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Rotation Numbers of Linear Schrödinger Equations with Almost Periodic Potentials and Phase Transmissions

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Abstract. In this paper we study the linear Schrödinger equation with an almost periodic potential and phase transmission. Based on the extended unique ergodic theorem by Johnson and Moser, we will show for such an equation the existence of the rotation number. This extends the work of Johnson and Moser (in Commun Math Phys 84:403–438, 1982; Erratum Commun Math Phys 90:317–318, 1983) where no phase transmission is considered. The continuous dependence of rotation numbers on potentials and transmissions will be proved.

1. Introduction

In this paper we study the following generalized linear Schrödinger equations with almost periodic potentials and phase transmissions:

$$\begin{cases} y'' + Q(t)y = 0 & \text{for } t \in \mathbb{R}\backslash\Gamma \\ \begin{pmatrix} y(t_i+) \\ y'(t_i+) \end{pmatrix} = A_i \begin{pmatrix} y(t_i-) \\ y'(t_i-) \end{pmatrix} & \text{for } t_i \in \Gamma, \end{cases} \tag{1.1}$$

where $Q(t)$ is an almost periodic function, Γ is a periodic lattice, and $A_i \in \mathrm{SL}(\mathbb{R}^2)$ are symplectic matrices, i.e., $A_i^T J A_i = J$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and A_i^T denotes the transpose of A_i. These equations are introduced by Kronig and Penney [9] to describe the quantum effect. A typical spectrum on (1.1) is

$$y'' + \beta \left(\sum_{i\in\mathbb{Z}} \delta(t - 2i\pi) \right) y = \lambda y,$$

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where $\delta(t)$ is the unit Dirac measure located at $t = 0$, and the parameter β describes the tense of quantum effects. This corresponds to Eq. (1.1) with

$$Q(t) \equiv -\lambda, \qquad \Gamma = 2\pi\mathbb{Z}, \qquad A_i = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.$$

In recent years, Niikuni [13,14] has established the existence of the rotation number for Eq. (1.1) when the potential $Q(t)$, the lattice Γ, and the transmissions $\{A_i\}$ are periodic of the same period. By the rotation number approach, the spectrum of (1.1) has been established, extending the works of [6,17]. For explanation to differential equations with general measures, see [11].

The main purpose of this paper is to initiate a study for Eq. (1.1) with almost periodic potentials $Q(t)$, following the classical work [7] where no phase transmission is considered. Though it is desired that Eq. (1.1) will admit a well-defined rotation number for general almost periodic lattice Γ and general almost periodic transmission $\{A_i\}$, as an initial step, we will show this is true for a periodic lattice Γ and a choice of transmissions as rigid rotations

$$A_i = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix},$$

where the sequence $S := \{\alpha_i\}$ is almost periodic. For the corresponding definitions, see Sect. 2.

With such a choice for Eq. (1.1), its dynamics can be described mainly by the argument equation with phase transmissions

$$\begin{cases} \theta' = \cos^2\theta + Q(t)\sin^2\theta & \text{for } t \in \mathbb{R}\backslash\Gamma \\ \theta(t_i+) = \theta(t_i-) + \alpha_i & \text{for } t_i \in \Gamma, \end{cases} \tag{1.2}$$

which results from Eq. (1.1) in the polar coordinates $y = r\sin\theta,\ y' = r\cos\theta$. For definiteness, we always understand solutions $\theta(t)$ of Eq. (1.2) to be right-continuous, i.e., $\theta(t+) = \theta(t)$. Let us use $\theta(t) = \theta(t; Q, \Gamma, S, \vartheta)$, $t \in \mathbb{R}$, to denote the unique solution of Eq. (1.2) satisfying the initial value $\theta(0) = \vartheta \in \mathbb{R}$. The main result of this paper is as follows:

Theorem 1.1. *Let $Q(t)$ and $S = \{\alpha_i\}$ be almost periodic and Γ be periodic. Then for any $\vartheta \in \mathbb{R}$, the following limit*

$$\lim_{t\to+\infty} \frac{\theta(t; Q, \Gamma, S, \vartheta) - \vartheta}{t} \tag{1.3}$$

does exist. Moreover, the limit of (1.3) is independent of the choice of the initial value $\vartheta \in \mathbb{R}$.

The complete proof of Theorem 1.1 is given in Sect. 4. The limit of (1.3) is called the *rotation number* of Eq. (1.2) and is denoted by $\varrho_\Gamma(Q, S)$. It can be used to describe the dynamics of the corresponding Eq. (1.1). For more precise statements of the results, see Theorem 4.2. It can be expected that such a rotation number is also useful in studying the spectrum structure of the operators from Eq. (1.1), as in [7,13,14] for the case of no phase transmission.

When no phase transmission is considered, i.e., $A_i = I$ in (1.1), or $S = \{0\}$ in (1.2), Theorem 1.1 has been established by Johnson and Moser in a classical paper [7]. One extension to the so-called p-Laplacian with almost periodic potentials is given in [4]. For linear Hamiltonian systems of many degrees of freedom, see also Novo et al. [15] for the ergodic representation of rotation numbers. For the extension to random dynamical systems, see [1,10]. However, when transmissions are taken into account, the usual approaches in these works do not work since the arguments are no longer continuous in time $t \in \mathbb{R}$.

Therefore, in Sect. 3, inspired by the classical Poincaré maps for periodic systems, we will construct for Eq. (1.1) a discrete skew-product dynamical system without discontinuity. See formula (3.5). Moreover, limits (1.3) will be reduced some Birkhoff sums of the skew-product dynamical system. See formula (3.13).

In Sect. 4, based on the extended unique ergodic theorem by Johnson and Moser [7], we will show that limits (1.3) do exist and are independent of ϑ. This yields the rotation number of Eq. (1.1). By making use of the ergodic representation of rotation numbers, we will prove that rotation numbers are continuous in potentials and phase transmissions. See Theorem 4.4.

