Weighted Eigenvalue Problems, Nonuniform Nonresonance, Stability and Arnold Tongues

Meirong Zhang†
Department of Applied Mathematics, Tsinghua University
Beijing 100084, People’s Republic of China
E-mail: mzhang@math.tsinghua.edu.cn

Weigu Li‡
Department of Mathematics, Peking University
Beijing 100871, People’s Republic of China
E-mail: weigu@sxx0.math.pku.edu.cn

July 18, 1998

† Supported by the National Natural Science Foundation of China and the Tsinghua University Education Foundation.
‡ Supported by the National Natural Science Foundation of China.
1991 Mathematics Subject Classification. 34L05, 34B30, 34D20, 34B15, 34C25, 34L15, 58F03.
Key words and phrases: Weighted eigenvalue problem, periodic problem, Dirichlet problem, Neumann problem, anti-periodic problem, rotation number, characteristic multiplier, characteristic number, nonuniform nonresonance, stability, parametric resonance, Arnold tongue.
Abstract

In this article, we will study the following weighted eigenvalue problems

\[ \ddot{x} + \lambda \varphi(t)x = 0, \quad t \in [0, T], \quad (E) \]

with respect to the Dirichlet, the Neumann, the periodic, and the anti-periodic boundary conditions:

\[ x(0) = x(T) = 0, \quad (D) \]
\[ \dot{x}(0) = \dot{x}(T) = 0, \quad (N) \]
\[ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \quad (P) \]
\[ x(0) + x(T) = \dot{x}(0) + \dot{x}(T) = 0, \quad (A) \]

where the weight function \( \varphi \) is assumed to be in \( L^1(0, T) \) and \( \varphi(t) \geq 0 \) for a.e. \( t \) and \( \varphi(t) \neq 0 \). After extending the weight function \( \varphi(t) \) to the whole \( \mathbb{R} \) by \( T \)-periodicity, we will use the rotation numbers in dynamical systems theory to introduce for (E) two sequences \( \{\lambda_k(\varphi) : k \in \mathbb{Z}^+\} \) and \( \{\bar{\lambda}_k(\varphi) : k \in \mathbb{Z}^+\} \) such that

\[ 0 = \lambda_0(\varphi) = \bar{\lambda}_0(\varphi) < \lambda_1(\varphi) \leq \cdots < \lambda_k(\varphi) \leq \bar{\lambda}_k(\varphi) < \cdots \]

and call them the characteristic numbers of (E). The relationship between characteristic numbers and all of above four classes of weighted eigenvalues is established. We will prove that \( \lambda_k(\varphi) \) and \( \bar{\lambda}_k(\varphi) \) are actually either the periodic eigenvalues when \( k \) is even, or the anti-periodic eigenvalues when \( k \) is odd. It is also proved that all characteristic numbers, in particular the periodic eigenvalues, can be recovered from the Dirichlet or the Neumann eigenvalues in the following way: For any \( k \in \mathbb{N} \), we have

\[ \lambda_k(\varphi) = \min\{\lambda_k^*(\varphi_s) : s \in \mathbb{R}\} \quad \text{and} \quad \bar{\lambda}_k(\varphi) = \max\{\lambda_k^*(\varphi_s) : s \in \mathbb{R}\}, \]

where \( \varphi_s(t) \equiv \varphi(t + s) \), and \( * = D \) or \( N \), and \( \lambda_k^*(\varphi) \) are eigenvalues of (E) with respect to the boundary conditions \( * \).

The comparison results of the weighted eigenvalues with respect to the weight functions are also proved. As a result, we can consider the nonuniform nonresonance problems of the following nonlinear nonautonomous differential equations

\[ \ddot{x} + f(t, x) = 0, \quad (NE) \]

when \( f(t, x) \) satisfies the following semilinearity conditions: There exist \( a(\cdot) \), \( b(\cdot) \in L^1(0, T) \) such that

\[ a(t) \leq \liminf_{|x| \to \infty} f(t, x)/x \leq \limsup_{|x| \to \infty} f(t, x)/x \leq b(t) \]

uniformly in a.e. \( t \in [0, T] \). If \( b(t) \geq a(t) \geq 0 \) and \( a(t) \neq 0 \), we can obtain the following nonresonance conditions of (NE) with respect to boundary conditions \( * \):

\[ \lambda_{k-1}^*(a) < 1 < \lambda_k^*(b) \]

for some \( k \in \mathbb{N} \). These nonuniform nonresonance conditions have generalized most nonresonance results for nonautonomous equations in the literature and are also necessary in some sense.

As another application of weighted eigenvalues, we can use characteristic numbers to describe the stability of the Hill’s equation (E), namely, Eq. (E) is stable if \( \lambda \in I_k := \]
\((\overline{\lambda}_{k-1}(\varphi), \underline{\lambda}_k(\varphi))\) for some \(k \in \mathbb{N}\), and Eq. (E) is unstable if \(\lambda \in J_k := (\underline{\lambda}_k(\varphi), \overline{\lambda}_k(\varphi))\) for some \(k \in \mathbb{N}\). Consequently, when the following parameterized Hill’s equation

\[ \ddot{x} + \lambda \varphi(t)x = 0 \quad (E_\varepsilon) \]

is considered, where the weight function \(\varphi_\varepsilon(t)\) is \(T\)-periodic and satisfies \(\varphi_\varepsilon(t) \geq 0\) and \(\varphi_\varepsilon(t) \neq 0\) for each parameter \(\varepsilon\). Then the Arnold tongues resulted from the parametric resonance of \((E_\varepsilon)\) are given by

\[ \underline{\lambda}_k(\varphi_\varepsilon) < \lambda < \overline{\lambda}_k(\varphi_\varepsilon), \]

whenever the family \(\varphi_\varepsilon\) satisfies \(\underline{\lambda}_k(\varphi_\varepsilon) < \overline{\lambda}_k(\varphi_\varepsilon)\).

The characteristic numbers also play an important role in the nonresonance problems of periodic solutions to some differential equations with singularities.
## Contents

1 Introduction 5

2 Weighted Eigenvalue Problems 12
   2.1 A Functional Analysis Approach to Dirichlet Problems 12
   2.2 A Functional Analysis Approach to Periodic Problems 13
   2.3 A Dynamics Approach and Characteristic Numbers 15
   2.4 Periodic Eigenvalues and Characteristic Numbers 20

3 Dirichlet, Neumann Eigenvalues and Characteristic Numbers 22
   3.1 Dirichlet Eigenvalues and Characteristic Numbers 22
   3.2 Neumann Eigenvalues and Characteristic Numbers 25

4 Nonuniform Nonresonance of Nonlinear Differential Equations 26
   4.1 Nonresonance of Periodic Problems 26
   4.2 Nonresonance of Other Problems 29

5 Stability and Parametric Resonance for Hill’s Equations 29
   5.1 Stability of Hill’s Equations 29
   5.2 Parametric Resonance and Arnold Tongues 32

6 Estimates for Eigenvalues and Numerical Results 34
   6.1 Lower Bounds for the First Eigenvalues 35
   6.2 Examples and Numerical Results 37

7 Concluding Remarks 38
1 Introduction

Let us begin with the nonresonance problem of the following scalar autonomous differential equation

\[ \ddot{x} + f(x) = h(t), \quad t \in [0, T], \]  

with respect to some boundary conditions, say the Dirichlet boundary conditions:

\[ x(0) = x(T) = 0, \]  \hspace{1cm} (D)

where \( T > 0 \) is fixed. By the nonresonance of problem (1.1)+(D), it means that for any \( h(t) \) in a certain class of functions, say \( h \in L^1(0, T) \), Eq. (1.1) has at least one solution satisfying (D).

When \( f = f(x) : \mathbb{R} \to \mathbb{R} \) is continuous and is asymptotically linear at infinity, i.e.,

\[ \lim_{|x| \to \infty} f(x)/x = a \]

for some \( a \in \mathbb{R} \), it is well known that one can obtain the nonresonance if \( a \) is not an eigenvalue of the following eigenvalue problem

\[ \ddot{x} + \lambda x = 0 \]  \hspace{1cm} (1.2)

with respect to (D). That is,

\[ a \neq \lambda^D_k = (k\pi/T)^2, \quad \forall k \in \mathbb{N}. \]  \hspace{1cm} (1.3)

When \( f(x) \) oscillates between two linear functions \( ax \) and \( bx \), i.e.,

\[ a \leq \liminf_{|x| \to \infty} f(x)/x \leq \limsup_{|x| \to \infty} f(x)/x \leq b, \]  \hspace{1cm} (1.4)

the corresponding nonresonance conditions are: There exists some \( k \in \mathbb{N} \) such that

\[ \lambda^D_{k-1} < a \leq b < \lambda^D_k. \]  \hspace{1cm} (1.5)

Nonresonance conditions (1.5) are sharp when \( f \) is autonomous and satisfies the semilinearity conditions (1.4) because they are also necessary in some sense. Inspired by these simple nonresonance conditions (1.3) and (1.5), a large amount of nonresonance results has been developed in the literature for different kinds of nonlinearities, autonomous or nonautonomous, asymptotically linear or asymmetric, the standard second order equations or equations with nonlinear principal parts like the \( p \)-Laplacian, and for different kinds of boundary conditions such as the periodic boundary conditions:

\[ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \]  \hspace{1cm} (P)

and the Neumann boundary conditions:

\[ \dot{x}(0) = \dot{x}(T) = 0. \]  \hspace{1cm} (N)

It is impossible to give a complete description for all these nonresonance results existed in the literature. However, in order to illustrate the idea in this article, it is worth to mention some of these.
Let us first confine ourselves to the autonomous case. Suppose now \( f(x) \) is asymmetric at \( x = \pm \infty \), or in the other word, \( f(x) \) has “jumping nonlinearity”, i.e.,

\[
\lim_{x \to \pm \infty} f(x)/x = a_{\pm}
\]

for some \( a_{\pm} \in \mathbb{R} \). Then the nonresonance must exclude those \( a_{\pm} \) such that the following positively homogeneous equation

\[
\ddot{x} + a_+ x_+ + a_- x_- = 0
\]

has nonzero solutions satisfying (D), where \( x_+ = \max\{x, 0\} \) and \( x_- = \min\{x, 0\} \). Currently, those \( (a_+, a_-) \) are called the Fučík spectrum (with respect to (D)), a generalization of eigenvalues to asymmetric nonlinearities. Such an idea is inspired by the early works of Fučík [17], [18] and Dancer [9], [10] and has been well developed in the literature. A functional analysis framework to deal with the nonresonance of those nonlinearities and an extensive bibliography on this subject are given by the first author of the present article in [42].

Now we assume that \( f(x) \) is inhomogeneous in \( x \), which means that \( f(x) \) may vary without any linear bounds like (1.4). Observe that the eigenvalues of (1.2)+(D) may be introduced using the time map of Eq. (1.2). A method of time maps is developed in the literature. We may give a high credit to this approach because the nonresonance of (1.1)+(D) is not necessarily assumed the semilinearity conditions (1.4). In fact one needs only to exclude nonlinear functions \( f(x) \) from being the resonant case \( f(x) = \lambda D k x \) in some sense of space averages. This is the main point of time map approach. Such an approach is fruitful when one deals with the periodic problems, see [14], [16] and the expository article [39].

Now we consider the second order differential equations with nonlinear principal parts, say the \( p \)-Laplacian,

\[
(\phi_p(x'))' + f(x) = h(t),\tag{1.6}
\]

where \( \phi_p(x) = |x|^{p-2}x, 1 < p < \infty \). The corresponding eigenvalue problem of Eq. (1.6) is

\[
(\phi_p(x'))' + \lambda \phi_p(x) = 0.\tag{1.7}
\]

Problem (1.7)+(D) has also a sequence of eigenvalues. Now the corresponding nonresonance conditions like that of (1.3) and (1.5) can be obtained by replacing \( \lambda D k \) by the eigenvalues of (1.7)+(D), see [12] and [41] and the references therein. This idea can even be generalized to more general nonlinear principal parts. For example, one may consider the nonresonance problems of the following equation

\[
(\phi(x'))' + f(x) = h(t)
\]

by introducing appropriate eigenvalues and Fučík spectrum, where \( \phi : \mathbb{R} \to \mathbb{R} \) is an odd homeomorphism with some growth limitation. We refer the readers to [19] and [20] for this subject.

Now we consider nonautonomous differential equations

\[
\ddot{x} + f(t, x) = 0,\tag{1.8}
\]
where \( f = f(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function, i.e., \( f(t,x) \) satisfies (i) for a.e. \( t \in [0,T] \), the map \( x \to f(t,x) \) is continuous; (ii) for each \( x \in \mathbb{R} \), the map \( t \to f(t,x) \) is measurable; and (iii) for each \( r > 0 \), there is an integrable function \( h_r(t) \) such that \( |f(t,x)| \leq h_r(t) \) for a.e. \( t \in [0,T] \) and for all \( x \) with \( |x| \leq r \). By a solution \( x(\cdot) \) of Eq. (1.8), it means that \( x(\cdot) \) is \( C^1 \) and \( \dot{x}(\cdot) \) is absolutely continuous on \([0,T] \), and Eq. (1.8) is satisfied for a.e. \( t \in [0,T] \).

We introduce the following notations. For any \( 1 \leq \alpha \leq \infty \), let \( L^\alpha(0,T) \) be the usual Lebesgue space with the corresponding norm \( \| \cdot \|_\alpha \). For \( x \in L^1(0,T) \), the mean value is \( \bar{x} = (1/T) \int_0^T x(t)dt \). For \( \varphi_1, \varphi_2 \in L^1(0,T) \), we write \( \varphi_1 \leq \varphi_2 \) if \( \varphi_1(t) \leq \varphi_2(t) \) for a.e. \( t \in [0,T] \) and \( \varphi_1 < \varphi_2 \).

In this article, we assume that \( f(t,x) \) in Eq. (1.8) satisfies the following semilinearity conditions: There exist \( a(\cdot), b(\cdot) \in L^1(0,T) \) such that
\[
a(t) \leq \liminf_{|x| \to \infty} f(t,x)/x \leq \limsup_{|x| \to \infty} f(t,x)/x \leq b(t)
\] (1.9) uniformly in a.e. \( t \in [0,T] \).

Inspired by the nonresonance results for autonomous equations, the corresponding nonresonance conditions for Eq. (1.8) are given in the literature. For example, the conditions (1.5) now read as: There exists \( k \in \mathbb{N} \) such that
\[
\lambda_{k-1}^D \leq a(\cdot) \leq b(\cdot) \leq \lambda_k^D,
\] (1.10) see [11], [12] and [34]. Conditions (1.10) are usually called the nonuniform (with respect to \( t \)) nonresonance conditions. They have also been generalized to the case that \( f(t,x) \) are nonuniformly asymmetric at \( x = \pm \infty \), and to the case that the principal parts are the \( p \)-Laplacian. See [7], [8], [12], [23], [32], [34] and the references therein.

These results can also be given for other boundary value problems such as the periodic problems (1.8)+(P), see [7], [21], [23], [31] and [33]. One may notice that the Neumann problems (1.8)+(N) and the relatively less considered anti-periodic problem (1.8)+(A):
\[
x(0) + x(T) = \dot{x}(0) + \dot{x}(T) = 0 \tag{A}
\] can also be dealt with similarly.

Besides the Dirichlet eigenvalues \( \lambda_k^D \), we will use \( \lambda_k^P \), \( \lambda_k^N \), and \( \lambda_k^A \) to denote eigenvalues of problems (1.2)+(P), (1.2)+(N), and (1.2)+(A) respectively. It is known that \( \lambda_k^D \) are given by (1.3) and the others are: \( \lambda_k^P = (2k\pi/T)^2 \) for \( k \in \mathbb{Z}^+ \), \( \lambda_k^N = (k\pi/T)^2 \) for \( k \in \mathbb{Z}^+ \), and \( \lambda_k^A = ((2k - 1)\pi/T)^2 \) for \( k \in \mathbb{N} \), where \( \mathbb{N} \) and \( \mathbb{Z}^+ \) are the sets of positive integers and nonnegative integers, respectively. We remark that all eigenvalues \( \lambda_k^D \) and \( \lambda_k^N \) are simple, while \( \lambda_0^P \) is simple and all \( \lambda_k^P \) and \( \lambda_k^A \) \( (k \in \mathbb{N}) \) have multiplicity 2.

The nonresonance conditions (1.10) look very good from the viewpoint of autonomous equations. However, as observed in [44] and [45], these nonuniform nonresonance conditions have some disadvantages when nonautonomous equations are considered.

