An Introduction to Theory of Dynamical Spectrum

Meirong Zhang

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People’s Republic of China.
E-mail: mzhang@math.tsinghua.edu.cn

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1 Basics to Topological Dynamics

Let $X$ be a topological space and $\mathcal{T}$ be a topological group. We usually take $\mathcal{T} = \mathbb{R}$ (the reals) or $\mathcal{T} = \mathbb{Z}$ (the integers).

**Definition 1.1** By a flow on $X$, we mean a mapping $f : X \times \mathcal{T} \to X$ such that (i) $f$ is continuous, and (ii) $f(f(x, t), s) = f(x, t + s)$ for all $x \in X$ and all $t, s \in \mathcal{T}$.

Note that for any $t \in \mathcal{T}$, the mapping $f_t(x) := f(x, t)$ is a self-homeomorphism of $X$ with the inverse $f_{-t}$. Now condition (ii) in Definition 1.1 is the following group property:

$$f_s \circ f_t = f_{s+t} \quad t, s \in \mathcal{T}.$$

Sometimes we also write the flow as $f(x, t) = x \cdot t$.

For a given $x \in X$, we define

- $\gamma(x) = \{x \cdot t : t \in \mathcal{T}\}$ = trajectory through $x$,
- $\gamma^+(x) = \{x \cdot t : t \in \mathcal{T}^+\}$ = positive semi-trajectory through $x$,
- $\gamma^-(x) = \{x \cdot t : t \in \mathcal{T}^-\}$ = negative semi-trajectory through $x$,
- $H(x) = \text{cl} \, \gamma(x)$ = hull of $x$,
- $H^+(x) = \text{cl} \, \gamma^+(x)$,
- $H^-(x) = \text{cl} \, \gamma^-(x)$,
- $\Omega(x) = \bigcap_{s \in \mathcal{T}} H^+(x \cdot s)$ = omega limit set,
- $A(x) = \bigcap_{s \in \mathcal{T}} H^-(x \cdot s)$ = alpha limit set,

where $\mathcal{T}^+$ and $\mathcal{T}^-$ denote the nonnegative and nonpositive elements in $\mathcal{T}$ respectively. For subsets $K \subset X$ and $I \subset \mathcal{T}$, we write

$$f(K, I) = \{f(x, t) : x \in K, t \in I\}.$$

**Definition 1.2** (i) A set $M \subset X$ is said to be invariant if $\gamma(x) \subset M$ whenever $x \in M$.

(ii) A set $M \subset X$ is said to minimal if $M$ is nonempty, closed and invariant and $M$ has no proper subset with these properties.

(iii) A closed invariant set $M$ is said to be isolated if there is a neighborhood $U$ of $M$ such that every invariant set $K$ contained in $U$ must be contained in $M$.

(iv) A set $M \subset X$ is said to be an attractor (repeller) if $M$ is a closed invariant set and there is an open neighborhood $U$ of $M$ with the property that $\Omega(x) \subset M$ ($A(x) \subset M$) for all $x \in U$.

(v) An attractor (repeller) $M$ is said to be stable (in the sense of Lyapunov) if for every neighborhood $U$ of $M$ there is a neighborhood $V$ of $M$ such that $f(V, t) \subset U$ for all $t \in T^+$ (for all $t \in T^-$), where $f(V, t) = \{f(x, t) : x \in V\}$.

**Proposition 1.1** Assume that $X$ is a compact Hausdorff space. Then every closed invariant set contains at least one minimal set.
The following is a characterization of stability of attractors.

**Proposition 1.2** Assume that $X$ is a compact space and $M$ is an attractor in $X$. Let

$$G(M) = \{ x \in X : \Omega(x) \subset M \}$$

be the region of attraction for $M$. Then $M$ is stable if and only if for all $x \in G(M) \setminus M$, one has $A(x) \cap M = \emptyset$.