2. Basics on Almost Periodicity

Let us first recall some facts on almost periodicity. For more details, see [5]. We say that a function $Q = Q(t) : \mathbb{R} \to \mathbb{R}$ is *almost periodic*, if Q is continuous and for any $\varepsilon > 0$, the set of the ε-periods

$$\mathcal{T}(Q,\varepsilon) := \{\tau \in \mathbb{R} : |Q(t+\tau) - Q(t)| < \varepsilon \ \forall\, t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$. That is, there exists some $l_\varepsilon > 0$ such that

$$[t, t + l_\varepsilon) \cap \mathcal{T}(Q,\varepsilon) \neq \emptyset \qquad \forall\, t \in \mathbb{R}.$$

Denote by $\mathcal{APF}$ the space of almost periodic functions, endowed with the uniform norm

$$\|Q\|_\infty := \sup_{t\in\mathbb{R}} |Q(t)|, \qquad Q \in \mathcal{APF}.$$

Then we have the following results:

Lemma 2.1. [5] *Any $Q \in \mathcal{APF}$ is bounded and uniformly continuous in $\mathbb{R}$. Moreover, $(\mathcal{APF}, \|\cdot\|_\infty)$ is a Banach space.*

Naturally, one has the following flow in $\mathcal{APF}$ defined by translations

$$\mathcal{APF} \times \mathbb{R} \ni (Q,\tau) \to Q\cdot\tau \in \mathcal{APF}, \qquad Q\cdot\tau := Q(\cdot + \tau).$$

In order to describe the phase transmissions, let us introduce

$$\mathcal{APF}^2 := \{(Q,W) \ : \ (Q,W) : \mathbb{R} \to \mathbb{R}^2 \text{ is such that } Q,\ W \in \mathcal{APF}\},$$

which is equipped with the uniform topology

$$\|(Q,W)\|_\infty := \max\{\|Q\|_\infty, \|W\|_\infty\}.$$

The translations in $\mathcal{APF}^2$ are simply

$$(Q,W) \to (Q\cdot\tau, W\cdot\tau).$$

Given $(Q,W)\in\mathcal{APF}^2$. The *hull* of $(Q,W)\in\mathcal{APF}^2$ is defined by

$$E(Q,W) := \mathrm{closure}_{(\mathcal{APF}^2,\|\cdot\|_\infty)}\{(Q\cdot\tau, W\cdot\tau) : \tau\in\mathbb{R}\}.$$

It is well known that $E(Q,W)$ is compact. For our purpose, we need to introduce the discrete hull as follows:

Definition 2.2. Given $T>0$. The T-hull of $(Q,W)\in\mathcal{APF}^2$ is defined by

$$E_T(Q,W) := \mathrm{closure}_{(\mathcal{APF}^2,\|\cdot\|_\infty)}\{(Q\cdot nT, W\cdot nT) : n\in\mathbb{Z}\}.$$

Then $E_T(Q,W)\subset E(Q,W)$ is also compact in $(\mathcal{APF}^2,\|\cdot\|_\infty)$. Like the case of continuous time, we can equip $E_T(Q,W)$ with a group structure by

$$(Q_1,W_1)\cdot(Q_2,W_2) := \lim_{m\to\infty}(Q(\cdot+n_m^1T+n_m^2T), W(\cdot+n_m^1T+n_m^2T)), \tag{2.1}$$

whenever

$$(Q_i,W_i) = \lim_{m\to\infty}(Q(\cdot+n_m^iT), W(\cdot+n_m^iT))\in E_T(Q,W), \qquad i=1,2. \tag{2.2}$$

Note that the definition of product of $E_T(Q,W)$ does make sense. That is, the limit of (2.1) does exist and is independent of the choice of times n_m^i so long as (2.2) hold. Thus, $E_T(Q,W)$ with the metric induced by $\|\cdot\|_\infty$ is a compact Abelian topological group.

In the space $\mathcal{APF}^2$, the following time-T translation is continuous:

$$\varphi_T(Q,W) := (Q\cdot T, W\cdot T), \quad (Q,W)\in\mathcal{APF}^2. \tag{2.3}$$

By the definition of T-hulls, one sees that $E_T(Q,W)$ is φ_T-invariant. We have the following results:

Lemma 2.3. *Given $(Q,W)\in\mathcal{APF}^2$.*

(i) *The transformation φ_T is minimal on $E_T(Q,W)$. That is, $E_T(q,w) = E_T(Q,W)$ for any $(q,w)\in E_T(Q,W)$.*

(ii) *The transformation φ_T on $E_T(Q,W)$ is uniquely ergodic with the unique invariant Borel probability measure being the Haar measure ν of the compact Abelian group $E_T(Q,W)$.*

Proof. (i) Let $(q,w)\in E_T(Q,W)$. Since $E_T(Q,W)$ is φ_T-invariant, we have

$$(q\cdot nT, w\cdot nT)\in E_T(Q,W) \qquad \forall\, n\in\mathbb{Z}.$$

By the compactness of $E_T(Q,W)$, we have

$$E_T(q,w)\subseteq E_T(Q,W).$$

On the other hand, suppose that

$$(Q\cdot n_mT, W\cdot n_mT)\to(q,w)\in E_T(Q,W),$$

where $n_m \in \mathbb{Z}$. Then by reversing times, we have

$$(q \cdot (-n_m T), w \cdot (-n_m T)) \to (Q, W).$$

Thus $(Q, W) \in E_T(q, w)$. One has then

$$E_T(Q, W) \subseteq E_T(q, w).$$

(ii) By (2.1) and (2.3), we have

$$\varphi_T(q, w) = (Q \cdot T, W \cdot T) \cdot (q, w) \qquad \forall\ (q, w) \in E_T(Q, W).$$

Thus, φ_T is a rotation on the group $E_T(Q, W)$. Then combining (i) with [16, Theorem 6.20], we can obtain the desired result. $\square$

We say that a sequence $S = \{\alpha_i\}_{i \in \mathbb{Z}} \subset \mathbb{R}$ is *almost periodic*, if for any $\varepsilon > 0$, the set

$$\mathcal{T}(S, \varepsilon) := \{k \in \mathbb{Z} : |\alpha_{i+k} - \alpha_i| < \varepsilon\ \forall i \in \mathbb{Z}\} \tag{2.4}$$

is relatively dense (in $\mathbb{Z}$) in the sense that there exists some $k_\varepsilon \in \mathbb{N}$ such that

$$\mathcal{T}(S, \varepsilon) \cap [i, i + k_\varepsilon) \neq \emptyset \qquad \forall\, i \in \mathbb{Z}.$$

Such a k_ε is called an ε-period of the sequence S. The set of all almost periodic sequences is denoted by $\mathcal{APS}$.