The first disadvantage is that coincidence degree theory (see [30]) can deal with \( L^1 \)-Carathéodory functions \( f(t,x) \), while conditions (1.10) naturally imply that \( a(\cdot) \) and \( b(\cdot) \) are in \( L^\infty(0,T) \).
The second one is that conditions (1.10) have no persistence, which means that (1.10) may no longer be satisfied when \(a(t)\) and \(b(t)\) have small perturbations. However, due to the Property \(P\) introduced in [15] and [22], and to a very general result concerning with positively homogeneous perturbations of the identity in Banach spaces in [42], the nonresonance problems (1.8)+(D) have some persistence, which means that if \(a(\cdot)\) and \(b(\cdot)\) satisfy (1.10), then the existence to (1.8)+(D) can also be guaranteed when \(f(t,x)\) only satisfies

\[
a(t) - \varepsilon_0 \leq \liminf_{|x| \to \infty} f(t,x)/x \leq \limsup_{|x| \to \infty} f(t,x)/x \leq b(t) + \varepsilon_0
\]

uniformly in a.e. \(t \in [0,T]\), where \(\varepsilon_0 > 0\) is small.

The third one is that conditions (1.10) cannot even deal with linear equations with time dependent coefficients. For example, for the following linear equation

\[
\ddot{x} + c(1 + \cos t)x = h(t), \quad (T = 2\pi)
\]

the nonresonance conditions (1.10) (when \(k \geq 2\)) give no any results because \(a(t) = b(t) = c(1 + \cos t)\) has zeros in \(t\).

These three observations show that it is needful to develop new nonresonance theory for nonautonomous equations. The first author has made some progress in [44] and [45] for this problem. It is now clear that these disadvantages come from that one always tries to compare nonautonomous equations (1.8) with autonomous eigenvalue problem (1.2). In [44], the nonuniform nonresonance of the Dirichlet problems for nonautonomous equations was considered. The main point in that paper is that one may compare nonautonomous equation (1.8) with the following weighted eigenvalue problem

\[
\ddot{x} + \lambda \varphi(t)x = 0 \tag{1.11}
\]

with respect to (D), where the weight function \(\varphi\) is in \(L^1(0,T)\) and \(\varphi \gtrsim 0\). These weighted eigenvalue problems have been considered in many textbooks such as [36]. The principal eigenvalue for higher dimensional Dirichlet problems with indefinite weight functions is also a topics in many papers such as [5], [25] and [29]. However, the role of these weighted eigenvalues in nonresonance problems seems to be seldom noticed, although this idea is very natural. Let

\[
0 < \lambda_1^D(\varphi) < \lambda_2^D(\varphi) < \cdots < \lambda_k^D(\varphi) < \cdots
\]

be the eigenvalues of problem (1.11)+(D). It is proved in [44] that nonresonance conditions (1.10) can be improved as: If the functions \(a(\cdot)\) and \(b(\cdot)\) in (1.9) satisfy either

\[
b \gtrsim 0 \quad \text{and} \quad \lambda_1^D(b) > 1, \tag{1.12}
\]

or,

\[
b \geq a \gtrsim 0 \quad \text{and} \quad \lambda_k^D(a) < 1 < \lambda_{k+1}^D(b) \quad \text{for some} \ k \in \mathbb{N}, \tag{1.13}
\]

then problem (1.8)+(D) has at least one solution. The proof in [44] is simply based on the comparison results of eigenvalues on weight functions and on the perturbation results in [42]. Conditions (1.12) and (1.13) not only have generalized (1.10), but also have overcome the disadvantages described above.
A similar idea for the periodic solution problems is also developed in [45]. As the comparison results for the first nonzero eigenvalue of the periodic problem (1.11)+(P) can be obtained from functional analysis, the classical nonuniform nonresonance conditions between the first two periodic eigenvalues $\lambda_0^P = 0$, $\lambda_1^P = (2\pi/T)^2$ (see [33]) can be improved as: The functions $a(\cdot)$ and $b(\cdot)$ in (1.9) satisfy that

$$\bar{a} > 0 \quad \text{and} \quad L b > 0, \quad \lambda_1^P(b) > 1.$$  \hfill (1.14)

However, the corresponding comparison result for other periodic eigenvalues seems not to be easily proved from functional analysis, see Remark 2.5 (i) in the next section. Thus the nonresonance results in other nonresonance straps like (1.13) are not given in [45] for the periodic problems.

In this article, we aim at using a dynamics approach to give a unified consideration for weighted eigenvalues of the Dirichlet problem (1.11)+(D), the periodic problem (1.11)+(P), the Neumann problem (1.11)+(N), and the anti-periodic problem (1.11)+(A), although they can be studied using either the ODE approach as in [36] or the functional analysis approach as in [44] and [45]. Our dynamics idea is as follows. Extend the domain of $\varphi(t)$ to the whole $\mathbb{R}$ by the $T$-periodicity. Eq. (1.11) is now equivalent to the following planar linear system

$$\dot{x} = -\sqrt{\lambda} y \quad \text{and} \quad \dot{y} = \sqrt{\lambda} \varphi(t)x,$$

where $\lambda > 0$. In the polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$, $\theta$ satisfies the following nonlinear equation

$$\dot{\theta} = \sqrt{\lambda} (\varphi(t) \cos^2 \theta + \sin^2 \theta) =: \Theta(t, \theta; \lambda).$$ \hfill (1.15)

As $\Theta(t, \theta; \lambda)$ is $T$-periodic in $t$ and is $2\pi$-periodic in $\theta$, we know from Arnold [2] and Hale [24] that the rotation number of (1.15)

$$\rho(\lambda) = \lim_{t \to +\infty} \frac{\theta(t; \theta_0, \lambda)}{t}$$

exists and is independent of $\theta_0$, where $\theta(t; \theta_0, \lambda)$ is the solution of (1.15) satisfying the initial condition: $\theta(0; \theta_0, \lambda) = \theta_0$.

In Section 2, we introduce two sequences $\{\lambda_k^D(\varphi) : k \in \mathbb{Z}^+\}$ and $\{\lambda_k^P(\varphi) : k \in \mathbb{Z}^+\}$ by

$$\lambda_k^D(\varphi) = \min\{\lambda : \rho(\lambda) = k/2\} \quad \text{and} \quad \lambda_k^P(\varphi) = \max\{\lambda : \rho(\lambda) = k/2\}.$$

We called them the characteristic numbers of (1.11). As $\rho(\lambda)$ is nondecreasing when $\lambda$ increases, we have the following order for characteristic numbers:

$$0 = \lambda_0^D(\varphi) = \lambda_0^P(\varphi) < \lambda_1^D(\varphi) \leq \lambda_1^P(\varphi) < \cdots < \lambda_k^D(\varphi) \leq \lambda_k^P(\varphi) < \cdots$$

The relationship between characteristic numbers and all of the above four classes of eigenvalues is established in Sections 2 and 3. Let the eigenvalues of (1.11) with respect to with (D), (P), (N) and (A) be respectively denoted by $\{\lambda_k^D(\varphi) : k \in \mathbb{N}\}$, $\{\lambda_k^P(\varphi) : k \in \mathbb{Z}^+\}$, $\{\lambda_k^N(\varphi) : k \in \mathbb{Z}^+\}$ and $\{\lambda_k^A(\varphi) : k \in \mathbb{N}\}$, by counting the multiplicity and by labelling from small to big. When the weight function $\varphi$ satisfies $\varphi^L > 0$, one sees from Eq. (1.11) that all these eigenvalues are nonnegative. Summarily, the following are proved in Sections 2 and 3.
(1) Both $\lambda_k(\varphi)$ and $\overline{\lambda}_k(\varphi)$ tend to infinity quadratically when $k \to \infty$.
(2) If $k$ is even, then $\lambda_k(\varphi)$ and $\overline{\lambda}_k(\varphi)$ are eigenvalues of the periodic problem.
(3) If $k$ is odd, then $\lambda_k(\varphi)$ and $\overline{\lambda}_k(\varphi)$ are eigenvalues of the anti-periodic problem.
(4) For any $k \in \mathbb{N}$, the interval $[\lambda_k(\varphi), \overline{\lambda}_k(\varphi)]$ contains a unique eigenvalue of the Dirichlet problem.
(5) For any $k \in \mathbb{Z}^+$, the interval $[\lambda_k(\varphi), \overline{\lambda}_k(\varphi)]$ contains a unique eigenvalue of the Neumann problem.

Conversely, all eigenvalues of the above four classes are given in the way described in (2)–(5).

Besides these, we will prove in Section 3 that all characteristic numbers, in particular eigenvalues of the periodic problem, can be recovered from eigenvalues of the Dirichlet and the Neumann problems. Namely, we have

(6) For any $k \in \mathbb{N}$,
\[ \lambda_k(\varphi) = \min\{\lambda_k^D(\varphi_s) : s \in \mathbb{R}\} \quad \text{and} \quad \overline{\lambda}_k(\varphi) = \max\{\lambda_k^D(\varphi_s) : s \in \mathbb{R}\}, \]
where $\varphi_s(t)$ is the translation of $\varphi(t)$: $\varphi_s(t) \equiv \varphi(t + s)$.

(7) For any $k \in \mathbb{N}$,
\[ \lambda_k(\varphi) = \min\{\lambda_k^N(\varphi_s) : s \in \mathbb{R}\} \quad \text{and} \quad \overline{\lambda}_k(\varphi) = \max\{\lambda_k^N(\varphi_s) : s \in \mathbb{R}\}. \]

In Section 3, the comparison results for all weighted eigenvalues are also proved, i.e.,

(8) If $\varphi \geq \psi > 0$ then $\lambda_k^*(\varphi) \leq \lambda_k^*(\psi)$ for all $k \in \mathbb{N}$, and if $\varphi > \psi > 0$ then $\lambda_k^*(\varphi) < \lambda_k^*(\psi)$ for all $k \in \mathbb{N}$, where $* = D, P, N, A$.

After these properties for weighted eigenvalues have been established, we will give some applications.

The first application is the nonresonance problems. As a result of the comparison results (8) and the nonresonance results in [42], the nonuniform nonresonance conditions like (1.12) and (1.13) can be given to other boundary value problems, see Section 4.

As a second application of weighted eigenvalues, we will prove in Section 5 that the stability and unstability of the Hill’s equation (1.11) (with the parameter $\lambda > 0$) can also be obtained using characteristic numbers:

(9) If $\lambda \in I_k := (\overline{\lambda}_k(\varphi), \lambda_k(\varphi))$ for some $k \in \mathbb{N}$, then Eq. (1.11) is stable.
(10) If $\lambda \in J_k := (\lambda_k(\varphi), \overline{\lambda}_k(\varphi))$ for some $k \in \mathbb{N}$, then Eq. (1.11) is unstable.

Consequently, when $1 \in I_k$ for some $k \in \mathbb{N}$, the following Hill’s equation
\[ \ddot{x} + \varphi(t)x = 0 \quad (1.16) \]
is stable. Using a lower bound for $\lambda_1(\varphi)$ in Section 6, the first stability interval $I_1$ is studied in Section 5. In particular, the classical condition for the first stability interval of Borg [4]
\[ \int_0^T |\varphi(t)|dt \leq \frac{4}{T} \]
is generalized using the $L^\alpha$ norms of $\varphi$, where $1 \leq \alpha \leq \infty$. 

10
When the following parameterized Hill's equation
\[ \ddot{x} + \lambda \varphi_\varepsilon(t)x = 0 \quad (1.17) \]
is considered, where \( \varphi_\varepsilon(\cdot) \in L^1(0,T) \) satisfies \( \varphi_\varepsilon > 0 \) for each parameter \( \varepsilon \), the Arnold tongues [1] resulted from parametric resonance of (1.17) can now be determined using characteristic numbers:
\[ \lambda_k(\varphi_\varepsilon) < \lambda < \overline{\lambda}_k(\varphi_\varepsilon), \]
whenever the family \( \varphi_\varepsilon \) satisfies
\[ \lambda_k(\varphi_\varepsilon) < \lambda < \overline{\lambda}_k(\varphi_\varepsilon). \]

Finally, as a third application of weighted eigenvalues, we will prove in a forthcoming paper [38] that the nonresonance problems of positive periodic solutions for differential equations with singularities like
\[ \ddot{x} + \lambda \varphi(t)x = c \gamma - \gamma + h(t), \quad (1.18) \]
where \( \varphi(\cdot) \) is periodic with \( \varphi_\varepsilon > 0 \) and \( c > 0, \gamma \geq 1 \), is also closely related with characteristic numbers, namely, the nonresonance to (1.18) can be obtained when \( \lambda \) is in the stability intervals \( I_k = (\overline{\lambda}_{k-1}(\varphi), \overline{\lambda}_k(\varphi)). \)

It is worth to mention the theory of Hale [24] for stability problems of linear equations. In [24], Hale introduced a different eigenvalue problem. Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be continuous and be \( T \)-periodic. \( \varphi(\cdot) \) is not necessarily assumed that \( \varphi \geq 0 \). He considered the following linear equations
\[ \ddot{x} + (\lambda + \varphi(t))x = 0 \quad (1.19) \]
with parameter \( \lambda \in \mathbb{R} \) and introduced in Theorem 8.1 of [24] two sequences \( \{a_k = a_k(\varphi) : k \in \mathbb{Z}^+\} \) and \( \{a_k^* = a_k^*(\varphi) : k \in \mathbb{N}\} \) such that
\[ a_0 < a_1^* \leq a_2^* \leq \cdots \leq a_{2k-1}^* < a_{2k}^* < a_{2k-1} > a_{2k} < \cdots \]
The relationship between these sequences \( \{a_k^*\}, \{a_k\} \) and the periodic, the anti-periodic eigenvalues of Eq. (1.19) is similar to the properties (2) and (3) for characteristic numbers. The stability and unstability intervals for Eq. (1.19) are now given by
\[ (a_0, a_1^*), \quad (a_2^*, a_1), \quad (a_2, a_3^*), \quad (a_4^*, a_3), \quad \cdots \]
and
\[ (-\infty, a_0], \quad (a_1^*, a_2^*), \quad (a_1, a_2), \quad (a_3^*, a_4^*), \quad (a_3, a_4), \quad \cdots \]
Thus Eq. (1.16) is stable when 0 is in some stable interval.

Consequently, when the stability problems are considered, our approach is much similar to that of Hale’s. However, the difference between these two approaches is also obvious. On the one hand, Hale’s theory can deal with any function \( \varphi(\cdot) \) without the assumption \( \varphi > 0 \), while our results in this article are mainly based on the positiveness assumption \( \varphi_\varepsilon \geq 0 \). On the other hand, the proof for the existence of \( \{a_k(\varphi)\} \) and \( \{a_k(\varphi)\} \) in [24] is based on some
zeros theorem for entire functions of fractional orders and the role of \( \{ a_k^*(\varphi) \} \) and \( \{ a_k(\varphi) \} \) in

the other problems such as the nonresonance problems of Eq. (1.8) and the Arnold tongues of

Eq. (1.17) is obscure, while our dynamics approach is more geometrical and is easily read.

In the last section of this article, we will briefly sketch how to unify these two approaches.

2 Weighted Eigenvalue Problems

Throughout this article, the weight function \( \varphi \) is always assumed to be in \( L^1(0, T) \) and \( \varphi^L > 0 \).

We always extend the domain of \( \varphi(\cdot) \) to the whole \( \mathbb{R} \) by the \( T \)-periodicity. We will consider

in this section and the next section the following weighted eigenvalue problems

\[
\ddot{x} + \lambda \varphi(t)x = 0 \tag{2.1}
\]

with respect to boundary conditions (D), (P), (A), or (N).

2.1 A Functional Analysis Approach to Dirichlet Problems

Let us first consider, using a functional analysis approach, eigenvalues of (2.1)+(D). To this

end, let \( H^D := H^1_0(0, T) \) be the usual Sobolev space, i.e.,

\[
H^D = \{ x \in L^2(0, T) : x \text{ satisfies (D) and has a weak derivative } \dot{x} \in L^2(0, T) \}.
\]

A convenient inner product on \( H^D \) is

\[
\langle x, y \rangle = \int_0^T \dot{x}(t)\dot{y}(t)dt.
\]

We introduce a symmetric bi-linear function \( A_\varphi : H^D \times H^D \to \mathbb{R} \) as

\[
A_\varphi(x, y) = \int_0^T \varphi(t)x ydt.
\]

It is easy to check that

\[
|A_\varphi(x, y)| \leq T\|\varphi\|_1\|\dot{x}\|_2\|\dot{y}\|_2, \quad \forall x, y \in H^D.
\]

Thus \( A_\varphi : H^D \times H^D \to \mathbb{R} \) is continuous. As a result of the Riesz representation theorem, there exists a bounded linear operator \( \tilde{A}_\varphi : H^D \to H^D \) such that

\[
A_\varphi(x, y) = \langle \tilde{A}_\varphi x, y \rangle, \quad \forall x, y \in H^D.
\]

It is not difficult to prove that the operator \( \tilde{A}_\varphi : H^D \to H^D \) is a bounded, symmetric, nonnegative, compact, linear operator. The following two propositions are proved in [44].

