infty] \), because this will lead to fourth-order differential equations. However, for some exponents \( \gamma \), the constants \( M(\gamma) \) can be computed explicitly. For example, \( M(2) = (2\pi)^4 \), which is the first positive 1-periodic eigenvalue of \( u^{(4)} = \lambda u \). Another constant is \( M(\infty) \approx 720 \), which can obtained using Fourier expansions and the variational method, as was done in [5] for second-order equations. For other kinds of Sobolev constants concerning with the fourth-order ordinary differential operators, see [2].

Now we give the following non-degeneracy result for problem (1) and (2).

**Theorem 2.** Suppose that \( q(t) \) is in \( L^q(S_T) \) for some \( \alpha \in [1, \infty] \). If

\[
\bar{q} > 0 \quad \text{and} \quad \| q_+ \|_\alpha < M(2\alpha^*, T) = M(2\alpha^*) / T^{4-1/\alpha},
\]

then (1) and (2) is non-degenerate. Here \( \alpha^* = \alpha / (\alpha - 1) \in [1, \infty] \) and \( q_+(t) = \max(q(t), 0) \).

**Proof.** We argue by contradiction. Assume that (1) and (2) has a non-trivial solution \( \phi \in W^{4,\alpha}(S_T) \). Let us write \( \phi = \bar{\phi} + \tilde{\phi} \), where \( \tilde{\phi} := \phi - \bar{\phi} \in H^2(S_T) \). Now Eq. (1) for \( \phi \) is

\[
\bar{\phi}^{(4)} = q(t)\bar{\phi} + q(t)\tilde{\phi}.
\]

Integrating this equation over one period, we have, by the \( T \)-periodicity of \( \bar{\phi} \),

\[
\int_0^T q(t)\bar{\phi} + \int_0^T q(t)\tilde{\phi} = 0.
\]

Since we have assumed in (5) that \( \bar{q} > 0 \), one has \( \bar{\phi} = -\left( \int_0^T q(t)\tilde{\phi} / (T\bar{q}) \right) \). Multiplying (6) by \( \tilde{\phi} - \bar{\phi} \), we have \( \tilde{\phi}\bar{\phi}^{(4)} - \tilde{\phi}\bar{\phi}^{(4)} = q(t)\tilde{\phi}^2 - q(t)\bar{\phi}^2 \). Integrating this equation over one period and making use of the \( T \)-periodicity of \( \tilde{\phi} \), we get

\[
-\int_0^T \tilde{\phi}^2 = \int_0^T \tilde{\phi}\bar{\phi}^{(4)} - \int_0^T \bar{\phi}\tilde{\phi}^{(4)} = \int_0^T q(t)\tilde{\phi}^2 - \int_0^T q(t)\bar{\phi}^2 = T\bar{q}\tilde{\phi}^2 - \int_0^T q(t)\bar{\phi}^2.
\]
Hence, using the Hölder inequality,
\[
\|\tilde{\varphi}''\|_2^2 = \int_0^T q(t)\tilde{\varphi}'^2 - T\tilde{\varphi}'^2 \leq \int_0^T q(t)\tilde{\varphi}'^2 \leq \int_0^T q_+(t)\tilde{\varphi}'^2 \\
\leq \|q_+\|_\sigma \|\tilde{\varphi}'\|_{\sigma^*} = \|q_+\|_\sigma \|\tilde{\varphi}'\|_{\sigma^*} \leq \frac{\|q_+\|_\sigma}{M(2\sigma^*, T)} \|\tilde{\varphi}''\|_2,
\]
where the Sobolev inequalities (3) and (4) are used. Under assumption (5), it is necessary that \(\|\tilde{\varphi}''\|_2 = 0\). Thus \(\tilde{\varphi}'\) is a constant. Since \(\tilde{\varphi} \in H^2(S_T)\), one has necessarily \(\tilde{\varphi}'(t) \equiv 0\). Now \(\tilde{\varphi} = -\left(\int_0^t q(t)\tilde{\varphi}\right)/T\tilde{q} = 0\). Thus \(\varphi = 0\), which contradicts the assumption \(\varphi \neq 0\). \(\square\)

3. Classes of nonlinearities

As a direct application of general non-degenerate potentials, one can obtain reasonable existence results for periodic solutions of nonlinear beam equation \(u^{(4)} = f(t, u)\), when \(f(t, u)\) grows semilinearly when \(|u| \to \infty\). We will not develop this here and refer readers to [5,10,13].