By a *lattice* Γ, it means that $\Gamma = \{\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots\} \subset \mathbb{R}$ is a set of discrete points. We say that Γ is *periodic* if there exist $T > 0$ and $n \in \mathbb{N}$ such that

$$t_{i+n} = t_i + T \qquad \forall\, i \in \mathbb{Z}. \tag{2.5}$$

The numbers T and n are called the *period* and the *length* of the periodic lattice Γ, respectively. In the following, we always assume that (T, n) in (2.5) is minimal. In this case, Γ is called a (T, n)-periodic lattice. The set of all (T, n)-lattices is denoted by $\mathcal{PL}_{T,n}$. Note that any $\Gamma \in \mathcal{PL}_{T,n}$ can be represented as

$$\Gamma = \{t_0, t_1, \ldots, t_{n-1}\} + T\mathbb{Z}, \qquad t_0 < t_1 < \cdots < t_{n-1}\ (< t_0 + T).$$

Lemma 2.4. *Given $\Gamma \in \mathcal{PL}_{T,n}$.*

(i) *Let $W \in \mathcal{APF}$. Then, the sequence $\{W(t_i) : t_i \in \Gamma\} \in \mathcal{APS}$.*
(ii) *Conversely, let $S = \{\alpha_i\}_{i \in \mathbb{Z}} \in \mathcal{APS}$. There exists $W \in \mathcal{APF}$ such that $W(t_i) = \alpha_i$ for all $t_i \in \Gamma$.*

Proof. (i) This follows from the definitions of two kinds of almost periodicity. For the detailed proof, one can see [5, p. 163].

(ii) Since $S \in \mathcal{APS}$, we know that

$$S_j := \{\beta_i := \alpha_{in+j}\}_{i \in \mathbb{Z}} \in \mathcal{APS}, \qquad j = 0, 1, \ldots, n-1. \tag{2.6}$$

In fact, for any $\varepsilon > 0$, if $k \in \mathcal{T}(S, \varepsilon/n)$, then for any $i \in \mathbb{Z}$, by (2.4) we have

$$|\alpha_{i+k} - \alpha_i| < \varepsilon/n.$$

Furthermore,

$$\begin{aligned}|\beta_{i+k} - \beta_i| &= |\alpha_{in+kn+j} - \alpha_{in+j}| \\ &\le |\alpha_{in+kn+j} - \alpha_{in+k(n-1)+j}| + \cdots + |\alpha_{in+k+j} - \alpha_{in+j}| \\ &< \varepsilon.\end{aligned}$$

This implies that $\mathcal{T}(S, \varepsilon/n) \subset \mathcal{T}(S_j, \varepsilon)$. Thus, $\mathcal{T}(S_j, \varepsilon)$ is also a relatively dense set.

Define auxiliary functions for $j = 0, 1, \ldots, n-1$,

$$W_j(t) = \begin{cases} \dfrac{\beta_i}{t_{in+j} - t_{in+j-1}}(t - t_{in+j-1}) & \text{when } t \in [t_{in+j-1}, t_{in+j}], \\ \dfrac{-\beta_i}{t_{in+j+1} - t_{in+j}}(t - t_{in+j}) & \text{when } t \in [t_{in+j}, t_{in+j+1}], \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where i runs over $\mathbb{Z}$, we know that $W_j(t)$ is almost periodic. In fact,it follows from (2.5) and (2.7) that

$$\|W_j \cdot kT - W_j\|_\infty \le \sup_{i \in \mathbb{Z}} |\beta_{i+k} - \beta_i|.$$

Combining this with (2.6), we conclude that

$$\{kT : k \in \mathcal{T}(S_j, \varepsilon)\} \subset \mathcal{T}(W_j, \varepsilon),$$

completing the assertion. Moreover, we observe that

$$W_j(t_{in+j}) = \beta_i = \alpha_{in+j}, \quad W_j(t_k) = 0 \ \text{ for } k \ne in + j.$$

Lemma 2.1 implies that $W(t) := \sum_{j=0}^{n-1} W_j(t)$ is a desired representation for (Γ, S). □

3. Reduction to Discrete Skew-Product Flows

Given $\Gamma \in \mathcal{PL}_{T,n}$, $S \in \mathcal{APS}$ and $Q \in \mathcal{APF}$. By Lemma 2.4, we can rewrite Eq. (1.2) as

$$\begin{cases} \theta' = \cos^2\theta + Q(t)\sin^2\theta & \text{for } t \in \mathbb{R}\backslash\Gamma \\ \theta(t_i) = \theta(t_i-) + W(t) & \text{for } t_i \in \Gamma, \end{cases} \tag{3.1}$$

where $W(t) = W_{\Gamma,S}(t)$ is determined by the periodic lattice Γ and the almost periodic sequence S.

From now on we fix a $\Gamma \in \mathcal{PL}_{T,n}$. Without loss of generality, we assume that $t_0 = 0$. Denote solutions of Eq. (3.1) by $\theta(t) = \theta(t; Q, \Gamma, W, \vartheta) = \theta_\Gamma(t; Q, W, \vartheta)$.

Lemma 3.1. *Given $\vartheta \in \mathbb{R}$. There holds the following relation*:

$$\lim_{t\to+\infty} \frac{\theta_\Gamma(t; Q, W, \vartheta) - \vartheta}{t} = \lim_{k\to+\infty} \frac{\theta_\Gamma(kT; Q, W, \vartheta) - \vartheta}{kT}. \tag{3.2}$$

That is, if one of the limits exists, then another exists as well and they are equal.

Proof. Let $t \in [kT, (k+1)T),\ k \in \mathbb{N}$. Then $t_{kn+j} \le t < t_{kn+j+1}$ for some $j,\ 0 \le j \le n-1$. Therefore,

$$\begin{aligned}
&|\theta(t) - \theta(kT)| \\
&= \left| \left\{ \int_{kT}^{kT+t_1} + \cdots + \int_{kT+t_{j-1}}^{kT+t_j} + \int_{kT+t_j}^{t} \right\} (\cos^2\theta(s) + Q(s)\sin^2\theta(s))\,\mathrm{d}s \right. \\
&\qquad \left. + \sum_{i=kn+1}^{kn+j} W(t_i) \right| \qquad \text{(by (2.5) and (3.1))} \\
&\le T(1 + \|Q\|_\infty) + n\|W\|_\infty.
\end{aligned}$$

It implies that

$$\lim_{t\to+\infty} \frac{\theta(t) - \vartheta}{t} = \lim_{k\to+\infty} \frac{kT}{t} \frac{\theta(kT) - \vartheta}{kT} = \lim_{k\to+\infty} \frac{\theta(kT) - \vartheta}{kT}.$$

The proof is complete. □

In order to prove the existence of these limits, we will embed Eq. (3.1) into the following family of equations:

$$\begin{cases} \theta' = \cos^2\theta + q(t)\sin^2\theta & \text{for } t \in \mathbb{R}\backslash\Gamma \\ \theta(t_i+) = \theta(t_i-) + w(t) & \text{for } t_i \in \Gamma, \end{cases} \tag{3.3}$$

where (q, w) runs over the hull $E_T(Q, W)$, as it did in [7].