**Proposition 2.1** (i) Eigenvalue problem (2.1)+(D) is equivalent to the following eigenvalue problem of the operator \( \tilde{A}_\varphi \):

\[
x = \lambda \tilde{A}_\varphi x, \quad x \in H^D.
\]
Problem (2.1)+(D) has a sequence of eigenvalues $\lambda_k^D(\varphi)$, $k \in \mathbb{N}$, such that

$$0 < \lambda_1^D(\varphi) < \lambda_2^D(\varphi) < \cdots < \lambda_k^D(\varphi) < \cdots$$

and $\lim_{k \to \infty} \lambda_k^D(\varphi) = +\infty$. Moreover, all $\lambda_k^D(\varphi)$ are simple.

(iii) The first eigenvalue $\lambda_1^D(\varphi)$ has the following variational characterization:

$$\lambda_1^D(\varphi) = \inf_{x \in H^D \setminus \{0\}} \frac{\int_0^T \dot{x}^2 dt}{\int_0^T \varphi(t)x^2 dt}.$$ 

Note that all operators $\tilde{A}_\varphi$ for different weight functions $\varphi$ are defined on the same underlying space $H^D$. As $\varphi \geq \psi$ implies $\tilde{A}_\varphi \geq \tilde{A}_\psi$ on $H^D$, we have the following comparison results for eigenvalues $\lambda_k^D(\varphi)$ (see, e.g., Reid [35]).

**Proposition 2.2**

(i) When $\varphi(t) \equiv 1$, we obtain the usual eigenvalues $\lambda_k^D(1) = (k\pi/T)^2$, $k \in \mathbb{N}$.

(ii) $\lambda_k^D(c\varphi) = c^{-1}\lambda_k^D(\varphi)$ for all $c > 0$ and all $k \in \mathbb{N}$.

(iii) Assume that $\varphi, \psi \in L^1(0,T)$ with $\varphi \geq 0$ and $\psi \geq 0$. Then $\lambda_k^D(\varphi) \leq \lambda_k^D(\psi)$ for all $k \in \mathbb{N}$ if $\varphi \geq \psi$. Moreover, $\lambda_k^D(\varphi) < \lambda_k^D(\psi)$ for all $k \in \mathbb{N}$ if $\varphi > \psi$.

We remark that this approach can be extended to the higher dimensional Dirichlet eigenvalue problem with weight $\Delta u + \lambda \varphi(x)u = 0$, $x \in \Omega$, $u = 0$, $x \in \partial \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain.

### 2.2 A Functional Analysis Approach to Periodic Problems

Now we consider the periodic eigenvalue problem (2.1)+(P). Obviously, $\lambda = 0$ is always an eigenvalue of (2.1)+(P) with constant eigenfunctions. Other eigenvalues $\lambda$ are positive because $\varphi \geq 0$.

A functional analysis approach to nonzero eigenvalues of (2.1)+(P) is given in [45]. Let $H^P$ be the Sobolev space of $T$-periodic functions:

$$H^P = \{x \in H^1(0,T) : x(0) = x(T)\}.$$ 

The usual inner product on $H^P$ is

$$(x,y) = \int_0^T (x(t)y(t) + \dot{x}(t)\dot{y}(t))dt.$$ 

We have the following important observation. If $x(\cdot)$ is an eigenfunction associated with a nonzero eigenvalue $\lambda \neq 0$, then integration of (2.1) on $[0,T]$ yields

$$\int_0^T \varphi(t)x(t)dt = 0. \quad (2.2)$$
As a result, we introduce a closed subspace of $H^P$ as

$$H^P_\varphi = \{ x \in H^P : x \text{ satisfies (2.2)} \} .$$

When $x \in H^P_\varphi$, we know from (2.2) that $x(t_0) = 0$ for some $t_0$. Thus, on the space $H^P_\varphi$, the inner product $(\cdot, \cdot)$ is equivalent to $(\cdot, \cdot)$ introduced above. Similarly, we introduce a symmetric bi-linear function $A_\varphi : H^P_\varphi \times H^P_\varphi \to \mathbb{R}$ as

$$A_\varphi(x, y) = \int_0^T \varphi(t)x'ydt.$$ 

As $A_\varphi$ is continuous on $H^P_\varphi$, there exists a bounded linear operator $\widetilde{A}_\varphi : H^P_\varphi \to H^P_\varphi$ such that

$$A_\varphi(x, y) = \langle \widetilde{A}_\varphi x, y \rangle, \quad \forall x, y \in H^P_\varphi.$$ 

The operator $\widetilde{A}_\varphi : H^P_\varphi \to H^P_\varphi$ is a bounded, symmetric, nonnegative, compact, linear operator. The following two propositions are proved in [45].

**Proposition 2.3** (i) When $\lambda \neq 0$, eigenvalue problem (2.1)+(P) is equivalent to the following eigenvalue problem of the operator $\widetilde{A}_\varphi$:

$$x = \lambda \widetilde{A}_\varphi x, \quad x \in H^P_\varphi. \quad (2.3)$$

(ii) Besides the zero eigenvalue $\lambda^0_\varphi(\varphi) = 0$, problem (2.1)+(P) has also a sequence of nonzero eigenvalues

$$0 < \lambda^1_\varphi(\varphi) \leq \lambda^2_\varphi(\varphi) \leq \cdots \leq \lambda^k_\varphi(\varphi) \leq \cdots$$

with $\lim_{k \to \infty} \lambda^k_\varphi(\varphi) = +\infty$.

(iii) The eigenvalue $\lambda^1_\varphi(\varphi)$ has the following variational characterization:

$$\lambda^1_\varphi(\varphi) = \inf_{x \in H^P_\varphi \backslash \{0\}} \frac{\int_0^T \dot{x}^2 dt}{\int_0^T \varphi(t)x^2 dt}. \quad (2.4)$$

Using variational characterization (2.4) and some Wirtinger-like inequalities, we have the following comparison result for eigenvalues $\lambda^1_\varphi(\varphi)$.

**Proposition 2.4** (i) When $\varphi(t) \equiv 1$, we obtain the usual eigenvalues $\lambda^0(\varphi) = 0$ and $\lambda^k_{2k-1}(1) = \lambda^k_{2k}(1) = (2k\pi/T)^2$, $k = 1, 2, \cdots$

(ii) $\lambda^k_{2k}(c\varphi) = c^{-1}\lambda^k_{2k}(\varphi)$ for all $c > 0$ and $k \geq 0$.

(iii) Assume that $\varphi, \psi \in L^1(0, T)$ with $\varphi \overset{L}{\geq} 0$ and $\psi \overset{L}{\geq} 0$ then $\lambda^1_\varphi(\varphi) \leq \lambda^1_\varphi(\psi)$ if $\varphi \overset{L}{\geq} \psi$. Moreover, $\lambda^1_\varphi(\varphi) < \lambda^1_\varphi(\psi)$ if $\varphi \overset{L}{\geq} \psi$.

**Remark 2.5** (i) The comparison result in Proposition 2.4 (iii) plays an important role in obtaining the nonuniform nonresonance conditions, for the periodic problems, between the first eigenvalue $\lambda^0_\varphi(\varphi) = 0$ and the second eigenvalue $\lambda^1_\varphi(\varphi)$, see (1.14). We do not know at this moment whether the comparison results as in Proposition 2.2 also hold for $\lambda^k_\varphi(\varphi)$ ($k \geq 2$). The main reason is that, not only the operators $\widetilde{A}_\varphi$ but also the underlying spaces $H^P_\varphi$ for periodic eigenvalue problems (2.3), are dependent on the weight function $\varphi$. 

14
(ii) The Neumann problem (2.1)+(N) and the anti-periodic problem (2.1)+(A) can also be discussed using the functional analysis approach. One may notice that the Neumann problem is much similar to the periodic problem, while the anti-periodic problem is much similar to the Dirichlet problem.

### 2.3 A Dynamics Approach and Characteristic Numbers

Now we use the rotation numbers to introduce two sequences \( \{ \lambda_k(\varphi) : k \in \mathbb{N} \} \) and \( \{ \bar{\lambda}_k(\varphi) : k \in \mathbb{N} \} \), which are called the characteristic numbers of Eq. (2.1).

Without loss of generality, assume that \( T = 2\pi \). Assume that \( \lambda > 0 \) in (2.1). Let \( y = -\dot{x}/\sqrt{\lambda} \). Then (2.1) is equivalent to the following planar linear system:

\[
\begin{align*}
\dot{x} &= -\sqrt{\lambda} y, \\
\dot{y} &= \sqrt{\lambda} \varphi(t)x.
\end{align*}
\]

Note that the existence and uniqueness for initial value problems of (2.5) hold. Thus we can define the Poincaré map \( P_\lambda : \mathbb{R}^2 \to \mathbb{R}^2 \) of (2.5) by

\[
(x_0, y_0) \mapsto (x(2\pi; x_0, y_0, \lambda), y(2\pi; x_0, y_0, \lambda)),
\]

where \( (x(t; x_0, y_0, \lambda), y(t; x_0, y_0, \lambda)) \) is the solution of (2.5) satisfying

\[
(x(0; x_0, y_0, \lambda), y(0; x_0, y_0, \lambda)) = (x_0, y_0).
\]

As (2.5) is a linear Hamiltonian system, we know that \( P_\lambda \) is a linear area-preserving transformation: \( \det P_\lambda = 1 \). The two eigenvalues \( \mu_{1,2}(\lambda) \) of \( P_\lambda \) are called the characteristic multipliers of system (2.5). As \( \det P_\lambda = 1 \), we see that \( \mu_1(\lambda) \cdot \mu_2(\lambda) = 1 \).

In order to consider eigenvalue problems, we rewrite (2.5) in the polar coordinates: \( x = r \cos \theta, y = r \sin \theta \). Then we have

\[
\begin{align*}
\dot{r} &= \sqrt{\lambda} (\varphi(t) - 1) r \cos \theta \sin \theta, \\
\dot{\theta} &= \sqrt{\lambda} (\varphi(t) \cos^2 \theta + \sin^2 \theta) =: \Theta(t, \theta; \lambda).
\end{align*}
\]

We are now only interested in Eq. (2.7). For any \( \theta_0 \in \mathbb{R} \), let \( \theta(t; \theta_0, \lambda) \) be the (unique) solution of (2.7) satisfying the initial condition: \( \theta(0; \theta_0, \lambda) = \theta_0 \). As the vector field \( \Theta(t, \theta; \lambda) \) is \( 2\pi \)-periodic in both \( t \) and \( \theta \), it is well known, at least for the case that \( \varphi(t) \) is continuous (see Arnold [2] and Hale [24]), that the following limit

\[
\rho(\lambda) = \rho_\varphi(\lambda) = \lim_{t \to +\infty} \frac{\theta(t; \theta_0, \lambda)}{t}
\]

exists and is independent of \( \theta_0 \). \( \rho(\lambda) \) is called the rotation number of system (2.7). For the general case that \( \varphi \in L^1(0, 2\pi) \), it can be proved using the similar idea.

Rotation numbers \( \rho(\lambda) \) can also be introduced using families of homeomorphisms of the circle \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \). From the uniqueness of solutions and the periodicity of the vector field \( \Theta(t, \theta; \lambda) \), we have

\[
\theta(t; \theta_0 + 2m\pi, \lambda) \equiv \theta(t; \theta_0, \lambda) + 2m\pi, \quad \forall m \in \mathbb{Z}.
\]

15
Define a map $M_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$M_\lambda(\theta_0) = \theta(2\pi; \theta_0, \lambda).$$

Then

$$M_\lambda(\theta_0 + 2m\pi) \equiv M_\lambda(\theta_0) + 2m\pi, \quad \forall m \in \mathbb{Z}. \quad (2.8)$$

Thus $M_\lambda$ yields a map of the circle $S^1$ for each $\lambda > 0$. Actually, by the comparison results for solutions of ODEs we know that $\{M_\lambda : \lambda > 0\}$ is a family of orientation-preserving homeomorphisms of $S^1$. Now, for each $\lambda > 0$, the rotation number $\rho(\lambda)$ is

$$\rho(\lambda) = \lim_{n \to +\infty} \frac{M^n_\lambda(\theta_0)}{2n\pi}.$$ 

Note that the vector field $\Theta(t, \theta; \lambda)$ is increasing when $\lambda$ or the weight function $\varphi$ increases.

We have the following monotonicity.

**Lemma 2.6** (i) If $\theta_1 > \theta_2$, then $\theta(t; \theta_1, \lambda) > \theta(t; \theta_2, \lambda)$ for all $t \geq 0$ and all $\lambda > 0$.

(ii) If $\lambda_1 > \lambda_2$, then $\theta(t; \theta_0, \lambda_1) \geq \theta(t; \theta_0, \lambda_2)$ for all $t \geq 0$ and all $\theta_0$. Moreover, $\theta(t; \theta_0, \lambda_1) > \theta(t; \theta_0, \lambda_2)$ for all $t \geq 2\pi$ and all $\theta_0$. In particular, $M_{\lambda_1}(\theta_0) > M_{\lambda_2}(\theta_0)$ for all $\theta_0$.

**Proof** (i) It can be obtained from the comparison results for solutions of ODEs.

(ii) For any given $\theta_0$ and $\lambda$, let $\theta(t) = \theta(t; \theta_0, \lambda)$ and $\theta_\lambda(t) = \frac{\partial}{\partial \lambda}\theta(t; \theta_0, \lambda)$. We get from (2.7) that $\theta_\lambda(t)$ satisfies, for a.e. $t$,

$$\frac{d\theta_\lambda}{dt} = a(t)\theta_\lambda + b(t)$$

and $\theta_\lambda(0) = 0$, where

$$a(t) = \sqrt{\lambda}(1 - \varphi(t))\sin 2\theta(t),$$

$$b(t) = \frac{1}{2\sqrt{\lambda}}(\varphi(t)\cos^2 \theta(t) + \sin^2 \theta(t)).$$

Thus we have

$$\theta_\lambda(t) = \int_0^t b(s) \exp \left( \int_s^t a(\tau)d\tau \right) ds.$$

As $b(s) \geq 0$, we know that $\theta_\lambda(t) \geq 0$ for all $t \geq 0$. As a result, $\theta(t; \theta_0, \lambda)$ is nondecreasing with respect to $\lambda$.

Now assume that $t \geq 2\pi$. As

$$a(\tau) \geq -\sqrt{\lambda}|1 - \varphi(\tau)|,$$

$$b(s) \geq \frac{1}{2\sqrt{\lambda}}\varphi_-(s),$$

where $\varphi_-(s) = \min\{\varphi(s), 1\}$. It is proved in the following Lemma 2.8 that $\varphi_- > 0$. Thus we have

$$\int_s^t a(\tau)d\tau \geq -\sqrt{\lambda}\int_0^t |1 - \varphi(\tau)|d\tau$$
for all $0 \leq s \leq t$. Therefore, for $t \geq 2\pi$,

$$
\theta_\lambda(t) = \int_0^t b(s) \exp \left( \int_s^t a(\tau) d\tau \right) ds \\
\geq \exp \left( -\sqrt{\lambda} \int_0^t |1 - \varphi(\tau)| d\tau \right) \int_0^t b(s) ds \\
\geq \frac{\pi \varphi^-}{\sqrt{\lambda}} \exp \left( -\sqrt{\lambda} \int_0^t |1 - \varphi(\tau)| d\tau \right) > 0.
$$

As a result, when $t \geq 2\pi$, we have $\theta(t; \theta_0, \lambda_1) > \theta(t; \theta_0, \lambda_2)$ if $\lambda_1 > \lambda_2$. □

In order to obtain the comparison results for $\lambda_L^L(\varphi)$ with different weight functions $\varphi$, we need the following monotonicity of $\theta(t; \theta_0, \lambda)$ with respect to the weight functions $\varphi$. Thus we write $\theta(t; \theta_0, \lambda)$ as $\theta(t; \theta_0, \varphi)$.