In the following we will use the classes of non-degenerate potentials constructed above to study the nonlinear beam equations
\[
u(t) = f(u) - s + \bar{h}(t),
\]
where \(s \in \mathbb{R}, \bar{h} \in L^1(\mathbb{R})\), and the nonlinearity \(f: \mathbb{R} \to \mathbb{R}\) is a continuous function. The parameter \(s\) is the negative mean value of the external term \(-s + \bar{h}(t)\).

In order for Eq. (8) to have \(T\)-periodic solutions \(u(t)\), by integrating (8) over one period, it is necessary that
\[
s = T^{-1} \int_{0}^{T} f(u(t))dt = f(u(T)) \in \mathcal{R}(f) := \{f(x): x \in \mathbb{R}\}.
\]
If, in addition, \(f: \mathbb{R} \to \mathbb{R}\) is monotone and is one-sided bounded, condition (9) is, in some sense, also sufficient for the existence of \(T\)-periodic solutions. More precisely, if \(s \in \text{int} \mathcal{R}(f)\), then (8) has at least one \(T\)-periodic solution. Here the one-sided boundedness is
\[
either \inf_{u \in \mathbb{R}} f(u) > -\infty, or \sup_{u \in \mathbb{R}} f(u) < +\infty,
\]
and \(\text{int} I\) is the interior of an interval \(I\). In this case, Eq. (8) is of the Landesman–Lazer type and these results are well known. See, for example, [12].

In order to study the uniqueness of the periodic solution of (8), we introduce another condition on the nonlinearities \(f(u)\).

**Definition 3.** Given \(\sigma \in [1, \infty)\) and \(A, B \in [0, \infty)\), we say that a continuous function \(f: \mathbb{R} \to \mathbb{R}\) belongs to the class \(\mathcal{C}(\sigma; A, B)\) if, for all \(x_1, x_2 \in \mathbb{R}, x_1 \neq x_2\), one has
\[
\left(\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right)^\sigma \leq A \left(\frac{f(x_1) + f(x_2)}{2}\right)^\sigma + B.
\]
Here \(y_+ = \max(y, 0)\) for \(y \in \mathbb{R}\).

Some properties on classes \(\mathcal{C}(\sigma; A, B)\) of functions are as follows. They can be verified without difficulty.

- It is obvious that
  \[
f \in \mathcal{C}(\sigma; A, B) \implies kf \in \mathcal{C}(\sigma; k^{\sigma-1}A, k^\sigma B), \quad k \geq 0.
  \]

- For \(A = 0\), the class \(\mathcal{C}(\sigma; 0, B)\) contains all Lipschitz continuous functions. In fact,
  \[
  |f(x_1) - f(x_2)| \leq L|x_1 - x_2| \implies f \in \mathcal{C}(\sigma; 0, L^\sigma), \quad \sigma \geq 1.
  \]

- In the case \(A > 0\), any \(f \in \mathcal{C}(\sigma; A, B)\) is bounded from below. In fact, \(f(x) \geq -B/A, x \in \mathbb{R}\).