Due to the 2π-periodic in θ of the vector field of (3.3) and the uniqueness of solutions of ODEs, we have the following result:

Lemma 3.2. *For any $(q, w) \in E_T(Q, W),\ \vartheta \in \mathbb{R},\ t \in \mathbb{R},\ k \in \mathbb{Z}$, there holds*

$$\theta_\Gamma(t; q, w, \vartheta + 2k\pi) - (\vartheta + 2k\pi) = \theta_\Gamma(t; q, w, \vartheta) - \vartheta. \tag{3.4}$$

Let us introduce the product space $Z := E_T(Q, W) \times \mathbb{S}_{2\pi}$ with the distance

$$d((q_1, w_1, \vartheta_1), (q_2, w_2, \vartheta_2)) := \max\{\|q_1 - q_2\|_\infty, \|w_1 - w_2\|_\infty, |\vartheta_1 - \vartheta_2|_{\mathbb{S}_{2\pi}}\}.$$

Note that Eq. (3.3) can yield a 'dynamical system' on $E_T(Q, W) \times \mathbb{S}_{2\pi}$, although $\theta(t)$ has discontinuity in time t because of the transmissions on the lattice.

In order to overcome this discontinuity, by noticing that Γ is a periodic lattice, we can introduce $\Phi_\Gamma := \{\Phi_\Gamma^i\}_{i\in\mathbb{Z}}$ by

$$\Phi_\Gamma^i(q, w, \vartheta) := (\varphi_T^i(q, w), \theta_\Gamma(iT; q, w, \vartheta)), \quad (q, w, \vartheta) \in Z,\ i \in \mathbb{Z}, \tag{3.5}$$

where φ_T is defined by (2.3), and both ϑ and $\theta_\Gamma(iT; q, w, \vartheta)$ are taken modulo 2π. Due to (3.4), it is well defined. In some sense, $\{\Phi_\Gamma^i\}$ is the Poincaré map of (3.3) (of the period T). Some observations on $\{\Phi_\Gamma^i\}$ are as follows:

Lemma 3.3. (i) *For any given $i \in \mathbb{Z}$, $\theta_\Gamma(iT; q, w, \vartheta)$ is Lipschitz continuous on Z.*

(ii) *For any given $i \in \mathbb{Z}$, $\Phi_\Gamma^i(q, w, \vartheta)$ is continuous on Z. Moreover,*

$$\Phi_\Gamma^{n_1+n_2} = \Phi_\Gamma^{n_1} \circ \Phi_\Gamma^{n_2} \quad \textit{for } n_1,\ n_2 \in \mathbb{Z}.$$

Proof. (i) Without loss of generality, we can assume that $i = 1$ and $n = 2$. That is,

$$\Gamma = \{0, t_1\} + T\mathbb{Z}, \qquad 0 < t_1 < T.$$

Let $\theta_i(t) = \theta_\Gamma(t; q_i, w_i, \vartheta_i)$, $i = 1, 2$, be two solutions of (3.3). Due to (3.4), we only consider the case $0 \le \vartheta_2 - \vartheta_1 \le \pi$. Then, $|\vartheta_2 - \vartheta_1| = |\vartheta_2 - \vartheta_1|_{\mathbb{S}_{2\pi}}$. For $t \in [0, t_1)$, we have

$$\begin{cases} \theta_i'(t) = \cos^2\theta_i(t) + q_i(t)\sin^2\theta_i(t), & t \in [0, t_1) \\ \theta_i(0) = \vartheta_i. \end{cases}$$

Denote $D(t) := \theta_2(t) - \theta_1(t)$. Then we have

$$\begin{aligned} D'(t) &= \cos^2\theta_2(t) + q_2(t)\sin^2\theta_2(t) - (\cos^2\theta_1(t) + q_1(t)\sin^2\theta_1(t)) \\ &= (q_2(t) - q_1(t))\sin^2\theta_2(t) + (\cos\theta_2(t) - \cos\theta_1(t))(\cos\theta_2(t) + \cos\theta_1(t)) \\ &\quad + q_1(t)(\sin\theta_2(t) - \sin\theta_1(t))(\sin\theta_2(t) + \sin\theta_1(t)). \end{aligned}$$

One has some $\zeta(t)$, $\eta(t)$ which are between $\theta_1(t)$ and $\theta_2(t)$ and satisfy

$$\cos\theta_2(t) - \cos\theta_1(t) = -D(t)\sin\zeta(t), \qquad \sin\theta_2(t) - \sin\theta_1(t) = D(t)\cos\eta(t).$$

Thus,

$$\begin{aligned} D'(t) &= (q_1(t)(\sin\theta_2(t) + \sin\theta_1(t))\cos\eta(t) - (\cos\theta_2(t) + \cos\theta_1(t)) \\ &\quad \times \sin\zeta(t)D(t) + (q_2(t) - q_1(t))\sin^2\theta_2(t) \\ &:= A(t)D(t) + B(t). \end{aligned} \tag{3.6}$$

Note that $(q_1, w_1) \in E_T(Q, W)$ and $\|q_1\|_\infty = \|Q\|_\infty$. Therefore,

$$\|A\|_\infty \le 2\|Q\|_\infty + 2 =: M \qquad \|B\|_\infty \le \|q_1 - q_2\|_\infty. \tag{3.7}$$

The solution of (3.6) with the initial value $D(0) = \vartheta_2 - \vartheta_1$ is

$$D(t) = \exp\left(\int_0^t A(s)\,\mathrm{d}s\right)\left(\vartheta_2 - \vartheta_1 + \int_0^t B(s)e^{-\int_0^s A(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right), \quad t \in [0, t_1).$$