**Lemma 2.7** (i) If $\varphi \geq \psi$, then $\theta(t; \theta_0, \lambda, \varphi) \geq \theta(t; \theta_0, \lambda, \psi)$ for all $t \geq 0$ and all $\theta_0$.

(ii) If $\varphi > \psi$, then $\theta(t; \theta_0, \lambda, \varphi) > \theta(t; \theta_0, \lambda, \psi)$ for all $t \geq 2\pi$ and all $\theta_0$.

**Proof** Let $\varphi \geq \psi \geq 0$ be fixed. Denote $\theta_1(t) = \theta(t; \theta_0, \lambda, \varphi)$ and $\theta_2(t) = \theta(t; \theta_0, \lambda, \psi)$ and set $\theta(t) = \theta_1(t) - \theta_2(t)$. Then $\theta(0) = 0$ and, for a.e. $t$,

$$
\frac{d\theta(t)}{dt} = \sqrt{\lambda} \left( (\varphi(t) \cos^2 \theta_1 + \sin^2 \theta_1) - (\psi(t) \cos^2 \theta_2 + \sin^2 \theta_2) \right) \\
= \sqrt{\lambda} \left( \psi(t) \cos^2 \theta_1 + \sin^2 \theta_1 - (\psi(t) \cos^2 \theta_2 + \sin^2 \theta_2) \right) \\
+ \sqrt{\lambda} \left( \varphi(t) - \psi(t) \right) \cos^2 \theta_1 \\
= a(t) \theta(t) + b(t),
$$

where

$$
a(t) = \sqrt{\lambda} (1 - \psi(t)) \sin 2\xi, \\
b(t) = \sqrt{\lambda} (\varphi(t) - \psi(t)) \cos^2 \theta_1,
$$

where $\xi = \xi(t)$ is between $\theta_1(t)$ and $\theta_2(t)$. As $b(t) \geq 0$, we have

$$
\theta(t) = \int_0^t b(s) \exp \left( \int_s^t a(\tau) d\tau \right) ds \geq 0
$$

for all $t \geq 0$. This proves (i).

Now assume that $\varphi > \psi$ and $t \geq 2\pi$. We need to prove that $\theta(t) > 0$. Otherwise, assume that $\theta(t_0) = 0$ for some $t_0 \geq 2\pi$. We get from (2.9) that

$$
\int_0^{t_0} b(s) \exp \left( \int_s^{t_0} a(\tau) d\tau \right) ds = 0.
$$

As $b(s) \geq 0$, we see that

$$
b(t) = \sqrt{\lambda} (\varphi(t) - \psi(t)) \cos^2 \theta_1(t) = 0
$$
for a.e. $t \in [0, t_0]$. As $t_0 \geq 2\pi$, we have

$$(\varphi(t) - \psi(t)) \cos^2 \theta_1(t) = 0$$

for a.e. $t \in [0, 2\pi]$. As $\theta_1(t)$ satisfies (2.7), it is not difficult to check from (2.7) that the function $\cos \theta_1(t)$ has only isolated zeros. Thus we get $\varphi(t) = \psi(t)$ for a.e. $t \in [0, 2\pi]$, a contradiction with our assumption $\varphi \overset{L}{>} \psi$. □

**Lemma 2.8** The rotation number function $\rho(\lambda)$ is continuous in $\lambda$ and is nondecreasing when $\lambda$ increases. Furthermore, there exist constants $A_{\pm} > 0$ (independent of $\lambda$) such that

$$A_- \sqrt{\lambda} \leq \rho(\lambda) \leq A_+ \sqrt{\lambda}, \quad \forall \lambda > 0. \tag{2.10}$$

**Proof** As $M_{\lambda}$ is continuously dependent on $\lambda$, thus $\rho(\lambda)$ is continuous in $\lambda$, see Hale [24].

The monotonicity of $\rho(\lambda)$ with respect to $\lambda$ directly follows from Lemma 2.6 (ii).

Now we are going to prove (2.10). Let $\theta(t) = \theta(t; \theta_0, \lambda)$. As

$$\varphi_-(t) \leq \varphi(t) \cos^2 \theta + \sin^2 \theta \leq \varphi_+(t)$$

for all $\theta$ and all $t$, where

$$\varphi_-(t) = \min\{\varphi(t), 1\} \quad \text{and} \quad \varphi_+(t) = \max\{\varphi(t), 1\},$$

we have

$$\sqrt{\lambda} \varphi_-(t) \leq \frac{d\theta}{dt} \leq \sqrt{\lambda} \varphi_+(t) \tag{2.11}$$

for a.e. $t$. We assert that

$$\int_0^{2\pi} \varphi_+ dt \geq \int_0^{2\pi} \varphi_- dt > 0.$$ 

Otherwise, assume that $\varphi_- = 0$. Then

$$\int_0^{2\pi} \varphi_-(t) dt = \int_{\varphi(t) > 1} \varphi(t) dt + \int_{\varphi(t) \leq 1} \varphi(t) dt = 0.$$ 

From $\int_{\varphi(t) > 1} dt = 0$ we know that $\varphi(t) \leq 1$ for a.e. $t \in [0, 2\pi]$. Thus we have

$$\int_{\varphi(t) \leq 1} \varphi(t) dt = \int_0^{2\pi} \varphi(t) dt = 0,$$

a contradiction with $\varphi \overset{L}{>} 0$.

Note that both $\varphi_-(t)$ and $\varphi_+(t)$ are $2\pi$-periodic. We get from (2.11) that

$$\theta(t) \geq \theta_0 + \sqrt{\lambda} \int_0^t \varphi_-(t) dt \geq \theta_0 + \sqrt{\lambda} 2\pi \bar{\varphi}_- [t/2\pi]$$

for all $t \geq 0$, where $[x]$ means the largest integer $\leq x$. Similarly, we have

$$\theta(t) \leq \theta_0 + \sqrt{\lambda} 2\pi \bar{\varphi}_+ ([t/2\pi] + 1)$$

for all $t \geq 0$. Consequently, we may take $A_{\pm} = \bar{\varphi}_{\pm} > 0$. □

The following is a relationship between characteristic multipliers and rotation numbers.
Theorem 2.9 (i) $\mu_{1,2}(\lambda)$ are positive real numbers if and only if $\rho(\lambda) = k/2$ for some even integer $k \in \mathbb{N}$.

(ii) $\mu_{1,2}(\lambda)$ are negative real numbers if and only if $\rho(\lambda) = k/2$ for some odd integer $k \in \mathbb{N}$.

(iii) $\mu_{1,2}(\lambda)$ are conjugate complex numbers if and only if $\rho(\lambda) \neq k/2$ for all $k \in \mathbb{N}$. In this case, we have $\mu_{1,2}(\lambda) = \exp(\pm i2\pi \rho(\lambda))$.

Proof (i) Suppose that $\mu_{1,2}(\lambda)$ are positive real numbers. Then there is $0 \neq v = (v_1, v_2) \in \mathbb{R}^2$ such that $P_\lambda v = \mu_1(\lambda)v$. Denote $(v_1, v_2) = (r_0 \cos \theta_0, r_0 \sin \theta_0)$. Then

$$\theta(2\pi; \theta_0, \lambda) = \theta_0 + 2n_0\pi$$

for some $n_0 \in \mathbb{N}$. Inductively, we can use (2.8) to obtain

$$M_\lambda^n(\theta_0) = \theta_0 + 2nn_0\pi, \quad \forall n \in \mathbb{N}.$$ 

Thus $\rho(\lambda) = \lim_{n \to +\infty} M_\lambda^n(\theta_0)/(2n\pi) = n_0 = (2n_0)/2$.

(ii) It can be proved similarly.

(iii) Now assume that $\mu_{1,2}(\lambda) = \exp(\pm i2\pi \alpha)$ are conjugate complex numbers. Then $P_\lambda$ is the rotation with angle $2\pi \alpha$ in some coordinates. This implies that $\rho(\lambda) = \alpha \mod \mathbb{Z}$. Thus $\mu_{1,2}(\lambda) = \exp(\pm i2\pi \rho(\lambda))$. \qed

Theorem 2.10 $\rho(\lambda)$ is strictly increasing at those $\lambda$ such that $\rho(\lambda) \notin \frac{1}{2}\mathbb{N}$.

Proof The proof is much similar to that in [28]. Suppose that $\lambda_0$ satisfies $\rho(\lambda_0) \notin \frac{1}{2}\mathbb{N}$. Then $\mu_{1,2}(\lambda_0) = \exp(\pm i2\pi \rho(\lambda_0))$ are conjugate complex numbers and $P_{\lambda_0}$ is the rotation with angle $2\pi \rho(\lambda_0)$ in some coordinates.

Case 1: $\rho(\lambda_0) = p/q$ is rational, where $p, q \in \mathbb{N}$ are relatively prime. Then all solutions of (2.7) with $\lambda = \lambda_0$ are periodic solutions of type $(p, q)$, i.e., $\theta(t; \theta_0, \lambda_0)$ satisfies

$$\theta(2q\pi; \theta_0, \lambda_0) - \theta_0 - 2p\pi \equiv 0, \quad \forall \theta_0.$$ 

By Lemma 2.6 (ii), $\theta(2q\pi; \theta_0, \lambda)$ is strictly increasing when $\lambda$ increases. Thus, for any $\lambda \neq \lambda_0$, we have

$$\theta(2q\pi; \theta_0, \lambda) - \theta_0 - 2p\pi \neq 0, \quad \forall \theta_0.$$ 

This implies that (2.7) with $\lambda \neq \lambda_0$ has no periodic solutions of type $(p, q)$. Therefore, $\rho(\lambda) \neq p/q = \rho(\lambda_0)$. As $\rho(\lambda)$ is nondecreasing, we know that $\rho(\lambda) > \rho(\lambda_0)$ if $\lambda > \lambda_0$ and $\rho(\lambda) < \rho(\lambda_0)$ if $\lambda < \lambda_0$. Namely, $\rho(\lambda)$ is strictly increasing at $\lambda_0$.

Case 2: $\rho(\lambda_0)$ is irrational. Then $\{(t, \theta(t; 0, \lambda_0)) : t \in \mathbb{R}\}$ is dense on torus. Therefore there are relatively prime integers $p_i, q_i \in \mathbb{N}$ such that $p_i, q_i \to +\infty$ and

$$\theta(2q_i\pi; 0, \lambda_0) - 2p_i\pi = h_i \to 0$$

as $i \to \infty$. This shows that $\rho(\lambda_0) < p_i/q_i$ for all $i$ because $\rho(\lambda_0)$ is irrational.

On the other hand, for any $\lambda > \lambda_0$,

$$\theta(2q_i\pi; 0, \lambda) - 2p_i\pi = \theta(2q_i\pi; 0, \lambda) - \theta(2q_i\pi; 0, \lambda_0) + h_i \geq \varepsilon_0/2$$
for all $i \gg 1$, where

$$
\varepsilon_0 = \min_{\theta_0} [\theta(2\pi; \theta_0, \lambda) - \theta(2\pi; \theta_0, \lambda_0)] > 0,
$$

because the function $\theta(t; \theta_0, \lambda) - \theta(t; \theta_0, \lambda_0)$ is $2\pi$-periodic in both $t$ and $\theta_0$ for any given $\lambda$ and $\lambda_0$. This shows that $\rho(\lambda) \geq \rho_i/q_i$ for all $i \gg 1$. Therefore $\rho(\lambda) > \rho(\lambda_0)$ if $\lambda > \lambda_0$. Similarly, $\rho(\lambda) < \rho(\lambda_0)$ if $\lambda < \lambda_0$. □

**Remark 2.11** In general, for a family $\{M_\lambda\}$ of circle homeomorphisms, the rotation numbers $\rho(M_\lambda)$ will have phase locking when $\rho(M_\lambda)$ takes rational numbers. This phenomenon will yield the “devil’s staircase” when one plots the graph of $\rho(M_\lambda)$, see Jensen, Bak, and Bohr [26] and Jonker [27]. However, for the family $\{M_\lambda\}$ constructed from system (2.7), we see from Theorem 2.10 that this phenomenon does not happen. Theorem 2.10 shows that phase locking can only occur when the rotation numbers $\rho(\lambda)$ take half integers.

For any $k \in \mathbb{N}$, we know from Lemma 2.8 that the preimage $\rho^{-1}(k/2)$ is a nonempty closed interval of $(0, \infty)$ which may shrink to a single point.

**Definition 2.12** For any given $k \in \mathbb{N}$, denote by $\lambda_k$ and $\lambda_k$ the left and the right endpoint of the interval $\rho^{-1}(k/2)$, respectively. For convenience, we also write $\lambda_0 = \lambda_0 = 0$. Thus we have two sequences $\{\lambda_k : k \in \mathbb{N}\}$ and $\{\lambda_k : k \in \mathbb{N}\}$ and call them the characteristic numbers of system (2.5).

When we want to emphasis their dependence on $\varphi$, we write $\lambda_k = \lambda_k(\varphi)$ and $\lambda_k = \lambda_k(\varphi)$. As $\rho(\lambda)$ is nondecreasing (Lemma 2.8), we have

$$
0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots
$$

From Lemma 2.8, we see that the characteristic numbers $\lambda_k$ and $\lambda_k$ grow quadratically when $k \to \infty$. Namely, we have

**Corollary 2.13** There are constants $B_+ > 0$ such that

$$
B_- k^2 \leq \lambda_k(\varphi) \leq \lambda_k(\varphi) \leq B_+ k^2, \quad \forall k \in \mathbb{N}.
$$

**2.4 Periodic Eigenvalues and Characteristic Numbers**

Now we give a differential equation explanation to characteristic numbers. The relationship between characteristic numbers and eigenvalues of the periodic and the anti-periodic problems will be established.

**Theorem 2.14** (i) When $\lambda > 0$, Eq. (2.1) has nonzero $2\pi$-periodic solutions if and only if either $\lambda = \lambda_k$ or $\lambda = \lambda_k$ for some even integer $k \in \mathbb{N}$.

(ii) Eq. (2.1) has nonzero $2\pi$- anti-periodic solutions if and only if either $\lambda = \lambda_k$ or $\lambda = \lambda_k$ for some odd integer $k \in \mathbb{N}$.
Proof  

(i) Let $\lambda = \lambda_k$ or $\overline{\lambda}_k$ for some even $k \in \mathbb{N}$. Then $\mu_{1,2}(\lambda)$ are positive numbers. We assert that $\mu_{1,2}(\lambda) = 1$. Otherwise, assume that both $\mu_{1,2}(\lambda) \neq 1$. Then there is a neighborhood of $V_\lambda$ of $\lambda$ such that for any $\lambda \in V_\lambda$, $\mu_{1,2}(\lambda)$ are positive real numbers and $\mu_{1,2}(\lambda) \neq 1$. By Theorem 2.9, $\rho(\lambda) = \overline{k}/2$ for some $k \in \mathbb{N}$. As $k$ is continuously dependent on $\lambda$, thus $k \equiv k$ for all $\lambda \in V_\lambda$. As a result, $V_\lambda \subset \rho^{-1}(\overline{k}/2)$ and $\lambda$ cannot be the endpoint of $\rho^{-1}(\overline{k}/2)$. Now the conclusion $\mu_{1,2}(\lambda) = 1$ implies that there is at least one nonzero vector $v \in \mathbb{R}$ such that $P_\lambda v = v$. Thus Eq. (2.5), or equivalently, Eq. (2.1), has at least one nonzero $2\pi$-periodic solution.

Conversely, assume that (2.1) has nonzero $2\pi$-periodic solutions. Then $\mu_{1,2}(\lambda) = 1$ and $\rho(\lambda) = \overline{k}/2$ for some even $k \in \mathbb{N}$. This implies that $\lambda \in [\lambda_k, \overline{\lambda}_k]$. We are going to prove that either $\lambda = \lambda_k$ or $\lambda = \overline{\lambda}_k$. Otherwise, assume that $\lambda_k < \lambda < \overline{\lambda}_k$. Consider the following function

$$G(\hat{\theta}, \dot{\lambda}) = \theta(2\pi; \dot{\theta}, \dot{\lambda}) - \dot{\theta} - k\pi, \quad \dot{\theta} \in \mathbb{R}, \quad \dot{\lambda} \in [\lambda_k, \overline{\lambda}_k].$$

(2.12)

Note that $G(\hat{\theta}, \dot{\lambda})$ is actually $\pi$-periodic in $\dot{\theta}$, because the vector field $\Theta(t; \theta, \dot{\lambda}) = \sqrt{\lambda}(\varphi(t) \cos^2 \theta + \sin^2 \theta)$ is $\pi$-periodic in $\theta$. We have the following facts.