- Using the trivial inequality \(x + y \geq 2^{-\sigma}(x^\sigma + y^\sigma)\) for \(\alpha, \beta \geq 0\) and \(\gamma \geq 1\), we have the following result:
  \[
f_i \in \mathcal{C}(\sigma; A_i, B_i), \quad A_i > 0, \quad i = 1, 2 \implies f_1 + f_2 \in \mathcal{C}(\sigma; 2\sigma^{-1} \max(A_1, A_2), 2\sigma^{-1}(B_1/A_1 + B_2/A_2) \max(A_1, A_2)).
  \]

From this fact and (11), for each \(\sigma \geq 1\), the class \(\mathcal{C}(\sigma) := \bigcup_{A>0,B>0} \mathcal{C}(\sigma; A, B)\) is a cone in the space \(C(\mathbb{R})\) of continuous functions of \(\mathbb{R}\).
More generally, if $f_i \in C(\sigma_i; A_i, B_i)$ with $A_i > 0$, $i = 1, 2$, one has

$$f_1 + f_2 \in C(\min(\sigma_1, \sigma_2); \hat{A}, \hat{B}) \text{ for some } \hat{A}, \hat{B} > 0.$$  \hfill (12)

It follows from (11) and (12) that the class $\hat{C} := \bigcup_{\sigma \geq 1} C(\sigma)$ is also a cone in the space $C(\mathbb{R})$.

Some typical examples are (i) $x^p \in C(p^\ast, 0)$ for $p \in (1, \infty)$; (ii) $x^1$, $|x| \in C(1, 0, 1)$; (iii) $\exp(x) \in C(1, 1, 0)$. The corresponding constants $(A, B) = (p^\ast, 0)$, $(A, B) = (1, 0)$ and $(A, B) = (0, 1)$ are optimal for (10) to hold.

The combination of these examples, together with Lipschitz continuous functions, can yield more examples. For example, $\exp(x) + \sin x \in C(1; 1, 2)$, and

$$x_+ + x^2_+ \in C(1; 1, 2),$$

for any $\epsilon > 0, x_+ + x^2_+ \in C(2; 4, 1)$. \hfill (13)

4. Uniqueness of periodic solutions of superlinear beam equations

Now we study the uniqueness for the $T$-periodic solution of Eq. (8). This will be obtained on the basis of the following observation.

**Proposition 4.** Let $f \in C(\sigma; A, B)$ be non-decreasing. Suppose that $s \in \mathcal{R}(f)$ satisfies

$$As + B < (M(2\sigma^*)/T^4)\sigma.$$  \hfill (14)

Then any two $T$-periodic solutions of Eq. (8) differ only by a constant.

**Proof.** Assume that $\psi_i(t), i = 1, 2$, are two different solutions of Eq. (8) satisfying (2). That is,

$$\psi_i^{(4)}(t) = f(\psi_i(t)) - s + \hat{h}(t) \text{ a.e. } t.$$  \hfill (15)

Integrating Eq. (15) over a period, we obtain

$$\int_0^T f(\psi_i(t))dt = Ts, \quad i = 1, 2.$$ \hfill (16)

Let $\psi(t) := \psi_1(t) - \psi_2(t)$ be the difference of two solutions. Then $\psi(t) \neq 0$. The difference of Eq. (15) gives

$$\psi^{(4)}(t) = f(\psi_1(t)) - f(\psi_2(t)) \text{ a.e. } t.$$ \hfill (17)

Let $I := \{t \in \mathbb{R} : \psi(t) \neq 0\}$, which is a non-empty open subset of $\mathbb{R}$. The function $q(t) = (f(\psi_1(t)) - f(\psi_2(t)))/(\psi_1(t) - \psi_2(t))$ is well defined for all $t \in I$. Obviously, $q(t) \in C(I)$. On the complement $J := \mathbb{R} \setminus I$, we always take $q(t) = 0$. Hence $q(t)$ is well defined on $\mathbb{R}$. It is obvious that $q(t)$ is measurable. Since we have assumed that $f(x)$ is non-decreasing in $x$, one has $q(t) \geq 0$ for all $t$. Moreover, condition (10) is now, for all $t \in I$,

$$q(t)^\sigma \leq A(f(\psi_1(t)) + f(\psi_2(t)))/2 + B \leq C$$ \hfill (18)

for some constant $C \geq 0$, because $f(x)$ is continuous and the $\psi_i(t)$ are $T$-periodic. Hence $q(t) \geq 0$ for all $t$ and $q \in L^\infty(S_T)$. Now Eq. (17) can be rewritten as saying that $\psi(t)$ is a non-trivial $T$-periodic solution of the linear beam equation (1) with such a choice of $q(t)$. Condition (18) implies