Then for any $t \in [0, t_1)$, we have

$$\begin{aligned} |D(t)| &\le \exp\left(\int_0^t |A(s)|\,\mathrm{d}s\right)\left(|\vartheta_2 - \vartheta_1| + \left|\int_0^t B(s)e^{-\int_0^s A(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right|\right) \\ &\le e^{Mt_1}\left(|\vartheta_2 - \vartheta_1| + \|q_1 - q_2\|_\infty \int_0^{t_1} e^{Ms}\,\mathrm{d}s\right) \le C_1 d, \end{aligned} \tag{3.8}$$

where (3.7) is used and

$$d := d((q_1, w_1, \vartheta_1), (q_2, w_2, \vartheta_2)). \tag{3.9}$$

By letting $t \to t_1-$, we conclude that

$$|\theta_2(t_1-) - \theta_1(t_1-)| \le C_1 d. \tag{3.10}$$

Now we consider $t \in [t_1, T)$. Then the solutions $\theta_i(t)$ satisfy

$$\begin{cases} \theta_i'(t) = \cos^2\theta_i(t) + q_i(t)\sin^2\theta_i(t), & t \in [t_1, T), \\ \theta_i(t_1) = \theta_i(t_1-) + w_i(t_1). \end{cases}$$

Again by the same argument, we consider $D(t)$ on $[t_1, T)$, and similarly as (3.8), we have

$$\begin{aligned} |D(t)| &\le e^{M(T-t_1)} \\ &\quad\times\left(|\theta_2(t_1-) + w_2(t_1) - \theta_1(t_1-) - w_1(t_1)| + \|q_1 - q_2\|_\infty \int_{t_1}^{T} e^{M(s-t_1)}\,\mathrm{d}s \right) \\ &\le e^{M(T-t_1)} \\ &\quad\times\left(|\theta_2(t_1-) - \theta_1(t_1-)| + \|w_1 - w_2\|_\infty + \|q_1 - q_2\|_\infty \int_{t_1}^{T} e^{M(s-t_1)}\,\mathrm{d}s \right) \\ &\le C_2 d. \qquad \text{(by (3.9) and (3.10))} \end{aligned}$$

Then by letting $t \to T-$, we have

$$|\theta_2(T-) - \theta_1(T-)| = |D(T-)| \le C_2 d$$

for some C_2. Finally,

$$\begin{aligned} &|\theta_\Gamma(T; q_2, w_2, \vartheta_2) - \theta_\Gamma(T; q_1, w_1, \vartheta_1)|_{\mathbb{S}_{2\pi}} \\ &\quad = |D(T-) + w_2(T) - w_1(T)|_{\mathbb{S}_{2\pi}} \le (C_2 + 1)d, \end{aligned}$$

proving the Lipschitz continuity.

(ii) We need only to prove that

$$\theta_\Gamma(n_1T; q\cdot n_2T, w\cdot n_2T, \theta_\Gamma(n_2T; q, w, \vartheta)) \equiv \theta_\Gamma(n_1T + n_2T; q, w, \vartheta). \tag{3.11}$$

Let us denote

$$\theta_1(t) := \theta_\Gamma(t + n_2T; q, w, \vartheta), \qquad \theta_2(t) := \theta_\Gamma(t; q\cdot n_2T, w\cdot n_2T, \theta_\Gamma(n_2T; q, w, \vartheta)).$$

Note that Γ is T-periodic. Then both $\theta_i(t)$ satisfy the following function:

$$\begin{cases} \theta' = \cos^2\theta + q(t + n_2T)\sin^2\theta, & t \in \mathbb{R}\backslash\Gamma \\ \theta(t) = \theta(t-) + w(t + n_2T), & t \in \Gamma. \end{cases}$$

Since $\theta_1(0) = \theta_2(0) = \theta_\Gamma(n_2T; q, w, \vartheta)$, we conclude that $\theta_1(t) \equiv \theta_2(t)$. By setting $t = n_1T$, we have the desired equality. Combining (i) with the definition (3.5), we have the continuity of $\Phi_\Gamma^i(q, w, \vartheta)$. □

Lemma 3.3 (ii) shows that Φ_Γ is a discrete skew-product flow on Z with the base flow $\{\varphi_T^i\}$. Let us introduce the following observation function:

$$F_\Gamma(q, w, \vartheta) := \frac{\theta_\Gamma(T; q, w, \vartheta) - \vartheta}{T}, \tag{3.12}$$

which can be considered as a function on $E_T(Q, W) \times \mathbb{S}_{2\pi}$. It follows from Lemma 3.3 (i) that $F_\Gamma(q, w, \vartheta)$ is Lipschitz continuous on Z.

Lemma 3.4. *There holds the following relation*:

$$\lim_{t\to+\infty}\frac{\theta_\Gamma(t;q,w,\vartheta)-\vartheta}{t}=\lim_{k\to+\infty}\frac{1}{k}\sum_{i=0}^{k-1}F_\Gamma\circ\Phi_\Gamma^i(q,w,\vartheta). \tag{3.13}$$

That is, if one of the limits exists, then another exists as well and they are equal.

Proof. For any $k\in\mathbb{N}$, we have

$$\begin{aligned}
&\frac{\theta_\Gamma(kT;q,w,\vartheta)-\vartheta}{kT}\\
&=\frac{1}{k}\sum_{i=0}^{k-1}\frac{\theta_\Gamma((i+1)T;q,w,\vartheta)-\theta_\Gamma(iT;q,w,\vartheta)}{T}\\
&=\frac{1}{k}\sum_{i=0}^{k-1}\frac{\theta_\Gamma(T;q\cdot iT,w\cdot iT,\theta_\Gamma(iT;q,w,\vartheta))-\theta_\Gamma(iT;q,w,\vartheta)}{T} \qquad \text{(by (3.11))}\\
&=\frac{1}{k}\sum_{i=0}^{k-1}F_\Gamma\circ\Phi_\Gamma^i(q,w,\vartheta) \qquad \text{(by (3.5), (3.4) and (3.12))}.
\end{aligned}$$

Now result (3.13) follows from (3.2). □

4. Proof of the Main Result

Define

$$F_\Gamma^*(q,w,\vartheta):=\lim_{k\to+\infty}\frac{1}{k}\sum_{i=0}^{k-1}F_\Gamma\circ\Phi_\Gamma^i(q,w,\vartheta) \tag{4.1}$$

whenever the limit exists. Due to uniqueness of solutions of ODEs, we know that

$$\theta_\Gamma(t;q,w,\vartheta_0)<\theta_\Gamma(t;q,w,\vartheta)<\theta_\Gamma(t;q,w,\vartheta_0+2\pi)\quad\text{when } \vartheta_0<\vartheta<\vartheta_0+2\pi.$$

Thus, by (3.13) and (4.1), if $F_\Gamma^*(q,w,\vartheta_0)$ exists for some ϑ_0, then $F_\Gamma^*(q,w,\vartheta)$ exists for any $\vartheta\in\mathbb{S}_{2\pi}$ and it is independent of ϑ.