Fact 1: As (2.1) has nonzero $2\pi$-periodic solutions, thus there is $\theta_0 \in [0, \pi)$ such that $G(\theta_0, \lambda) = 0$.

Fact 2: As we have proved that both $\lambda_k$ and $\overline{\lambda}_k$ are eigenvalues of (2.1)+(P), thus there are $\theta_1, \theta_2 \in [0, \pi)$ such that $G(\theta_1, \lambda_k) = 0$ and $G(\theta_2, \overline{\lambda}_k) = 0$. As $G(\hat{\theta}, \dot{\lambda})$ is strictly increasing when $\dot{\lambda}$ increases, thus $\lambda_k < \lambda < \overline{\lambda}_k$ implies that

$$G(\theta_1, \lambda) > G(\theta_1, \lambda_k) = 0 \quad \text{and} \quad G(\theta_2, \lambda) < G(\theta_2, \overline{\lambda}_k) = 0.$$

Since $G(\hat{\theta}, \lambda)$ is $\pi$-periodic in $\dot{\theta}$, we get from Facts 1 and 2 that, besides $\theta_0$, $G(\hat{\theta}, \lambda)$ has another zero $\theta_* \in [0, \pi)$ such that $\theta_* \neq \theta_0$. Thus $P_\lambda$ has two different eigen-directions $(\cos \theta_0, \sin \theta_0)$ and $(\cos \theta_*, \sin \theta_*)$. This implies that $P_\lambda$ is the identity because we have known that $P_\lambda$ has $1$ as its eigenvalue. Consequently, $G(\hat{\theta}, \lambda) \equiv 0$, which contradicts with Fact 2. This contradiction shows that either $\lambda = \lambda_k$ or $\lambda = \overline{\lambda}_k$.

(ii) It can be proved similarly. $\square$

As a result of Theorems 3.4 and 2.14, we see that the characteristic numbers $\lambda_k^P(\varphi)$ and $\overline{\lambda}_k(\varphi)$ are actually the eigenvalues of the periodic and the anti-periodic problems $\lambda_k^P(\varphi)$ and $\lambda_k^A(\varphi)$.

**Corollary 2.15** (i) The eigenvalues $\lambda_k^P(\varphi)$ ($k \geq 0$) of the periodic problem are given by

$$\lambda_k^P(\varphi) = \begin{cases} \lambda_{k+1}(\varphi), & \text{if } k \text{ odd}, \\ \overline{\lambda}_k(\varphi), & \text{if } k \text{ even}. \end{cases}$$

(ii) The eigenvalues $\lambda_k^A(\varphi)$ ($k \geq 1$) of the anti-periodic problem are given by

$$\lambda_k^A(\varphi) = \begin{cases} \lambda_k(\varphi), & \text{if } k \text{ odd}, \\ \overline{\lambda}_{k-1}(\varphi), & \text{if } k \text{ even}. \end{cases}$$

In the next section, we will use eigenvalues of the Dirichlet problems and the Neumann problems to characterize all characteristic numbers $\lambda_k(\varphi)$ and $\overline{\lambda}_k(\varphi)$.
When the parametric resonance and the Arnold tongues are considered in Section 5, we see that it is an important problem to know whether the eigenvalues $\lambda_k^P(\varphi)$ and $\lambda_k^A(\varphi)$ are simple. Now we use differential equations to give an answer to this problem.

**Theorem 2.16**

(i) If $k \in \mathbb{N}$ is even, then $\lambda_k^k(\varphi) = \lambda_k^k(\varphi)$ if and only if all solutions of (2.1) with $\lambda = \Delta_k(\varphi)$ are $T$-periodic.

(ii) If $k \in \mathbb{N}$ is odd, then $\lambda_k^k(\varphi) = \lambda_k^k(\varphi)$ if and only if all solutions of (2.1) with $\lambda = \Delta_k(\varphi)$ are $T$-anti-periodic.

**Proof**

(i) Assume $T = 2\pi$. Let $G(\hat{\theta}, \lambda)$ be as (2.12). If $\lambda_k = \lambda_k =: \lambda$, then the following equation

$$G(\hat{\theta}, \lambda) = \theta(2\pi; \hat{\theta}, \lambda) - \hat{\theta} - k\pi = 0 \quad (2.13)$$

has solutions $\hat{\theta}$ only when $\lambda = \lambda$. Thus, $G(\hat{\theta}, \lambda) < 0$ for all $\hat{\theta}$ if $\lambda < \lambda$, and $G(\hat{\theta}, \lambda) > 0$ for all $\hat{\theta}$ if $\lambda > \lambda$. As $G(\hat{\theta}, \lambda)$ is continuously dependent on $\lambda$, we see that $G(\hat{\theta}, \lambda) \equiv 0$. Thus all solutions of (2.1) are $2\pi$-periodic.

Conversely, if all solutions of (2.1) with $\lambda = \Delta_k$ are $2\pi$-periodic, then $G(\hat{\theta}, \lambda) \equiv 0$ for all $\hat{\theta}$. As $G(\hat{\theta}, \lambda)$ is strictly increasing with respect to $\lambda$ by Lemma 2.6, we see that Eq. (2.13) has solutions $\hat{\theta}$ only when $\lambda = \lambda$. This shows that $\rho(\lambda) \neq \rho(\lambda)$ when $\lambda \neq \lambda$. As a result, $\lambda_k = \lambda_k$.

(ii) It can be proved similarly. □

3 Dirichlet, Neumann Eigenvalues and Characteristic Numbers

In this section, we will discuss the relationship between the Dirichlet, the Neumann eigenvalues and characteristic numbers. It will be proved that all characteristic numbers, in particular all eigenvalues of the periodic problems, can be obtained using either eigenvalues of the Dirichlet problems or the Neumann problems. The comparison results for all four classes of weighted eigenvalues will be proved.

3.1 Dirichlet Eigenvalues and Characteristic Numbers

The following theorem shows that eigenvalues of the Dirichlet problems can be controlled by characteristic numbers.

**Theorem 3.1** For any $k \in \mathbb{N}$, we have

$$\lambda_k(\varphi) \leq \lambda_k^D(\varphi) \leq \lambda_k(\varphi). \quad (3.1)$$

**Proof** We assume $T = 2\pi$. Let $(x(t), y(t))$ be the solution of (2.5) satisfying $(x(0), y(0)) = (0, 1)$. Then $\lambda$ is an eigenvalue of problem (2.1)+(D) if and only if

$$x(2\pi) = 0 \quad \iff P_\lambda(0, 1) = c(0, 1) \quad (c = y(2\pi)) \quad \iff \theta(2\pi; \pi/2, \lambda) = \pi/2 + k\pi \quad \text{for some } k \in \mathbb{N}. \quad (3.2)$$
Note that $\theta(2\pi; \pi/2, 0) = \pi/2$ and
\[ \lim_{\lambda \to +\infty} \theta(2\pi; \pi/2, \lambda) \geq \lim_{\lambda \to +\infty} \left( \pi/2 + 2\pi \phi_\lambda \sqrt{\lambda} \right) = +\infty, \]
see the proof of Lemma 2.8. As $\theta(2\pi; \pi/2, \lambda)$ is strictly increasing with respect to $\lambda$, thus for each $k \in \mathbb{N}$, Eq. (3.2) has a unique solution $\lambda = \lambda_k$. We assert that $\rho(\lambda_k) = k/2$. Thus, $\lambda_k = \lambda_k^D(\varphi)$ and (3.1) is satisfied by the definition of characteristic numbers.

Now we prove the assertion that (3.2) implies that $\rho(\lambda) = k/2$. When $k = 2k'$ is even, (3.2) is
\[ M_\lambda(\pi/2) = \pi/2 + 2k'\pi. \]
Using equality (2.8), we can prove by induction that
\[ M_n^\lambda(\pi/2) = \pi/2 + 2nk'\pi \]
for all $n \in \mathbb{N}$. Thus we have
\[ \rho(\lambda) = \lim_{n \to \infty} \frac{M_n^\lambda(\pi/2)}{2n\pi} = \lim_{n \to \infty} \frac{\pi/2 + 2nk'\pi}{2n\pi} = k' = k/2. \]

When $k = 2k' + 1$ is odd, we consider also the solution $(\tilde{x}(t), \tilde{y}(t))$ of (2.5) satisfying $(\tilde{x}(0), \tilde{y}(0)) = (0, -1)$. As Eq. (2.5) is linear, we see that $x(t) \equiv -\tilde{x}(t)$ and $y(t) \equiv -\tilde{y}(t)$.

This implies that
\[ \frac{d\theta(t; 3\pi/2, \lambda)}{dt} = \frac{d\theta(t; \pi/2, \lambda)}{dt}. \]
Thus (3.2) gives the following two equalities:
\[ M_\lambda(\pi/2) = \pi/2 + (2k' + 1)\pi = 3\pi/2 + 2k'\pi, \]
\[ M_\lambda(3\pi/2) = 3\pi/2 + (2k' + 1)\pi = \pi/2 + 2(k' + 1)\pi. \]
Using these two equalities, we can obtain, again from (2.8), that
\[ M_n^\lambda(\pi/2) = \pi/2 + n(2k' + 1)\pi \]
for all $n \in \mathbb{N}$. Thus $\rho(\lambda) = (2k' + 1)/2 = k/2$. The assertion is proved. □

Conversely, all characteristic numbers can be recovered from eigenvalues of the Dirichlet problems. To this end, for any fixed $s$, let $\varphi_s : \mathbb{R} \to \mathbb{R}$ be defined by
\[ \varphi_s(t) = \varphi(t + s), \quad t \in \mathbb{R}. \]
We have the following important fact.

**Theorem 3.2** For any $s$, we have $\rho_{\varphi_s}(\lambda) = \rho_\varphi(\lambda)$ for all $\lambda$. As a result, $\Delta_{k}(\varphi_s) = \Delta_k(\varphi)$ and $\overline{\lambda}_k(\varphi_s) = \overline{\lambda}_k(\varphi)$ for all $s$ and all $k$. 


Proof Let $s$ be fixed. Denote by $\theta(t; \theta_0, \lambda, \varphi_s)$ the solution of the following equation

$$\frac{d\theta}{dt} = \sqrt{\lambda} (\varphi(t + s) \cos^2 \theta + \sin^2 \theta)$$

(3.3)
satisfying the initial condition: $\theta(0; \theta_0, \lambda, \varphi_s) = \theta_0$.

Let $\hat{\theta}(t) = \theta(t - s; \theta_0, \lambda, \varphi_s)$. Then $\hat{\theta}(t)$ satisfies Eq. (2.7) and the initial condition: $\hat{\theta}(0) = \theta(-s; \theta_0, \lambda, \varphi_s)$. Thus $\theta(t) \equiv \theta(t; -s; \theta_0, \lambda, \varphi_s, \lambda, \varphi)$. Namely, we have the following equality:

$$\theta(t; \theta(-s; \theta_0, \lambda, \varphi_s), \lambda, \varphi) \equiv \theta(t - s; \theta_0, \lambda, \varphi_s)$$

for all $t, s, \theta_0$, and $\lambda$. Hence

$$\rho_\varphi(\lambda) = \lim_{t \to +\infty} \frac{\theta(t; \theta(-s; \theta_0, \lambda, \varphi_s), \lambda, \varphi)}{t} = \lim_{t \to +\infty} \frac{\theta(t - s; \theta_0, \lambda, \varphi_s)}{t} = \rho_{\varphi_s}(\lambda),$$

because the definition of the rotation numbers are independent of the initial points. □

Now we have the following characterization of characteristic numbers using eigenvalues of the Dirichlet problems.

Theorem 3.3 (i) For any $s$, we have

$$\underline{\lambda}_k(\varphi) \leq \lambda_k^D(\varphi_s) \leq \overline{\lambda}_k(\varphi), \quad k \in \mathbb{N}.$$  

(ii) For any $k \in \mathbb{N}$, we have

$$\underline{\lambda}_k(\varphi) = \min_s \lambda_k^D(\varphi_s), \quad (3.4)$$

$$\overline{\lambda}_k(\varphi) = \max_s \lambda_k^D(\varphi_s). \quad (3.5)$$

Proof (i) It directly follows from Theorems 3.1 and 3.2.

(ii) Let us first consider the case that $k \in \mathbb{N}$ is even. By Theorem 2.14 (i), $\lambda = \underline{\lambda}_k(\varphi)$ is an eigenvalue of the periodic problem, i.e., there is a nonzero $T$-periodic function $x(\cdot)$ satisfying (2.1). As a result, $x(\cdot)$ satisfies $\int_0^T \varphi(t)x(t)dt = 0$. This implies that $x(s_1) = 0$ for some $s_1$. Now the nonzero function $y(t) := x(t + s_1)$ satisfies

$$\ddot{y}(t) + \lambda \varphi(t + s_1)y(t) = 0$$

(3.6)

and the Dirichlet boundary conditions:

$$y(0) = y(T) = x(s_1) = 0.$$

Thus $\underline{\lambda}_k(\varphi) = \lambda = \lambda_k^D(\varphi_{s_1})$. This shows that (3.4) holds and the minimum can be attained. Similarly, (3.5) also holds.

Now we consider the case that $k \in \mathbb{N}$ is odd. By Theorem 2.14 (ii), Eq. (2.1) with $\lambda = \underline{\lambda}_k(\varphi)$ has a nonzero $T$-anti-periodic solution $x(\cdot)$. As $\varphi(t)$ is $T$-periodic, $x(\cdot)$ must satisfy $x(t + T) \equiv -x(t)$. Since $x(0) = -x(T)$ implies that $x(s_1) = 0$ for some $s_1 \in [0, T]$, we know once again that $y(t) := x(t + s_1)$ satisfies Eq. (3.6) and the Dirichlet boundary conditions:

$$y(0) = x(s_1) = 0 \quad \text{and} \quad y(T) = x(T + s_1) = -x(s_1) = 0.$$
Now we can proceed the proof as before. □

In the following theorem, we consider dependence of characteristic numbers on weight functions \( \varphi \). In particular, comparison results for eigenvalues of the periodic problems also hold.

**Theorem 3.4**

(i) When \( \varphi(t) \equiv 1 \), then \( \lambda_k(1) = \overline{\lambda}_k(1) = (k\pi/T)^2 \) for all \( k \in \mathbb{N} \).

(ii) \( \lambda_k(c \varphi) = c^{-1} \lambda_k(\varphi) \) and \( \overline{\lambda}_k(c \varphi) = c^{-1} \overline{\lambda}_k(\varphi) \) for all \( c > 0 \) and all \( k \in \mathbb{N} \).

(iii) Assume that \( \varphi, \psi \in L^1(0, T) \) with \( \varphi \geq 0 \) and \( \psi \geq 0 \). Then \( \lambda_k(\varphi) \leq \lambda_k(\psi) \) and \( \overline{\lambda}_k(\varphi) \leq \overline{\lambda}_k(\psi) \) for all \( k \in \mathbb{N} \) if \( \varphi \geq \psi \). Moreover, \( \lambda_k(\varphi) < \lambda_k(\psi) \) and \( \overline{\lambda}_k(\varphi) < \overline{\lambda}_k(\psi) \) for all \( k \in \mathbb{N} \) if \( \varphi > \psi \).

**Proof**

These properties can be obtained from the corresponding properties for eigenvalues of the Dirichlet problems in Proposition 2.2 and from the characterization of characteristic numbers in Theorem 3.3.

We give the proof for comparison results (iii). If \( \varphi \geq \psi \), we know from Lemma 2.7 (i) that \( \rho_\varphi(\lambda) \geq \rho_\psi(\lambda) \) for all \( \lambda > 0 \). Thus

\[
\overline{\lambda}_k(\varphi) = \min\{\lambda : \rho_\varphi(\lambda) = k/2\} \leq \min\{\lambda : \rho_\psi(\lambda) = k/2\} = \overline{\lambda}_k(\psi).
\]

Similarly, \( \lambda_k(\varphi) \leq \lambda_k(\psi) \).