By a vector bundle $E$ with base $Y$, projection $p$, and fiber $X (= \mathbb{F}^n)$, we mean the following:

(i) $E$ and $Y$ are topological spaces and $p$ is a continuous mapping of $E$ onto $Y$,

(ii) for each $y \in Y$, $X_y := p^{-1}(y)$ is a vector space, and

(iii) for each $y \in Y$, there exists an open set $G \subset Y$ with $y \in G$ and a homeomorphism $\tau : p^{-1}(G) \to X \times G$ such that for each $\eta \in G$, $X_\eta$ is mapped onto $X \times \{\eta\}$ and $\tau : X_\eta \to X \times \{\eta\}$ is a linear homeomorphism.

A vector bundle $E$ can be given “coordinates” $(x, y)$, where $y \in Y$ and $x \in X_y$.

We consider in most case the trivial vector bundles $X \times Y$, where $X$ is a normed space and $Y$ a topological space. In this case, a subset $\mathcal{V}$ of $X \times Y$ is called a subbundle if the following hold.

(i) $\mathcal{V}$ is a closed subset of $X \times Y$.

(ii) For each $y \in Y$, $\mathcal{V}(y) = \{ x \in X : (x, y) \in \mathcal{V} \}$ is a linear subspace of $X$.

(iii) $\text{dim} \mathcal{V}(y)$ is a constant independent of $y \in Y$.

This means that the linear space $\mathcal{V}(y)$ varies continuously in $y$, i.e., when $y$ are restricted to small open sets, one can make a continuous choice of basis of $\mathcal{V}(y)$.

If $\mathcal{V}$ and $\mathcal{W}$ are subsets of $X \times Y$ we say that

$$X \times Y = \mathcal{V} \oplus \mathcal{W}$$

is a Whitney sum (also referred to as a splitting of $X \times Y$ into a Whitney sum) if the following hold.

(i) $\mathcal{V}$ and $\mathcal{W}$ are both subbundles of $X \times Y$.

(ii) $\mathcal{V}(y) \cap \mathcal{W}(y) = \{0\}$ for all $y \in Y$.

(iii) $X = \mathcal{V}(y) + \mathcal{W}(y)$ for all $y \in Y$.

One has the following characterization of Whitney sums.

**Proposition 1.3** Assume that $Y$ is a compact Hausdorff space. The following statements are equivalent:

(A) $X \times Y = \mathcal{V} \oplus \mathcal{W}$ is a Whitney sum.

(B) The mapping $\hat{P}(y) : X \times Y \to X \times Y$ given by

$$\hat{P}(x, y) = (P(y)x, y)$$

is jointly continuous in $(x, y)$, where $P(y)$ is the projection on $X$ with range $\mathcal{V}(y)$ and null space $\mathcal{W}(y)$.
Let $Y$ be a compact Hausdorff space with the uniform topology. As in metric spaces, one can define for each $y \in Y$ the uniform family of $\alpha$-neighborhoods $V_\alpha(y)$, where $\alpha$ are in some directed index set $J$. For example, if the topology on $Y$ is metrizable then $J$ would be the positive real line $(0, \infty)$ and $V_\alpha(y)$ would represent the family of $\alpha$-balls of $y$. In this setting, one can define a flow $\sigma$ on $Y$ to be chain-recurrent if for every $y \in Y$, $\tau > 0$ and every index $\alpha \in J$ there exist a finite number of $T$ values $t_i \geq \tau$ ($i = 1, \cdots, k$) and a finite number of points $y_i$ ($i = 0, 1, \cdots, k$) such that $y = y_0 = y_k$ and $\sigma(y_{i-1}, t_i) \in V_\alpha(y_i)$ ($i = 1, \cdots, k$). One has the following result.

**Proposition 1.4** Assume that $\sigma$ is a chain-recurrent flow on a compact Hausdorff space $Y$. Then the only stable attractor of $\sigma$ is the whole space $Y$.

## 2 Skew-product Flows

We begin with the simpler case for skew-product flows. Let $X$ and $Y$ be topological spaces and $T$ be as before. The product space $X \times Y$ can be thought as a bundle with base $Y$ and fibers $X \times \{y\}$, $y \in Y$.