$$\int_0^T q(t)^\sigma = \int_{t \in [0,T]} q(t)^\sigma \leq \int_{t \in [0,T]} (A(f(\psi_1(t)) + f(\psi_2(t)))/2 + B)$$

$$\leq \int_{t \in [0,T]} (A(f(\psi_1(t)) + f(\psi_2(t)))/2 + B) + \int_{t \in [0,T]} (A(f(\psi_1(t)) + f(\psi_2(t)))/2 + B)$$

$$= A \left( \int_0^T f(\psi_1(t)) + \int_0^T f(\psi_2(t)) \right) + BT = (As + B)T.$$ \hfill (19)

Thus $\|q\|^\sigma \leq ((As + B)T)^{1/\sigma}$. Now the condition $\|q\|^\sigma < M(2\sigma^*; T) = M(2\sigma^*; T^{4-1/\sigma})$ (see (5)) can be guaranteed by requiring $s$ so that $((As + B)T)^{1/\sigma} < M(2\sigma^*; T^{4-1/\sigma})$, which is the same as (14).

Under assumption (14), if we have $\hat{q} > 0$, by Theorem 2, one has $\psi(t) \equiv 0$, contradicting with the assumption $\psi_1 \neq \psi_2$. Hence we must have $\hat{q} = 0$. As $q(t) \geq 0$, we know that $q(t) \equiv 0$. Now Eq. (1) is simply $\psi^{(4)} = 0$ and hence we have necessarily $\psi(t) \equiv c \neq 0$ because $\psi(t)$ is periodic.

Now we consider beam equations of the Landesman–Lazer type.

**Theorem 5.** Let $f : \mathbb{R} \to \mathbb{R}$ be strictly increasing and $\inf_{x \in \mathbb{R}} f(x) = 0$. Suppose that $f \in C(\sigma; A, B)$ for some $\sigma \geq 1$ and $A$, $B \geq 0$. Then, for any $s \in \mathcal{R}(f)$ satisfying (14) and for any $\hat{h} \in L^1(S_T)$, Eq. (1) has exactly one $T$-periodic solution.
Proof. Since $f$ is strictly increasing, we have $\int R(f) = R(f)$. Hence, for any $s \in R(f)$, Eq. (1) has at least one $T$-periodic solution.

In order to prove the uniqueness, assume, by contradiction, that (1) has two different periodic solutions $\varphi_i, i = 1, 2$. When $s$ satisfies (14), we know from Proposition 4 that $\varphi_2(t) \equiv \varphi_1(t) + c$, where $c \neq 0$. Without loss of generality, we assume that $c > 0$. Hence one has $f(\varphi_2(t)) > f(\varphi_1(t))$ for all $t$, because $f$ is increasing. Now we have $\int_0^T f(\varphi_2(t)) > \int_0^T f(\varphi_1(t))$, which contradicts (16). \hfill \square

Example 6. Theorem 5 applies to the example $f(x) = \exp(x) \in C(1; 1, 0)$ in a direct way. For this case, one has $R(f) = (0, \infty)$. Hence the equation
\begin{equation}
 u^{(4)} = \exp(u) - s + \tilde{h}(t)
\end{equation}
has at least one $T$-periodic solution for each $s > 0$ and each $\tilde{h}$. Now condition (14) is
\begin{equation}
 s < (M(\infty)/T)^4 = 720/T^4.
\end{equation}
Theorem 5 asserts that for $s > 0$ satisfying (20), Eq. (19) has exactly one $T$-periodic solution for each $\tilde{h} \in L^1(S_T)$.