When \( \varphi > \psi \), we do not know whether the strict inequality \( \rho_\varphi(\lambda) > \rho_\psi(\lambda) \) holds for all \( \lambda > 0 \). However, \( \lambda = \lambda_k^D(\varphi) \) and \( \hat{\lambda} = \lambda_k^D(\psi) \) are determined by

\[
\theta(2\pi; \pi/2, \hat{\lambda}, \varphi) = \pi/2 + k\pi
\]

and

\[
\theta(2\pi; \pi/2, \hat{\lambda}, \psi) = \pi/2 + k\pi,
\]

respectively, see the proof of Theorem 3.1. By Lemma 2.7 (ii), we know that if \( \varphi \geq \psi \) then

\[
\theta(2\pi; \pi/2, \lambda, \varphi) > \theta(2\pi; \pi/2, \lambda, \psi), \quad \forall \lambda > 0.
\]

This implies \( \lambda_k^D(\varphi) = \hat{\lambda} < \lambda_k^D(\psi) \). As \( \varphi > \psi \) implies \( \varphi_s \geq \psi_s \) for all \( s \), we have, from what we have just proved,

\[
\lambda_k^D(\varphi_s) < \lambda_k^D(\psi_s)
\]

for all \( s \). Now the strict inequalities \( \lambda_k(\varphi) < \lambda_k(\psi) \) and \( \overline{\lambda}_k(\varphi) < \overline{\lambda}_k(\psi) \) follow from Theorem 3.3 (ii). □

In Theorem 6.2 of Section 6, a lower bound for the first characteristic number \( \lambda_1(\varphi) = \lambda_1^D(\varphi) \), which is useful in the stability problems we will consider in Section 5.

### 3.2 Neumann Eigenvalues and Characteristic Numbers

Another important eigenvalue problem is the Neumann problem \((2.1)+(N)\). As in the proof of Theorem 3.1 for the Dirichlet problem, eigenvalues of the Neumann problem \((2.1)+(N)\) are determined by

\[
\theta(2\pi; 0, \lambda) = k\pi, \quad k \in \mathbb{Z}^+.
\]
Thus problem (2.1)+(N) has a sequence of eigenvalues
\[ 0 = \lambda_0^N(\varphi) < \lambda_1^N(\varphi) < \cdots < \lambda_k^N(\varphi) < \cdots \]
All eigenvalues \(\lambda_k^D(\varphi)\) are simple and \(\lim_{k \to \infty} \lambda_k^N(\varphi) = +\infty\).

Similarly we can also use eigenvalues of the Neumann problems to characterize all characteristic numbers.

**Theorem 3.5** (i) For any \(k \in \mathbb{N}\), we have
\[ \underbrace{\lambda_k(\varphi)}_{\Delta_k(\varphi)} \leq \lambda_k^N(\varphi) \leq \overbrace{\lambda_k(\varphi)}^{\Lambda_k(\varphi)}. \]
(ii) For any \(k \in \mathbb{N}\), we have
\[ \Delta_k(\varphi) = \min_s \lambda_k^N(\varphi_s), \quad \Lambda_k(\varphi) = \max_s \lambda_k^N(\varphi_s). \]

**Proof** It is similar to that for Theorem 3.3. For the proof of (ii), we need only to notice that any periodic or anti-periodic solution \(x(\cdot)\) also satisfies \(\dot{x}(s_1) = 0\) for some \(s_1\). □

As in Theorem 3.4, we can get from Lemma 2.7, Theorem 3.5 and Eq. (3.7) the following comparison results.

**Theorem 3.6** (i) When \(\varphi(t) \equiv 1\), then we obtain the usual eigenvalues \(\lambda_k^N(1) = (k\pi/T)^2\) for all \(k \geq 0\).
(ii) \(\lambda_k^N(c \varphi) = c^{-1}\lambda_k^N(\varphi)\) for all \(c > 0\) and all \(k \in \mathbb{N}\).
(iii) Assume that \(\varphi, \psi \in L^1(0,T)\) with \(\varphi \geq 0\) and \(\psi \geq 0\). Then \(\lambda_k^N(\varphi) \leq \lambda_k^N(\psi)\) for all \(k \in \mathbb{N}\) if \(\varphi \geq \psi\). Moreover, \(\lambda_k^N(\varphi) < \lambda_k^N(\psi)\) for all \(k \in \mathbb{N}\) if \(\varphi > \psi\).

### 4 Nonuniform Nonresonance of Nonlinear Differential Equations

In this section, we use the weighted eigenvalues to give some nonuniform nonresonance results for nonlinear nonautonomous differential equations.

#### 4.1 Nonresonance of Periodic Problems

We consider the nonresonance of the following nonlinear differential equation
\[ \ddot{x} + f(t, x) = 0 \quad (4.1) \]
with respect to the periodic boundary conditions (P), where \(f(t, x) (\equiv f(t+T, x)) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is an \(L^1\)-Carathéodory function and satisfies the following semilinearity condition: There exist \(a(\cdot), b(\cdot) \in L^1(0,T)\) such that
\[ a(t) \leq \liminf_{|x| \to \infty} f(t, x)/x \leq \limsup_{|x| \to \infty} f(t, x)/x \leq b(t) \quad (4.2) \]
uniformly in a.e. \( t \in [0, T] \).

In [45], a nonuniform nonresonance result between the first two eigenvalues of the periodic problems is proved. Namely, we have

**Theorem 4.1** [45] Assume that functions \( a(\cdot) \) and \( b(\cdot) \) in (4.2) satisfy that \( b \geq L^1(0, T) \) and \( a(\cdot) \) is not necessarily \( a \geq 0 \). If

\[
0 < \bar{a} \quad \text{and} \quad 1 < \lambda_1^P(b),
\]

then Eq. (4.1) has at least one \( T \)-periodic solution. Actually, condition (4.3) also guarantees that for any continuous function \( g : \mathbb{R} \to \mathbb{R} \), the following damped equation

\[
\ddot{x} + g(x)\dot{x} + f(t, x) = 0
\]

also has at least one \( T \)-periodic solution.

Now we give nonuniform nonresonance results for Eq. (4.1) in other nonresonance straps.

**Theorem 4.2** Assume that \( a(\cdot) \) and \( b(\cdot) \) in (4.2) satisfy \( b \geq a \geq 0 \). If

\[
\lambda_k^P(a) < 1 < \lambda_{k+1}^P(b)
\]

for some \( k \in \mathbb{N} \), then Eq. (4.1) has at least one \( T \)-periodic solution.

**Proof** As \( b \geq a \), we know from Theorem 3.4 that \( \lambda_k^P(b) \leq \lambda_k^P(a) < 1 \) by condition (4.5). Thus condition (4.5) implies that \( 1 \neq \lambda_n^P(b) \) for all \( n \in \mathbb{N} \), i.e., the following equation

\[
\ddot{x} + b(t)x = 0
\]

has only the trivial \( T \)-periodic solution.

Now we can use coincidence degree (cf, Mawhin [30]) to deform Eq. (4.1) to Eq. (4.6). Due to the Property \( P \) introduced in [15] and [22], and a general result in [42] on positively homogeneous perturbations of the identity on Banach spaces, the existence to (4.1)+(P) can be obtained if one can check that for any \( c \in L^1(0, T) \) with

\[
a(t) \leq c(t) \leq b(t)
\]

the following linear equation

\[
\ddot{x} + c(t)x = 0
\]

has only the trivial \( T \)-periodic solution. This is equivalent to that \( \lambda_k^P(c) \neq 1 \) for all \( n \in \mathbb{N} \). From the comparison results for eigenvalues, we get from (4.5) and (4.7) that

\[
\lambda_k^P(c) \leq \lambda_k^P(a) < 1
\]

and

\[
\lambda_{k+1}^P(c) \geq \lambda_{k+1}^P(b) > 1.
\]

Thus \( \lambda_n^P(c) \neq 1 \) for all \( n \in \mathbb{N} \). Now the existence to (4.1)+(P) follows from Theorem 5.1 of [42]. □
Remark 4.3 (i) Assume that \( a(t) \equiv b(t) \), which means that \( f(t, x) \) is asymptotically linear:

\[
\lim_{|x| \to \infty} \frac{f(t, x)}{x} = b(t).
\]

Now conditions (4.5) mean that \( \lambda_n^P(b) \neq 1 \) for all \( n \in \mathbb{N} \).

(ii) By Theorem 3.4 (iii), we know that conditions (4.5) are the generalization of the following well-known nonuniform nonresonance conditions: There exists some \( k \in \mathbb{N} \) such that

\[
(2k\pi/T)^2 \frac{L}{\lambda} < a \leq b \leq (2(k+1)\pi/T)^2.
\]

(4.8)

Thus conditions (4.5) are the right generalization of conditions (1.5) to nonautonomous differential equations, because conditions (4.5) have overcome all disadvantages described in Section 1.

The nonuniform nonresonance conditions (4.3) and (4.5) are necessary in the following sense.

**Theorem 4.4** Let \( b \geq a > 0 \). If Eq. (4.1) has at least one \( T \)-periodic solution for any \( L^1 \)-Carathéodory function satisfying the semilinearity condition (4.2), then \( a(\cdot) \) and \( b(\cdot) \) satisfy

\[
\lambda_k^P(a) < 1 < \lambda_k^P(b) \quad (4.9)
\]

for some \( k \in \mathbb{N} \).

**Proof** If \( \lambda_k^P(b) > 1 \), then (4.9) is satisfied for \( k = 1 \). Now we assume that \( \lambda_k^P(b) \leq 1 \). Let \( k_0 = \min \{ k : \lambda_k^P(b) > 1 \} \). Then \( k_0 \geq 2 \). By our assumption, we know that for any \( c \in L^1(0, T) \) satisfying (4.7) and any \( h \in L^1(0, T) \), the following equation

\[
\ddot{x} + c(t)x = h(t)
\]

has at least one \( T \)-periodic solution. Thus \( c \) satisfies \( \lambda_n^P(c) \neq 1 \) for all \( n \). In particular, we have

\[
\Lambda_k(\tau) := \lambda_k^P(\tau a + (1 - \tau)b) \neq 1
\]

for all \( k \in \mathbb{N} \) and all \( \tau \in [0, 1] \). Since \( \Lambda_{k_0-1}(\tau) \) is continuous in \( \tau \) and \( \Lambda_{k_0-1}(0) = \lambda_{k_0-1}^P(b) \leq 1 \), we must have \( \lambda_{k_0-1}^P(b) < 1 \) and \( \Lambda_{k_0-1}(1) = \lambda_{k_0-1}^P(a) < 1 \). Namely, (4.9) is satisfied for \( k = k_0 \).

**Example 4.5** Let us consider the existence of \( 2\pi \)-periodic solutions of the following nonlinear equation

\[
\ddot{x} + \lambda ((1 + \cos t) \cos^2 x + (1.1 + \cos t) \sin^2 x) x = h(t),
\]

(4.10)

where the parameter \( \lambda > 0 \). For Eq. (4.10), we have

\[
a(t) = \lambda(1 + \cos t) \quad \text{ and } \quad b(t) = \lambda(1.1 + \cos t).
\]

As \( b \leq 2.1\lambda \), we see that the classical nonresonance result between the first two eigenvalues 0 and 1 gives the existence of \( 2\pi \)-periodic solutions to (4.10) when

\[
0 < \lambda \leq 1/2.1 \approx 0.47620
\]

28
The other classical nonresonance conditions (4.8) give no any results to Eq. (4.10). However, we can get from (4.3) and (4.5) the following nonresonance intervals of Eq. (4.10):
\[ \lambda \in I_k := (\lambda_{k-1}^P(1 + \cos t), \lambda_k^P(1.1 + \cos t)) \]
for \( k \in \mathbb{N} \). Numerically, we have
\[ I_1 = (0.00000, 0.85446) \]
\[ I_2 = (0.92911, 1.29272) \]
\[ I_3 = (1.51954, 3.89852) \]
\[ I_4 = (4.31580, 4.64281) \]
\[ I_5 = (5.52400, 9.09903) \]

4.2 Nonresonance of Other Problems

We may obtain similar nonresonance results for other boundary value problems.

**Theorem 4.6** Assume that \( f(t,x) \) satisfies (4.2). Then we have
(i) if \( b^L > 0 \) and \( \lambda_A^1(b) > 1 \), then Eq. (4.1)+(A) has at least one solution.
(ii) if \( b \geq a^L > 0 \) and \( \lambda_A^k(a) < 1 < \lambda_A^{k+1}(b) \)
for some \( k \in \mathbb{N} \), then Eq. (4.1)+(A) has at least one solution.

**Theorem 4.7** Assume that \( f(t,x) \) satisfies (4.2). Then we have
(i) if \( \bar{a} > 0 \) and \( b^L > 0 \), \( \lambda_N^1(b) > 1 \), then Eq. (4.1)+(N) has at least one solution.
(ii) if \( b \geq a^L > 0 \) and \( \lambda_N^k(a) < 1 < \lambda_N^{k+1}(b) \)
for some \( k \in \mathbb{N} \), then Eq. (4.1)+(N) has at least one solution.

5 Stability and Parametric Resonance for Hill’s Equations

As a direct application of characteristic numbers \( \lambda_k^P(\varphi) \) and \( \lambda_k^H(\varphi) \), we will consider stability and instability of the Hill’s equation:
\[ \ddot{x} + \varphi(t)x = 0, \quad (5.1) \]
where \( \varphi : \mathbb{R} \to \mathbb{R} \) is \( T \)-periodic such that \( \varphi^L > 0 \).

5.1 Stability of Hill’s Equations

The stability of the Hill’s equations is a basic and an important problem in theory of ordinary differential equations and Hamiltonian systems. A lot of theories and results for this problem has been developed in many textbooks such as [1] and [24]. We mention that the theory of
Hale [24] is very successful for the stability problems of Eq. (5.1), although his proof is less geometrical and is based on entire function theory, see Section 1 and the discussion in Section 7. We see from the dynamics approach in Section 2 that the introducing of characteristic numbers, or the weighted eigenvalues of the periodic and the anti-periodic problems, is very geometrical and is easily understood. Meanwhile, their role in stability problems of (5.1) is also clear and is given below.

Let us first mention a classical result of Borg [4] on the first stability region of (5.1), see also [24].

**Proposition 5.1** [24] Let \( \varphi \) be a \( T \)-periodic function such that \( \varphi(t) \not\equiv 0 \) and \( \int_0^T \varphi(t)dt \geq 0 \). Then Eq. (5.1) is stable if
\[
\int_0^T |\varphi(t)|dt \leq \frac{4}{T}.
\] (5.2)

The stability can be explained using characteristic multipliers. Let \( P : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( \mu_{1,2} \) be the Poincaré map and characteristic multipliers of the following planar system:
\[
\dot{x} = -y, \quad \dot{y} = \varphi(t)x.
\]
Then we have the following stability results.

**Proposition 5.2** [1] (i) If \( |\text{tr } P| = |\mu_1 + \mu_2| < 2 \), then (5.1) is stable.
(ii) If \( |\text{tr } P| > 2 \), then (5.1) is unstable.

As we have a relationship between rotation numbers and characteristic multipliers (see Theorem 2.9), now we can use \( \lambda_k \) and \( \bar{\lambda}_k \) to obtain stability of Eq. (5.1). We introduce a parameter \( \lambda > 0 \) in the Hill’s equation and consider the following equation:
\[
\ddot{x} + \lambda \varphi(t)x = 0,
\] (5.3)
where \( \varphi \) is as in Section 2.

**Theorem 5.3** (i) For any \( k \in \mathbb{N} \), Eq. (5.3) is stable for any \( \lambda \in (\bar{\lambda}_{k-1}(\varphi), \lambda_k(\varphi)) \).
(ii) Let \( k \in \mathbb{N} \) be such that \( \lambda_k(\varphi) < \lambda_k(\varphi) \). Then Eq. (5.3) is unstable for any \( \lambda \in (\lambda_k(\varphi), \bar{\lambda}_k(\varphi)) \).
(iii) For any \( k \in \mathbb{N} \), Eq. (5.3) with \( \lambda = \lambda_k(\varphi) \) or \( \lambda = \bar{\lambda}_k(\varphi) \) is stable if and only if \( \lambda_k(\varphi) = \bar{\lambda}_k(\varphi) \).