The extension of the unique ergodic theorem in [7] will be used to establish the existence of rotation numbers. For our purpose, we choose the discrete case.

Theorem 4.1. [7] *Let $\{\varphi^i\}_{i\in\mathbb{Z}}$ be a continuous dynamical system of discrete time on a compact metric space X. Then, for any $f\in C(X):=C(X,\mathbb{R})$ satisfying*

$$\int_X f\,\mathrm{d}\mu=0$$

for all invariant Borel probability measures μ *of* $\{\varphi^i\}$*, one has*

$$\lim_{k\to+\infty}\frac{1}{k}\sum_{i=0}^{k-1} f\circ\varphi^i(x)=0$$

uniformly in $x\in X$*.*

Now we concentrate on the ergodic limits (4.1). For simplicity, denote $E := E_T(Q,W)$ and $Z = E_T(Q,W)\times\mathbb{S}_{2\pi}$. The proof is along the line of [7]. For more details, see also [4]. Due to Lemma 3.3 (ii), $\{\Phi_\Gamma^i\}$ is a continuous dynamical system of discrete time which admits invariant Borel probability measures μ, see [16, Corollary 6.9.1] or [8,12].

Theorem 4.2. *Given* $(\Gamma,Q,W)\in\mathcal{PL}_{T,n}\times\mathcal{APF}^2$*. We assert that*

$$\varrho_\Gamma=\varrho_\Gamma(Q,W):=\int_Z F_\Gamma\,\mathrm{d}\mu \tag{4.2}$$

is independent of invariant Borel probability measures μ *of* $\{\Phi_\Gamma^i\}$*. Moreover, one has the convergence*

$$\lim_{k\to+\infty}\frac{1}{k}\sum_{i=0}^{k-1}F_\Gamma\circ\Phi_\Gamma^i(q,w,\vartheta)=\varrho_\Gamma, \tag{4.3}$$

which is uniform in the whole space $(q,w,\vartheta)\in Z$*.*

Proof. Let μ be any invariant Borel probability measure of $\{\Phi_\Gamma^i\}$. By the Birkhoff ergodic theorem, there exists a Borel set $Z_\mu\subset Z$, depending on the measure μ, such that $\mu(Z_\mu)=1$ and the limits $F_\Gamma^*(q,w,\vartheta)$ of (4.1) exist for all $(q,w,\vartheta)\in Z_\mu$. Furthermore, $F_\Gamma^*\in L^1(Z,\mu)$ is integrable and satisfies

$$\int_Z F_\Gamma^*\,\mathrm{d}\mu=\int_Z F_\Gamma\,\mathrm{d}\mu=:\varrho_{\Gamma,\mu}. \tag{4.4}$$

Due to the independence of $F_\Gamma^*(q,w,\vartheta)$ of ϑ, Z_μ can be taken as the form $Z_\mu=E_\mu\times\mathbb{S}_{2\pi}$, where E_μ is a Borel set of E.

Assertion 1. $\nu(E_\mu)=1$ where ν is the Haar measure on $E=E_T(Q,W)$.

To this end, denote the projection $Z=E\times\mathbb{S}_{2\pi}\to E$ by π_1. Then the push $(\pi_1)_*\mu=\mu\circ\pi_1^{-1}$ of μ on Z to E is an invariant Borel probability measure of φ_T. Hence, $(\pi_1)_*\mu$ must be the Haar measure ν on E. Thus, $\nu(E_\mu)=\mu(\pi_1^{-1}(E_\mu))=\mu(E_\mu\times\mathbb{S}_{2\pi})=1$.

Assertion 2. There exists $\hat{E}_\mu\subset E_\mu$ such that $\nu(\hat{E}_\mu)=1$ and

$$F_\Gamma^*(q,w,\vartheta)\equiv\varrho_{\Gamma,\mu},\qquad (q,w,\vartheta)\in\hat{E}_\mu\times\mathbb{S}_{2\pi}. \tag{4.5}$$

Since $F_\Gamma^*(q,w,\vartheta)=F_\Gamma^*(q,w)$ is independent of $\vartheta\in\mathbb{S}_{2\pi}$ for $(q,w)\in E_\mu$, $F_\Gamma^*(q,w)$ can be considered as an integrable function in $L^1(E,\nu)$ which is φ_T-invariant. As ν is the unique ergodic measure of φ_T, one has some $\hat{E}_\mu\subset E_\mu$ such that $\nu(\hat{E}_\mu)=1$ and $F_\Gamma^*(q,w)$ is constant on $\hat{E}_\mu$. By (4.4), this constant must be $\varrho_{\Gamma,\mu}$. This gives (4.5).

These two assertions show that (4.3) has the constant limit on $\hat{E}_\mu \times \mathbb{S}_{2\pi}$. We need to extend this result to the whole space Z.

(i) Let us prove that the constant $\varrho_{\Gamma,\mu}$ of (4.4) is independent of invariant measures μ of Φ_Γ. In fact, for another measure μ', arguing as above, one has the corresponding objects $\hat{E}_{\mu'}$, $\varrho_{\Gamma,\mu'}$. Since $\nu(\hat{E}_\mu) = \nu(\hat{E}_{\mu'}) = 1$, $\hat{E}_\mu \cap \hat{E}_{\mu'} \neq \emptyset$. Taking a point $(q, w) \in \hat{E}_\mu \cap \hat{E}_{\mu'}$, we have

$$\varrho_{\Gamma,\mu} = \lim_{k\to+\infty} \frac{1}{k} \sum_{i=0}^{k-1} F_\Gamma \circ \Phi_\Gamma^i(q, w, \vartheta) = \varrho_{\Gamma,\mu'} \qquad \forall\, \vartheta \in \mathbb{S}_{2\pi}.$$

See (4.1) and (4.5). Thus, $\varrho_{\Gamma,\mu}$ is independent of μ, which is denoted by ϱ_Γ as in (4.3).