**Definition 2.1** A flow $\pi$ on $X \times Y$ is called a skew-product flow (SPF, for short) if $\pi$ can be written in the form

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t)).$$

In this case $\sigma : Y \times T \to Y$ is itself a flow on $Y$. We call the flow $\sigma$ the base flow of $\pi$. If, in addition, $X$ is a finite-dimensional linear space and if $\varphi(x, y, t)$ is linear in $x$ for each $(y, t) \in Y \times T$, then $\pi$ is called a linear skew-product flow (LSPF, for short).

In the sequel, we always assume that $X = \mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $Y$ is a Hausdorff space (and for our purpose, is also compact). Let $\pi = (\varphi, \sigma)$ be a linear skew-product flow on $X \times Y$. Write $\sigma(y, t) = y \cdot t$. Denote the linear transformation $x \to \varphi(x, y, t)$ by $\Phi(y, t)x = \varphi(x, y, t)$. The group property for the flow $\pi$ then implies that

$$\Phi(y \cdot t, s)\Phi(y, t) = \Phi(y, t + s) \quad (2.1)$$

for all $y \in Y$ and all $t, s \in T$. Let $| \cdot |$ be a norm on $X$ and $\| \cdot \|$ be the corresponding operator norm for linear transformations from $X$ to $X$. Then $|\Phi x| \leq \|\Phi\||x|$ for any linear transformation $\Phi$ and any $x \in X$.

For a linear skew-product flow $\pi = (\varphi, \sigma)$ on $X \times Y$, we introduce the bounded set $\mathcal{B}$, the stable set $\mathcal{S}$ and the unstable set $\mathcal{U}$ by

$$\begin{align*}
\mathcal{B} &= \{(x, y) \in X \times Y : \sup_{t \in T} |\varphi(x, y, t)| < \infty\}, \\
\mathcal{S} &= \{(x, y) \in X \times Y : \lim_{t \to +\infty} |\varphi(x, y, t)| = 0\}, \\
\mathcal{U} &= \{(x, y) \in X \times Y : \lim_{t \to -\infty} |\varphi(x, y, t)| = 0\}.
\end{align*}$$

We also consider the fibers

$$\begin{align*}
\mathcal{B}(y) &= \{x \in X : (x, y) \in \mathcal{B}\}, \\
\mathcal{S}(y) &= \{x \in X : (x, y) \in \mathcal{S}\}, \\
\mathcal{U}(y) &= \{x \in X : (x, y) \in \mathcal{U}\}.
\end{align*}$$
By the linearity of $\varphi(x, y, t)$ in $x$, $\mathcal{B}(y)$, $\mathcal{S}(y)$ and $\mathcal{U}(y)$ are always linear subspaces of $X$ and

$$\mathcal{S}(y) \cap \mathcal{U}(y) \subset \mathcal{B}(y).$$

Moreover, these sets are invariant under $\pi$ in the following sense:

$$\pi(\mathcal{B}(y), t) = \mathcal{B}(y \cdot t), \quad \pi(\mathcal{S}(y), t) = \mathcal{S}(y \cdot t), \quad \pi(\mathcal{U}(y), t) = \mathcal{U}(y \cdot t),$$

for all $y \in Y$ and all $t \in T$.

Let us introduce some examples of skew-product flows.

**Example 2.1** Consider a nonlinear nonautonomous differential equation on $\mathbb{R}^n$:

$$\dot{x} = \frac{dx}{dt} = f(x, t), \quad x \in X := \mathbb{R}^n, \quad t \in \mathbb{R},$$

(2.2)

where $f : X \times \mathbb{R} \to X$ is continuous and has some regularity so that any initial value problem $x(s) = x_0$ of (2.2) has a unique solution $x = \varphi(x_0, s, t)$ which is defined on the whole $\mathbb{R}$. This can yield a (nonlinear, in general) skew-product on $X \times \mathbb{R}$. More precisely, we extend Eq. (2.2) to the extended space $X \times \mathbb{R}$ and consider differential equation

$$\begin{align*}
\frac{dx}{dt} &= f(x, s), \\
\frac{ds}{dt} &= 1.
\end{align*}$$

(2.3)

Then Eq. (2.3) induces a flow $\pi(x, s, t) = (\varphi(x, s, t), s + t)$ on $X \times \mathbb{R}$ which is a skew-product with the base flow $\sigma(s, t) = s + t$ on $\mathbb{R}$.