**Proof** (i) If \( \lambda \in (\bar{\lambda}_{k-1}, \lambda_k) \), then \((k-1)/2 < \rho(\lambda) < k/2\) by the definition of characteristic numbers. From Theorem 2.9, \( \mu_{1,2}(\lambda) = \exp(\pm 2\pi \rho(\lambda)) \) are conjugate complex numbers. Thus
\[
|\text{tr } P_\lambda| = |\exp(i2\pi \rho(\lambda)) + \exp(-i2\pi \rho(\lambda))| = |2\cos(2\pi \rho(\lambda))| < 2.
\]
Consequently, Eq. (5.3) is stable.
(ii) Assume that \( \lambda_k < \bar{\lambda}_k \). Then, for any \( \lambda \in (\lambda_k, \bar{\lambda}_k) \), we have \( \rho(\lambda) = k/2 \). Thus \( \mu_{1,2}(\lambda) \) are real numbers. If \( k \) is even, then \( \mu_{1,2}(\lambda) > 0 \). As \( \lambda \in (\lambda_k, \bar{\lambda}_k) \) is not an eigenvalue of the periodic problem, we know that \( \mu_{1,2}(\lambda) \neq 1 \). Therefore,
\[
\text{tr } P_\lambda = \mu_1(\lambda) + \mu_2(\lambda) = \mu_1(\lambda) + \mu_1(\lambda)^{-1} > 2.
\]
Thus, (5.3) is unstable. When $k$ is odd, we have $\text{tr} P_\lambda < -2$ and Eq. (5.3) is also unstable.

(iii) $\lambda_k(\varphi) = \overline{\lambda}_k(\varphi) = \lambda$ has multiplicity 2 is equivalent to that all solutions of Eq. (5.3) are periodic (when $k$ is even) or are anti-periodic (when $k$ is odd). This means that either $P_\lambda = I$ or $P_\lambda = -I$. Thus Eq. (5.3) is stable. □

Remark 5.4

(i) Theorem 5.3 gives a clear scenario for stability of Eq. (5.3) when parameter $\lambda$ ranges over $(0, \infty)$. More precisely, for any $k \in \mathbb{N}$, let

$$I_k = (\lambda_{k-1}(\varphi), \lambda_k(\varphi)) \quad \text{and} \quad J_k = (\lambda_k(\varphi), \overline{\lambda}_k(\varphi)).$$

Then we have

$$(0, \infty) \setminus \bigcup_{k \in \mathbb{N}} \{\lambda_k(\varphi), \overline{\lambda}_k(\varphi)\} = \bigcup_{k \in \mathbb{N}} (I_k \cup J_k).$$

Theorem 5.3 says that (5.3) is stable if $\lambda \in I_k$, and (5.3) is unstable if $\lambda \in J_k$. Clearly, for any given $k \in \mathbb{N}$, the appearance of unstable interval $J_k$ is equivalent to that $\lambda_k < \overline{\lambda}_k$. Thus it is an important problem to know whether eigenvalues $\lambda_k^P$ and $\lambda_k^A$ are multiple (cf. Theorem 2.16).

(ii) Using Theorem 2.9, Theorem 5.3 also gives a clear picture for change of characteristic multipliers of (5.3) when $\lambda$ runs from 0 to infinity.

As a result of Theorem 5.3, we can obtain a stability criteria for Hill’s equation (5.1).

Corollary 5.5

Eq. (5.1) is stable if $1 \in I_k$ for some $k \in \mathbb{N}$.

Using the lower bounds (6.3) for $\lambda_1(\varphi)$ in the next section, we see that the classical stability condition (5.2) implies that 1 is in the first stability interval $I_1$. More general, we have

Theorem 5.6

Assume that $\varphi \geq 0$ is in $L^\alpha(0, T)$ for some $1 \leq \alpha \leq \infty$. Then Eq. (5.1) is stable when

$$\|\varphi\|_\alpha < K_D(2\alpha^*), \quad \text{if} \quad 1 < \alpha \leq \infty, \quad (5.4)$$

or

$$\|\varphi\|_\alpha \leq K_D(\infty) = 4/T, \quad \text{if} \quad \alpha = 1, \quad (5.5)$$

where $K_D(\beta)$ are some best Sobolev constants which are explicitly given in (6.2).

Proof  By Theorem 6.2, both condition (5.4) and condition (5.5) imply that $\lambda_1(\varphi) > 1$, i.e., 1 is in the first stability interval $I_1$. Thus (5.1) is stable. Hence condition (5.4) is a generalization of the classical stability condition (5.2) by using $L^\alpha$ norms of $\varphi$. □

Remark 5.7 When $1 < \alpha \leq \infty$, the constant $K_D(2\alpha^*)$ in (5.4) is best possible to guarantee that 1 is in the first stability interval, see Remark 6.3 in the next section.
5.2 Parametric Resonance and Arnold Tongues

Now we consider parametric resonance. Let $\{\varphi_\varepsilon(t) : \varepsilon \in [-1, 1]\}$ be a family of $T$-periodic functions such that $\varphi_\varepsilon \not\equiv 0$ for each $\varepsilon$. We consider stable and unstable regions of the following family of Hill’s equations

$$\ddot{x} + \lambda \varphi_\varepsilon(t)x = 0, \quad (5.6)$$

where the parameter $\lambda \geq 0$. A typical example is

$$\varphi_\varepsilon(t) = 1 + \varepsilon \cos t, \quad \varepsilon \in [-1, 1], \quad (5.7)$$

by taking $T = 2\pi$.

For the family (5.7), when $\varepsilon = 0$, $\varphi_\varepsilon(t) \equiv 1$ is constant and $\Lambda_k(\varphi_\varepsilon) = \overline{\Lambda}_k(\varphi_\varepsilon) = (k/2)^2$. Observed by Arnold [1], when $0 < |\varepsilon| \ll 1$ is fixed, for each $k \in \mathbb{N}$, a very small unstable interval $\lambda \in (\alpha_k(\varepsilon), \beta_k(\varepsilon))$, which is near $(k/2)^2$, of (5.6) will appear. This is called parametric resonance, see Section 25 of [1]. Such a phenomenon is currently called the Arnold tongues when one plots unstable regions $\alpha_k(\varepsilon) < \lambda < \beta_k(\varepsilon)$ in the $(\lambda, \varepsilon)$ plane. It can be explained using the phase locking for the families of circle homeomorphisms such as

$$f_{\mu,\varepsilon}(x) = x + \mu + \varepsilon \sin x, \quad x \in S^1.$$  

The numerical simulation and the theoretical explanation to Arnold tongues are extensively considered, see, for example, [26] and [27].

Now we use characteristic numbers to give a complete description for the appearance of the Arnold tongues. As a result of Theorem 5.3, we have

**Theorem 5.8** Let $\varphi_\varepsilon$ be a family such that $\varphi_\varepsilon \not\equiv 0$ for each $\varepsilon$. Then the region of unstable parameters $(\lambda, \varepsilon)$ of system (5.6) is given by

$$\Lambda_k(\varphi_\varepsilon) < \lambda < \overline{\Lambda}_k(\varphi_\varepsilon), \quad (5.8)$$

if

$$\Lambda_k(\varphi_\varepsilon) < \overline{\Lambda}_k(\varphi_\varepsilon) \quad (5.9)$$

holds.

For the family (5.7), conditions (5.9) do hold for all $k \in \mathbb{N}$ when $\varepsilon \neq 0$. Actually, we can use the Fourier series and eigenvalue problems of infinite matrixes to prove the following results on periodic and anti-periodic solutions of the Mathieu’s equation

$$\ddot{x} + (\alpha + \beta \cos t)x = 0, \quad (5.10)$$

where $\alpha, \beta$ are constants.

**Proposition 5.9** (i) All solutions of Eq. (5.10) are $2\pi$-periodic if and only if $\beta = 0$ and $\alpha = (k/2)^2$ for some even integer $k \in \mathbb{N}$.

(ii) All solutions of Eq. (5.10) are $2\pi$-anti-periodic if and only if $\beta = 0$ and $\alpha = (k/2)^2$ for some odd integer $k \in \mathbb{N}$.
Now we have the following explanation to the appearance of the Arnold tongues.

For the family (5.7) and any given $k \in \mathbb{N}$, $\lambda_k(\varphi_\varepsilon) = \frac{k}{2}$ have multiplicity 2 when $\varepsilon = 0$. As $\varepsilon$ evolves, the characteristic numbers $\lambda_k(\varphi_\varepsilon)$ and $\lambda_k(\varphi_\varepsilon)$ split into two simple characteristic numbers, and an unstable interval $\lambda \in (\lambda_k(\varphi_\varepsilon), \lambda_k(\varphi_\varepsilon))$ appears. This gives an Arnold tongue. Now the $k^{th}$ Arnold tongue is given by (5.8), which is bounded by the following two eigenvalue curves in $(\lambda, \varepsilon)$ plane:

$$\lambda = \lambda_k(\varphi_\varepsilon) \quad \text{and} \quad \lambda = \lambda_k(\varphi_\varepsilon).$$

The width of the $k^{th}$ Arnold tongue, for any fixed $\varepsilon$, is

$$W_k(\varepsilon) = \lambda_k(\varphi_\varepsilon) - \lambda_k(\varphi_\varepsilon),$$

which is the gap of the periodic eigenvalues $\lambda_k^p(\varphi_\varepsilon)$ and $\lambda_k^{p-1}(\varphi_\varepsilon)$ when $k$ is even, or the gap of the anti-periodic eigenvalues $\lambda_k^{A+1}(\varphi_\varepsilon)$ and $\lambda_k^{A}(\varphi_\varepsilon)$ when $k$ is odd. It is observed in [1] that

$$W_k(\varepsilon) = O(|\varepsilon|^k) \quad \text{when} \quad |\varepsilon| \ll 1.$$

Thus, the Arnold tongues are not in practice observed when $k$ is large and $\varepsilon$ is small. However, when $\varepsilon$ is near $\pm 1$, the width $W_k(\varepsilon)$ will become large if $k$ is large, see the following figures. Thus the Arnold tongues will be easily observed in practice for any $k \in \mathbb{N}$ when $\varepsilon \approx \pm 1$.

Note that $\varphi_\varepsilon(t) = 1 + \varepsilon \cos t$ is even in $t$. We give a result concerning periodic solutions of (5.1) when $\varphi(t)$ is even.

**Proposition 5.10** Assume that $\psi(t)$ is $T$-periodic and is even in $t$. Then the following equation

$$\ddot{x} + \psi(t)x = 0 \quad (5.12)$$

has nonzero $T$-periodic (anti-periodic) solutions if and only if (5.12) has some nonzero even or odd $T$-periodic (anti-periodic) solutions.

**Proof** Let $x(t)$ be a nonzero $T$-periodic (anti-periodic) solution of (5.12). As $\psi(t)$ is even in $t$, we know that $x(-t)$ is also a solution of (5.12). As a result, $y_\pm(t) = x(t) \pm x(-t)$ are also $T$-periodic (anti-periodic) solutions of (5.12). Since $x(\cdot)$ is nonzero, we know that at least one of $y_\pm(\cdot)$ is nonzero, which is even or odd. □

As odd and even function satisfies the Dirichlet and the Neumann boundary conditions respectively, we have

**Corollary 5.11** Let $\varphi$ be $T$-periodic and be even in $t$ and $\varphi \not\equiv 0$. Then characteristic numbers coincide with eigenvalues of the Dirichlet and the Neumann problems, i.e.,

$$\{\lambda_k(\varphi), \lambda_k(\varphi)\} = \{\lambda_k^D(\varphi), \lambda_k^N(\varphi)\}$$

for any $k \in \mathbb{N}$.
For the family (5.7), the corresponding Arnold tongues have the following two features.

(i) As
\[ \varphi_{-\varepsilon,s}(t) = 1 - \varepsilon \cos(t + s) \equiv 1 + \varepsilon \cos(t + s + \pi) = \varphi_{\varepsilon,s+\pi}(t), \]
thus we know from Theorem 3.3 that
\[ \lambda_k(\varphi_{\varepsilon,s}) = \lambda_k(\varphi_{\varepsilon}) \quad \text{and} \quad \overline{\lambda}_k(\varphi_{-\varepsilon}) = \overline{\lambda}_k(\varphi_{\varepsilon}) \]
for all \( \varepsilon \). This means that all Arnold tongues are symmetric with respect to the \( \lambda \)-axis.

(ii) As \( \varphi_{\varepsilon}(t) = 1 + \varepsilon \cos t \) is even in \( t \), we know from Corollary 5.11 that the curves
\[ \lambda = \lambda^D_k(\varphi_{\varepsilon}) \quad \text{and} \quad \lambda = \lambda^N_k(\varphi_{\varepsilon}) \]
of eigenvalues of the Dirichlet and the Neumann problems coincide with boundary curves (5.11) of the Arnold tongues.

¿From Theorem 5.8, we see that the Arnold tongues will occur for any family \( \varphi_{\varepsilon} \) such that conditions (5.9) are satisfied.

Example 5.12 Instead of the family \( \varphi_{\varepsilon}(t) = 1 + \varepsilon \cos t \), we consider the family \( \psi_{\varepsilon}(t) = 1 + \varepsilon \sin t, \varepsilon \in [-1,1] \). Observe that
\[ \psi_{\varepsilon,s}(t) = 1 + \varepsilon \sin(t + s) \equiv 1 + \varepsilon \cos(t + s - \pi/2) = \varphi_{\varepsilon,s-\pi/2}(t). \]
By Theorem 3.3, the Arnold tongues for the family \( \psi_{\varepsilon} \) are completely same as that for the family (5.7). However, the curves of eigenvalues
\[ \lambda = \lambda^D_k(\psi_{\varepsilon}) \quad \text{and} \quad \lambda = \lambda^N_k(\psi_{\varepsilon}) \]
are strictly inside the tongues, which mean that eigenvalues \( \lambda^D_k(\psi_{\varepsilon}) \), \( \lambda^N_k(\psi_{\varepsilon}) \) and characteristic numbers \( \lambda_k(\psi_{\varepsilon}) \), \( \overline{\lambda}_k(\psi_{\varepsilon}) \) are all different when \( \varepsilon \neq 0 \).

In the following figures, the first three Arnold tongues and eigenvalue curves (5.13) for the family \( \psi_{\varepsilon} \) are plotted.

**INSERT THE FIRST 3 FIGURES HERE**

**Figure 5.1** The Arnold tongues for the family \( \psi_{\varepsilon}(t) = 1 + \varepsilon \sin t \), and eigenvalue curves \( \lambda = \lambda^D_k(\psi_{\varepsilon}) \) \((-·-)\), \( \lambda = \lambda^N_k(\psi_{\varepsilon}) \) \((-−−)\) inside the tongues.

### 6 Estimates for Eigenvalues and Numerical Results

Considering their role in the stability problems and nonresonance problems, we will give in this section some lower bounds for the first characteristic numbers \( \lambda_1(\varphi) \) and the first Dirichlet eigenvalues \( \lambda_1^D(\varphi) \), where \( \varphi \in L^1(0,T) \) with \( \varphi > 0 \).
6.1 Lower Bounds for the First Eigenvalues

We need the following best Sobolev constants. For any $1 \leq \alpha \leq \infty$, let $K_D(\alpha)$ be the best Sobolev constant in the following inequality:

$$C\|u\|_\alpha^2 \leq \|\dot{u}\|_2^2$$

for all $u \in H^D$, i.e.,

$$K_D(\alpha) = \inf_{u \in H^D \setminus \{0\}} \frac{\|\dot{u}\|_2^2}{\|u\|_\alpha^2}. \quad (6.1)$$

The explicit formula for $K_D(\alpha)$ is given in [37].

**Proposition 6.1** [37] (i) The constants $K_D(\alpha)$ are

$$K_D(\alpha) = \begin{cases} \frac{2\pi}{\alpha T^{1+2/\alpha}} \left( \frac{2}{2+\alpha} \right)^{1-2/\alpha} \left( \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{\alpha})} \right)^2, & \text{if } 1 \leq \alpha < \infty, \\ \frac{4}{T}, & \text{if } \alpha = \infty. \end{cases} \quad (6.2)$$

(ii) The infimum in (6.1) can only be attained by functions $u = c u_\alpha(t)$, where $c \neq 0$ and $u_\alpha(t)$ is given by

$$u_\alpha(t) = \begin{cases} F^{-1}_\alpha(2F_\alpha(1)t/T), & \text{if } t \in [0,T/2], \\ F^{-1}_\alpha(2F_\alpha(1)(1-t/T)), & \text{if } t \in [T/2,T], \end{cases}$$

where $F_\alpha : [0,1] \to \mathbb{R}$ is given by

$$F_\alpha(u) = \int_0^u \frac{du}{(1-u^\alpha)^{1/2}}.$$

Now we have the following lower bounds for $\lambda_1(\varphi)$.