(ii) Let us consider the function $f := F_\Gamma - \varrho_\Gamma$. By Lemma 3.3 (i), f is continuous on Z. From (4.4), f fulfills the requirement of Theorem 4.1. Thus, as $k \to +\infty$,

$$\frac{1}{k} \sum_{i=0}^{k-1} f \circ \Phi_\Gamma^i(q, w, \vartheta) = \frac{1}{k} \sum_{i=0}^{k-1} F_\Gamma \circ \Phi_\Gamma^i(q, w, \vartheta) - \varrho_\Gamma \to 0$$

uniformly in $(q, w, \vartheta) \in Z$. This gives (4.3). □

Definition 4.3. Given $(\Gamma, Q, W) \in \mathcal{PL}_{T,n} \times \mathcal{APF}^2$. We call the number $\varrho_\Gamma = \varrho_\Gamma(Q, W)$ of (4.2) the *rotation number* of (3.1) or that of (1.1).

Proof of Theorem 1.1. In the present terminology, the solutions $\theta(t; Q, \Gamma, S, \vartheta)$ of (1.3) are the same as $\theta_\Gamma(t; Q, W, \vartheta)$. Now the existence of the limit of (1.3) follows simply from (3.13), (4.2) and (4.3). In fact, one has

$$\lim_{t\to+\infty} \frac{\theta_\Gamma(t; q, w, \vartheta) - \vartheta}{t} = \varrho_\Gamma(Q, W) \tag{4.6}$$

uniformly in $(q, w, \vartheta) \in E_T(Q, W) \times \mathbb{S}_{2\pi}$. That is, the rotation number $\varrho_\Gamma(Q, W)$ describes the dynamics for not only Eq. (3.1), but also the family of Eqs. (3.3). □

Note that for any $(q, w) \in E_T(Q, W)$, one has $(\Gamma, q, w) \in \mathcal{PL}_{T,n} \times \mathcal{APF}^2$ with the same lattice Γ. The rotation number $\varrho_\Gamma(q, w)$ of Eq. (3.3) is defined as well. By (4.6), one has actually

$$\varrho_\Gamma(q, w) = \varrho_\Gamma(Q, W) \qquad \forall (q, w) \in E_T(Q, W).$$

In fact, one has the ergodic representation (4.2) for rotation number $\varrho_\Gamma(Q, W)$.

Given a periodic lattice $\Gamma \in \mathcal{PL}_{T,n}$. Let us consider rotation number $\varrho_\Gamma(Q, W)$ as a nonlinear functional of $(Q, W) \in (\mathcal{APF}^2, \|\cdot\|_\infty)$. We will show that $\varrho_\Gamma(Q, W)$ is continuous in (Q, W). Note that in the ergodic representation (4.2), even when $\Gamma \in \mathcal{PL}_{T,n}$ is fixed, all of these objects depend on the pair $(Q, W) \in \mathcal{APF}^2$. These include the space Z, the dynamical system $\{\Phi_\Gamma^i\}$ and its invariant measure μ, and the observation F_Γ. In the following, we will apply the monotonicity of rotation numbers to give a considerable reduction for the continuity problem.

Theorem 4.4. *Given a periodic lattice $\Gamma \in \mathcal{PL}_{T,n}$. Then rotation number $\varrho_\Gamma(Q,W)$ is continuous in $(Q,W) \in \mathcal{APF}^2$ with the uniform topology $\|\cdot\|_\infty$.*

Proof. At first, let us prove the following monotonicity for rotation numbers: Suppose that $Q_0 \le Q_1$ and $W_0 \le W_1$. By the comparison theorem of ODEs with respect to the vector field, one has $\theta_\Gamma(t;Q_0,W_0,\vartheta) \le \theta_\Gamma(t;Q_1,W_1,\vartheta)$ for all $t \ge 0$ and therefore,

$$\varrho_\Gamma(Q_0,W_0) \le \varrho_\Gamma(Q_1,W_1) \qquad \text{when} \;\; Q_0 \le Q_1,\; W_0 \le W_1.$$

In case $\|(Q_1,W_1)-(Q_0,W_0)\|_\infty = \max\{\|Q_1-Q_0\|_\infty, \|W_1-W_0\|_\infty\} < \delta$, we have

$$Q_0 - \delta < Q_1 < Q_0 + \delta, \qquad W_0 - \delta < W_1 < W_0 + \delta.$$

Hence,

$$\varrho_\Gamma(Q_0-\delta,W_0-\delta) \le \varrho_\Gamma(Q_1,W_1) \le \varrho_\Gamma(Q_0+\delta,W_0+\delta).$$

In order to show the continuity of $\varrho_\Gamma(Q,W)$ in (Q,W), it suffices to prove that for any fixed $(Q,W) \in \mathcal{APF}^2$, $\varrho_\Gamma(Q+\delta,W+\delta)$ is continuous in δ at $\delta = 0$. More generally, for any sequence $\delta_k \to \delta_0 \in \mathbb{R}$, by denoting

$$Q_k := Q + \delta_k, \qquad W_k := W + \delta_k, \qquad k \in \mathbb{N} \cup \{0\}, \tag{4.7}$$

we will show that

$$\varrho_\Gamma(Q_k,W_k) \to \varrho_\Gamma(Q_0,W_0) \qquad \text{as } k \to +\infty. \tag{4.8}$$

For $k \in \mathbb{N} \cup \{0\}$, denote

$$E_k = E_T(Q_k,W_k), \qquad Z_k = E_k \times \mathbb{S}_{2\pi}.$$

Note that E_k and Z_k depend on k. However, by denoting

$$E := E_T(Q,W), \qquad Z = E \times \mathbb{S}_{2\pi},$$

we know from (4.7) that

$$E_k = E + \delta_k(1,1) \subset \mathcal{APF}^2$$

are translations of E. Accordingly, one can introduce homeomorphisms

$$\sigma_k : Z = E \times \mathbb{S}_{2\pi} \to Z_k = (E + \delta_k(1,1)) \times \mathbb{S}_{2\pi}$$

by

$$\sigma_k(q,w,\vartheta) = (q+\delta_k, w+\delta_k, \vartheta).$$

Now, the flows $\Phi_\Gamma = \{\Phi_\Gamma^i\}$ on Z_k induce

$$\Phi_{\Gamma,k}^i(q,w,\vartheta) := \sigma_k^{-1} \circ \Phi_\Gamma^i \circ \sigma_k(q,w,\vartheta), \qquad i \in \mathbb{Z},\; (q,w,\vartheta) \in Z.$$