Example 7. Let $p \in (1, \infty)$. The function $f(x) = x^p \in C(p^*; p^*, 0)$ is non-decreasing, but is not strictly increasing. Theorem 5 can be applied to the following superlinear equation:
\begin{equation}
 u^{(4)} = u^{p}\_\_ - s + \tilde{h}(t)
\end{equation}
in an indirect way. For this case, one has $R(f) = [0, \infty)$. Eq. (21) has at least one $T$-periodic solution for each $s > 0$ and each $\tilde{h} \in L^1(S_T)$. Note that the function $f(x) = x^p$ is strictly increasing in $x \in (0, \infty)$. After a modification of the proof of Theorem 5, we conclude that if
\begin{equation}
 0 < s < (M(2p)/p)^{p^*} / T^{4p^*},
\end{equation}
then for each $\tilde{h} \in L^1(S_T)$, Eq. (21) has exactly one $T$-periodic solution. The reasons are as follows. Note that the second inequality of (22) corresponds to (14) for $f(x) = x^p$. If (21) has two different solutions $\varphi_i, i = 1, 2$, we have then $\varphi_2 = \varphi_1 + c$ for some $c \neq 0$. We assume that $c > 0$. The difference of Eq. (21) for $\varphi_i$ yields $(\varphi_1(t))^{p}\_\_ = (\varphi_2(t))^{p}\_\_ = (\varphi_1(t) + c)^{p}\_\_ \forall t$. This implies that $\varphi_1(t) < \varphi_2(t) + c \leq 0$ for all $t$. Now Eq. (21) for $\varphi_1(t)$ becomes $\varphi_1^{(4)} = -s + \tilde{h}(t)$. Integrating over one period, we get $s = 0$ because $\tilde{h} \in L^1(S_T)$. This is a contradiction with the first inequality $s > 0$ of (22).

Reasoning as in Example 7, we can give another example.

Example 8. Consider the following superlinear equation:
\begin{equation}
 u^{(4)} = u\_\_ + u^2\_\_ - s + \tilde{h}(t).
\end{equation}
We can use both of the characterizations in (13) for the function $x_{\_\_} + x^2\_\_ \in C(2; 4, 1)$, conditions (14) are now $s > 0$ and $4s + 1 < (M(4))^{1/4}/T$. In order to obtain reasonable conditions, $T$ should be less than $(M(4))^{1/4}$. We conclude that when
\begin{equation}
 0 < T < (M(4))^{1/4}, \quad 0 < s < ((M(4))^2/T^8 - 1)/4,
\end{equation}
Eq. (23) has exactly one $T$-periodic solution for each $\tilde{h} \in L^1(S_T)$. Different from the case for (20) and (22), we have now a restriction on the period $T$ in (24). This additional assumption on the period is quite natural, as in the uniqueness condition derived from the Lipschitz continuous nonlinearities.

For the corresponding second-order equations like (19) and (21), the uniqueness results have been obtained by Ortega and Zhang [6]. Moreover, the conditions like (20) and (22) are optimal for guaranteeing the uniqueness of the $T$-periodic solution when $\tilde{h}$ runs over $L^1(S_T)$. It is then an interesting problem to find the corresponding optimal bounds on $s$ in (20) and (22) for these superlinear beam equations.

As noted by Kunze and Ortega [3], those nonlinearities $f(x) \in C(\sigma; A, B)$ considered in this work have some connection with the classical Rozenblum–Lieb–Cwikel conditions for PDEs [11]. Since $f \in C(\sigma; A, B)$ is, in general, not differentiable, the uniqueness result of this work cannot be obtained by studying linearization equations, as we did in the usual considerations. The ideas of this work can be further extended to some higher-order superlinear equations.

Finally, we remark that the non-degeneracy condition (5) for problem (1) and (2) means that 0 is strictly between the first two eigenvalues of $u^{(4)} = (\lambda + q(t))\_\_u$. Another trivial non-degeneracy condition on (1) and (2) is $q(t) < 0$ for all $t$, because in this case the operator $u^{(4)} = q(t)\_\_u$ is positive in the usual sense and therefore 0 is left to the smallest eigenvalue. The latter can be applied to the uniqueness of solutions of (8) where $f(x)$ is decreasing, as was done in [1] for the second-order differential equations.
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