**Theorem 6.2** Assume that $\varphi \geq 0$ and $\varphi \in L^\alpha(0,T)$ for some $1 \leq \alpha \leq \infty$. If $1 < \alpha \leq \infty$, then

$$\lambda_1^D(\varphi) \geq \lambda_1(\varphi) \geq K_D(2\alpha^*)/\|\varphi\|_\alpha. \quad (\alpha^* = \alpha/(\alpha - 1)) \quad (6.3)$$

If $\alpha = 1$, we have the following strict inequality

$$\lambda_1(\varphi) > K_D(\infty)/\|\varphi\|_1 = 4/(T\|\varphi\|_1). \quad (6.4)$$

**Proof** Note that $\|\varphi_s\|_\alpha = \|\varphi\|_\alpha$ for all $s$ and all $\alpha$. By (3.4), it suffices to prove the following two inequalities:

$$\lambda_1^D(\varphi) \geq K_D(2\alpha^*)/\|\varphi\|_\alpha, \quad \text{if } 1 < \alpha \leq \infty, \quad (6.5)$$

$$\lambda_1^D(\varphi) > K_D(\infty)/\|\varphi\|_1, \quad \text{if } \alpha = 1. \quad (6.6)$$

To this end, let $x \in H^D$. Then we have

$$\int_0^T \varphi(t)x^2 dt \leq \|\varphi\|_\alpha \|x\|_{\alpha^*}^2 \leq (\|\varphi\|_\alpha/K_D(2\alpha^*)) \|\dot{x}\|_2^2. \quad (6.7)$$
Using Proposition 2.1 (iii), we have

$$\lambda_D^1(\varphi) = \inf_{x \in H^D \setminus \{0\}} \frac{\int_0^T \dot{x}^2 dt}{\int_0^T \varphi(t)x^2 dt} \geq \frac{K_D(2\alpha^*)}{\|\varphi\|_\alpha}$$

for all $1 \leq \alpha \leq \infty$. This gives (6.5) when $1 < \alpha \leq \infty$.

However, when $\alpha = 1$ we have the strict inequality (6.6). Otherwise, assume that $\lambda_D^1(\varphi) = K_D(\infty)/\|\varphi\|_1$. Namely, there is a nonzero $x \in H^D$ such that

$$\int_0^T \varphi(t)x^2 dt = (\|\varphi\|_1/K_D(\infty)) \int_0^T \dot{x}^2 dt.$$

From the proof for (6.7), we see that $x(\cdot)$ must simultaneously satisfy the following two equalities:

$$\int_0^T \varphi(t)x^2 dt = \|x\|_\infty^2 \int_0^T \varphi(t) dt, \quad (6.8)$$

$$K_D(\infty)\|x\|_\infty^2 = \|\dot{x}\|_2^2. \quad (6.9)$$

By Proposition 6.1 (ii), we know that (6.9) holds only when $x(t) = c u_\infty(t) = \left\{ \begin{array}{ll} 2c T/2, & t \in [0, T/2], \\ 2c (T - t), & t \in [T/2, T], \end{array} \right.$

for some nonzero constant $c$. Now (6.8) implies that

$$\varphi(t)(x^2(t) - \|x\|_\infty^2) = 0 \quad \text{a.e. } t \in [0, T].$$

As $|x(t)| = \|x\|_\infty$ holds only when $t = T/2$, thus we have $\varphi(t) = 0$ for a.e. $t$, which contradicts to the assumption $\varphi \not\equiv 0$. □

**Remark 6.3** When $1 < \alpha \leq \infty$, the equality in (6.3) may hold for some $\varphi \not\equiv 0$. Actually, let $x(t) = u_\alpha(t)$ and

$$\varphi(t) = c|u_\alpha(t)|^{2/(\alpha-1)}, \quad t \in [0, T],$$

where $c > 0$ is a constant. Now all equalities in (6.7) hold. As a result, for these $\varphi(t)$, we have

$$\lambda_D^1(\varphi) = K_D(2\alpha^*)/\|\varphi\|_\alpha.$$

Another lower bound of $\lambda_D^1(\varphi)$ is given in [44], which is proved using the Opial’s inequality in [3] and [6].

Let $\Phi(t)$ be a primitive of $\varphi(t)$ and let

$$\kappa_1(\mu) = \left( 2 \int_0^{T/2} t (\Phi(t) - \mu)^2 dt \right)^{1/2}, \quad (6.10)$$

$$\kappa_2(\mu) = \left( 2 \int_{T/2}^T (T - t)(\Phi(t) - \mu)^2 dt \right)^{1/2}. \quad (6.11)$$

**Proposition 6.4** [44] Let $\kappa_1(\mu)$ and $\kappa_2(\mu)$ be as in (6.10) and (6.11). Then

$$\lambda_D^1(\varphi) \geq \frac{1}{\min_{\mu \in \mathbb{R}} \max\{\kappa_1(\mu), \kappa_2(\mu)\}}. \quad (6.12)$$
6.2 Examples and Numerical Results

Let us consider the family
\[ \varphi_\varepsilon(t) = 1 + \varepsilon \cos t, \quad \varepsilon \in [-1, 1] \tag{6.13} \]
and the equation
\[ \ddot{x} + \lambda (1 + \varepsilon \cos t)x = 0. \tag{6.14} \]

The classical stability result gives the stability of Eq. (6.14) only when
\[ 0 < \lambda \leq \frac{4}{2\pi \|\varphi_\varepsilon\|_1} = \frac{1}{\pi^2} \approx 0.10132 \tag{6.15} \]
for all \( \varepsilon \in [-1, 1] \).

However, if we use the \( L^\alpha \) norms of \( \varphi_\varepsilon \), we can get from (6.2), (6.3) and (6.4) the following lower bounds for \( \lambda_1(\varphi_\varepsilon) \):
\[ \lambda_1(\varphi_\varepsilon) \geq K_D(2\alpha^*) \frac{\|\varphi_\varepsilon\|}{\|\varphi_\varepsilon\|_\alpha} =: H(\alpha, \varepsilon) \]
for all \( 1 \leq \alpha \leq \infty \). Thus we have
\[ \lambda_1(\varphi_\varepsilon) = \lambda_1^D(\varphi_\varepsilon) \geq H_1(\varepsilon) := \max_{1 \leq \alpha \leq \infty} H(\alpha, \varepsilon). \tag{6.16} \]

The graph of the function \( \lambda = H_1(\varepsilon) \) is plotted in the following figure. We see that even when \( \varepsilon = \pm 1 \), we have the following first stability interval:
\[ 0 < \lambda < \lambda_1(\varphi_{\pm 1}) \approx 0.16445 \]
Thus the first stability interval for (6.14) is much better than that given in (6.15).

Now we use Proposition 6.4 to obtain lower bounds for \( \lambda^D_1(\varphi_\varepsilon) \). For the family (6.13), let \( \Phi_\varepsilon(t) = t + \varepsilon \sin t \) be the primitive of \( \varphi_\varepsilon \). We have
\[ \kappa_1^2(\mu) = \frac{\pi^4}{2} - 16\varepsilon + 4\pi^2\varepsilon + \frac{\pi^2\varepsilon^2}{2} - \frac{4\pi^3\mu}{3} - 4\pi\varepsilon\mu + 2\mu^2, \]
\[ \kappa_2^2(\mu) = \frac{11\pi^4}{6} - 16\varepsilon - 4\pi^2\varepsilon + \frac{\pi^2\varepsilon^2}{2} - \frac{8\pi^3\mu}{3} + 4\pi\varepsilon\mu + 2\mu^2. \]

Now we can obtain from (6.12)
\[ \lambda^D_1(\varphi_\varepsilon) \geq \frac{1}{\min_{\mu \in \mathbb{R}} \max\{\kappa_1(\mu), \kappa_2(\mu)\}} = \frac{1}{\kappa_1(\pi)} \]
\[ = \left( \frac{6}{\pi^4 - 96\varepsilon + 3\pi^2\varepsilon^2} \right)^{1/2} =: H_2(\varepsilon). \tag{6.17} \]

The eigenvalue curves \( \lambda = \lambda_1(\varphi_\varepsilon) \) and \( \lambda = \lambda^D_1(\varphi_\varepsilon) \), and the lower bound curves \( \lambda = H_1(\varepsilon) \) and \( \lambda = H_2(\varepsilon) \) are plotted in the following figure. We see that \( H_1(\varepsilon) \) is very closed to \( \lambda_1(\varphi_\varepsilon) \), while \( H_2(\varepsilon) \) is closed to \( \lambda^D_1(\varphi_\varepsilon) \) for all \( \varepsilon \in [-1, 1] \).

INSERT THE LAST FIGURE HERE
Figure 6.1 The eigenvalue curves $\lambda = \lambda_A(\varphi_\varepsilon)(\cdots)$ and $\lambda = \lambda_D(\varphi_\varepsilon)$ (——), and their lower bound curves $\lambda = H_1(\varepsilon)$ (---) and $\lambda = H_2(\varepsilon)$ (---) for the family $\varphi_\varepsilon(t) = 1 + \varepsilon \cos t$, where the left dashed region is the first stability region for Eq. (6.14).

The following numerical results on weighted eigenvalues are obtained using the Fourier expansions and eigenvalues of matrices.

Table 6.1 The first 5 eigenvalues of the periodic, the anti-periodic, the Dirichlet, and the Neumann problems for $\psi(t) = 1 + \sin t$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda^P_k(\psi)$</th>
<th>$\lambda^A_k(\psi)$</th>
<th>$\lambda^D_k(\psi)$</th>
<th>$\lambda^N_k(\psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.92911</td>
<td>0.16450</td>
<td>0.22053</td>
<td>0.36855</td>
</tr>
<tr>
<td>2</td>
<td>1.51954</td>
<td>0.44490</td>
<td>1.23571</td>
<td>0.99139</td>
</tr>
<tr>
<td>3</td>
<td>4.31580</td>
<td>2.31282</td>
<td>3.05371</td>
<td>2.33784</td>
</tr>
<tr>
<td>4</td>
<td>5.52400</td>
<td>3.21221</td>
<td>5.50483</td>
<td>4.69132</td>
</tr>
<tr>
<td>5</td>
<td>10.17253</td>
<td>6.93424</td>
<td>8.09059</td>
<td>7.81726</td>
</tr>
</tbody>
</table>

7 Concluding Remarks

Firstly, when the weight functions $\varphi(\cdot)$ satisfy $\varphi \geq 0$, we have, from Sections 2 and 3, a clear relationship between all weighted eigenvalues $\lambda^D_k(\varphi)$, $\lambda^P_k(\varphi)$, $\lambda^A_k(\varphi)$, $\lambda^N_k(\varphi)$. Of particular interests, the characterization of all characteristic numbers using the Dirichlet and the Neumann eigenvalues

$$\underline{\lambda}_k(\varphi) = \min_s \lambda^D_k(\varphi_s) \quad \text{and} \quad \overline{\lambda}_k(\varphi) = \max_s \lambda^D_k(\varphi_s),$$

and

$$\underline{\lambda}_k(\varphi) = \min_s \lambda^N_k(\varphi_s) \quad \text{and} \quad \overline{\lambda}_k(\varphi) = \max_s \lambda^N_k(\varphi_s),$$

reveals the hidden relation between these eigenvalue problems and seems to be completely new.

The role of characteristic numbers in nonresonance problems and stability problems is clear. The result for the first stability region

$$\|\varphi\|_\alpha < K_D(2\alpha^*), \quad (1 < \alpha \leq \infty)$$

using the $L^\alpha$ norms, not the $L^1$ norm of $\varphi$, not only generalizes the classical result, but also gives much better stability results when the weight functions $\varphi_\varepsilon(t) = 1 + \varepsilon \cos t$.

Secondly, when $\varphi(\cdot)$ does not necessarily satisfy $\varphi \geq 0$, Hale [24] studied the stability problems for the following linear equation

$$\ddot{x} + (\lambda + \varphi(t))x = 0, \quad (7.1)$$

or equivalently, the following planar linear system

$$\dot{x} = y, \quad \dot{y} = -(\lambda + \varphi(t))x. \quad (7.2)$$
His introducing of the sequences \( \{ a^*_k(\varphi) \} \) and \( \{ a_k(\varphi) \} \) is by solving the following equations on trace of (7.2):

\[
\text{tr} \tilde{P}_\lambda = \pm 2, \tag{7.3}
\]

where \( \tilde{P}_\lambda \) is the Poincaré map of (7.2). However, the existence to solutions of (7.3) is seriously dependent on the fact that the function \( \text{tr} \tilde{P}_\lambda \) is an entire function of \( \lambda \) with order 1/2, a result from the fact that the corresponding weight function \( \varphi(t; \lambda) = \lambda + \varphi(t) \) is entire in \( \lambda \). When the stability problem for (7.2) is considered, their role of these numbers \( \{ a^*_k(\varphi) \} \) and \( \{ a_k(\varphi) \} \) is similar to that of the characteristic numbers in this article. Thus the Arnold tongues for Eq. (7.2) can be completely determined using these numbers.

It can be noticed that the parametric resonance and the Arnold tongues for the equation

\[
\ddot{x} + \lambda(1 + \varepsilon \cos t)x = 0 \tag{7.4}
\]

and the equation

\[
\ddot{x} + (\lambda + \varepsilon \cos t)x = 0 \tag{7.5}
\]

have some different features. In order to give a unified consideration for the Arnold tongues for (7.4) and (7.5), it is an important problem to consider the following weighted problem

\[
\ddot{x} + \varphi(t; \lambda)x = 0. \tag{7.6}
\]

When the weight function \( \varphi(t; \lambda) \) is not an entire function of \( \lambda \), the approach in [24] is not applicable. However, Eq. (7.6) can also be studied using the dynamics approach, regardless of whether \( \varphi(t; \lambda) \) is positive. In fact, write Eq. (7.6) as its equivalent planar system

\[
\dot{x} = -y, \quad \text{and} \quad \dot{y} = \varphi(t; \lambda)x. \tag{7.7}
\]

In the polar coordinates, we again get an equation on the circle \( S^1 \):

\[
\dot{\theta} = \varphi(t; \lambda) \cos^2 \theta + \sin^2 \theta. \tag{7.8}
\]

When \( \varphi(t; \lambda) \) is nondecreasing in \( \lambda \), a theory like that in this article can also be developed for Eq. (7.6) and Eq. (7.8). Thus the parametric resonance and the Arnold tongues for more general type of equations

\[
\ddot{x} + \varphi(t; \lambda, \varepsilon)x = 0 \tag{7.9}
\]

can be given. We will not develop this idea in this article.

Finally, we remark that characteristic numbers also play an important role in discussing the existence of periodic solutions to some class of differential equations with singularities. For example, consider the following equation:

\[
\ddot{x} + f(t, x) = c x^{-\gamma} + h(t), \tag{7.10}
\]

where \( f(t, x) \) is as in Section 4, and \( c > 0 \) (this means that Eq. (7.10) is of repulsive type at the singularity 0 and \( \gamma \geq 1 \) (this is the strong force condition at 0), see [13], [40] and [43].

In [43], the first author of the present article established a relationship between the periodic BVP of singular equations like (7.10) and the Dirichlet BVP of nonsingular equations,
and proved that Eq. (7.10) has at least one positive $T$-periodic solution if $a(\cdot)$, $b(\cdot)$ in the semilinearity conditions satisfy $\bar{a} > 0$ and $b(\cdot)$ satisfies the Property $(MD)$ in [43]. An analysis shows that the Property $(MD)$ in [43] is just equivalent to

$$\lambda_1(b) > 1.$$ 

As a result, for singular equations like

$$\ddot{x} + \lambda \varphi(t)x = c x^{-\gamma} + h(t), \quad (7.11)$$

where $c$, $\lambda > 0$ and $\gamma \geq 1$, we can obtain the existence of positive $T$-periodic solutions to (7.11) when $0 < \lambda < \lambda_1(\varphi)$. This interesting phenomenon will be considered in a further paper [38] and it will be proved that one can obtain the existence of positive $T$-periodic solutions of singular equations (7.11) when $\lambda$ is in the stability region of the Hill’s equation (5.3).

**Acknowledgement**

During the preparation of this article, many conservations with our colleagues in the Dynamical Systems Seminar of Peking University are helpful. We would like to express here our thanks to all of them, especially to Tongren Ding, Zhenxi Dong, Shaobo Gan, Bin Liu and Zhifen Zhang.

**References**


M. del Pino, M. Elgueta, and R. Manásevich, A homotopic deformation along $p$ of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')'+f(t,u)=0$, $u(0)=u(T)=0$, $p>1$, *J. Differential Equations*, **80** (1989), 1–13.