One sees that these $\Phi_{\Gamma,k} := \{\Phi_{\Gamma,k}^i\}$ are flows on the same space Z. Explicitly, one has

$$\Phi_{\Gamma,k}^i(q,w,\vartheta) = (q \cdot iT, w \cdot iT, \theta_\Gamma(iT; q+\delta_k, w+\delta_k, \vartheta)) \tag{4.9}$$

for $i \in \mathbb{Z}$ and $(q,w,\vartheta) \in Z$. Similarly, the corresponding observation functions F_Γ on Z_k are transformed into

$$F_{\Gamma,k}(q,w,\vartheta) := F_\Gamma \circ \sigma_k(q,w,\vartheta), \qquad (q,w,\vartheta) \in Z.$$

By (3.12), one has

$$F_{\Gamma,k}(q,w,\vartheta) = \frac{\theta_\Gamma(T;q+\delta_k,w+\delta_k,\vartheta)-\vartheta}{T}, \qquad (q,w,\vartheta)\in Z. \quad (4.10)$$

For any $k \in \mathbb{N}\cup\{0\}$, let $\hat\mu_k$ be any Φ_Γ-invariant probability measure on Z_k. Then, for each $k\in\mathbb{N}$, $\mu_k := \hat\mu_k \circ \sigma_k$ is a $\Phi_{\Gamma,k}$-invariant measure on Z. Now the ergodic representation (4.2) for $\varrho_\Gamma(Q_k,W_k)$ can be rewritten as

$$\varrho_\Gamma(Q_k,W_k) = \int\limits_{Z_k} F_\Gamma \,\mathrm{d}\hat\mu_k = \int\limits_{Z} F_{\Gamma,k}\,\mathrm{d}\mu_k. \quad (4.11)$$

The last representations are now on the same space Z.

According to [16, Theorem 6.5], we may assume that $\mu_k \rightharpoonup \mu_\infty$ in the weak topology for some measure μ_∞. That is,

$$\int\limits_Z g\,\mathrm{d}\mu_k \to \int\limits_Z g\,\mathrm{d}\mu_\infty \qquad \forall\, g\in C(Z)=C(Z,\mathbb{R}).$$

We have actually

$$g_k \to g_\infty \text{ in } (C(Z),\|\cdot\|_\infty) \Longrightarrow \int\limits_Z g_k\,\mathrm{d}\mu_k \to \int\limits_Z g_\infty\,\mathrm{d}\mu_\infty, \quad (4.12)$$

because

$$\begin{aligned}\int\limits_Z g_k\,\mathrm{d}\mu_k - \int\limits_Z g_\infty\,\mathrm{d}\mu_\infty &= \int\limits_Z (g_k-g_\infty)\,\mathrm{d}\mu_k + \left(\int\limits_Z g_\infty\,\mathrm{d}\mu_k - \int\limits_Z g_\infty\,\mathrm{d}\mu_\infty\right)\\ &= O(\|g_k-g_\infty\|_\infty) + \left(\int\limits_Z g_\infty\,\mathrm{d}\mu_k - \int\limits_Z g_\infty\,\mathrm{d}\mu_\infty\right),\end{aligned}$$

where the two terms tend to 0, following the definitions of $g_k \to g_\infty$ and the weak convergence $\mu_k \rightharpoonup \mu_\infty$.

At first, we assert that μ_∞ is a $\Phi_{\Gamma,0}$-invariant probability measure. Indeed, let $f\in C(Z)$. It follows from Lemma 3.3 (i) and (4.9) that

$$g_k := f\circ\Phi^1_{\Gamma,k} \to g_\infty := f\circ\Phi^1_{\Gamma,0} \text{ in } (C(Z),\|\cdot\|_\infty).$$

As μ_k is $\Phi_{\Gamma,k}$-invariant, we have

$$\int\limits_Z f\,\mathrm{d}\mu_k = \int\limits_Z f\circ\Phi^1_{\Gamma,k}\,\mathrm{d}\mu_k = \int\limits_Z g_k\,\mathrm{d}\mu_k.$$

By letting $k\to+\infty$, we obtain

$$\int\limits_Z f\,\mathrm{d}\mu_\infty = \int\limits_Z g_\infty\,\mathrm{d}\mu_\infty = \int\limits_Z f\circ\Phi^1_{\Gamma,0}\,\mathrm{d}\mu_\infty. \quad (4.13)$$

See (4.12). As $f\in C(Z)$ is arbitrary, by [16, Theorem 6.8], (4.13) means that μ_∞ is $\Phi_{\Gamma,0}$-invariant.

Next, it follows from Lemma 3.3 (i) and (4.10) that

$$F_{\Gamma,k} \to F_{\Gamma,0} \text{ in } (C(Z),\|\cdot\|_\infty).$$

Now it follows from the ergodic representations (4.11) that

$$\varrho_\Gamma(Q_k, W_k) = \int_Z F_{\Gamma,k}\, \mathrm{d}\mu_k \to \int_Z F_{\Gamma,0}\, \mathrm{d}\mu_\infty. \tag{4.14}$$

See also (4.12). Since μ_∞ is $\Phi_{\Gamma,0}$-invariant, we know from Theorem 4.2 that

$$\int_Z F_{\Gamma,0}\, \mathrm{d}\mu_\infty = \varrho_\Gamma(Q_0, W_0).$$

Hence, (4.14) proves the desired result (4.8). □

We end the paper with some remarks.

(i) When $S = \{0\}$, i.e., $W = 0$, the results of this paper reduce to those for classical Schrödinger equations with almost periodic potentials [7,4].

(ii) For general almost periodic, symplectic transmissions $A = \{A_i\}$, it is also possible to deduce the existence of rotation numbers by considering the hull of transmissions.

(iii) Following the reduction of Eq. (1.1) to multiplication of matrices as in [2,3], it is possible to prove the existence of Lyapunov exponents for Eq. (1.1) under the assumptions on (Q, Γ, S) above.

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