In particular, if $f(x, t) = A(t)x$ is linear in $x$, the induced flow is a LSPF with the base $\mathbb{R}$.

In the next example, we present another class of LSPFs deduced from linear nonautonomous differential systems, which will be the objective of many works.

**Example 2.2** Let $\mathcal{A}$ be a space of matrix-valued functions from $\mathbb{R}$ to $\mathbb{R}^{n^2}$ with some topology. We assume that $\mathcal{A}$ is invariant under the translations $A \to A_\tau$, where $A_\tau(t) = A(\tau + t)$, $\tau, t \in \mathbb{R}$.

For any $x_0 \in X := \mathbb{R}^n$, any $A \in \mathcal{A}$, and any $\tau \in \mathbb{R}$, let $x = \varphi(x_0, A, \tau)$ be the solution at time $\tau$ of the following problem

$$\dot{x} = A(t)x, \quad x(0) = x_0.$$  

(2.4)

Then

$$\pi(x_0, A, \tau) := (\varphi(x_0, A, \tau), A_\tau)$$

gives a linear skew-product flow on $X \times \mathcal{A}$. In the sequel, we are most interested in those linear skew-product flows generated by equations (2.4). In order to ensure that $\mathcal{A}$ is a compact Hausdorff space, a careful choice of the set $\mathcal{A}$ and the corresponding topology on $\mathcal{A}$ will be the content of the next section.

A more natural source for skew-product flows are the tangent flows of differentiable flows or diffeomorphisms on differentiable manifolds. However, such skew-product flows are modelled on tangent bundles of manifolds, not on the trivial bundles $X \times Y$ as considered in the first two examples. As a result we generalize the concept of skew-product flows to vector bundles.
3. Topologies on Spaces of Matrix-valued Functions

**Definition 2.2** Let \( p : E \to Y \) be a vector bundle. We call a flow \( \pi \) on \( E \) a linear skew-product flow if \( \pi \) is such a flow that \( \pi_t \) maps linearly fibers to fibers, i.e., for any \( y \in Y \) and any \( t \in T \), there exists \( \sigma(y, t) \in Y \) such that \( \pi(E_y, t) \subset E_{\sigma(y,t)} \).

For a linear skew-product flow \( \pi \) on \( E \), it is not difficult to verify that \( \sigma \) is itself a flow on \( Y \) and for each \( y \in Y \) and each \( t \in T \), the mapping \( E_y \ni x \mapsto \pi(x, t) \in E_{\sigma(y,t)} \) is a linear homeomorphism for which we denote also by \( \Phi(y, t) \). The equality (2.1) is also satisfied in this case. Now the bounded set, stable set, unstable set of the flow \( \pi \) and their fibers can be defined word by word. We remark here that all fibers \( B(y), S(y) \) and \( U(y) \) are now subspaces of the fiber \( E_y \) for each \( y \in Y \).

Now we can give another important class of linear skew-product flows.

**Example 2.3** Let \( M^n \) be a compact smooth manifold of dimension \( n \). Let \( \sigma : M^n \times T \to M^n \) be a smooth flow. (In fact, it suffices to assume that \( \sigma \) is of class \( C^1 \).) Then the tangent flow \( D\sigma : TM^n \times T \to TM^n \),

\[
D\sigma(v, t) = D\sigma_t(y)v, \quad v \in T_yM^n, \ y \in M^n, \ t \in T,
\]

is a linear skew-product flow on the tangent bundle \( TM^n \) of \( M^n \) in the sense of Definition 2.2. In this case the base flow is just the flow \( \sigma \) on \( M^n \).

### 3 Topologies on Spaces of Matrix-valued Functions

As mentioned in Example 2.2, we are most interested in Eq. (2.4). As a result, we need to discuss whether the spaces \( A \) of matrix-valued functions are compact. In this section, we discuss some topologies on \( A \) such that one can obtain compact spaces.

Recall that for a matrix-valued function \( A(t) \) we always write the translations as \( \sigma(A, \tau) = A(\tau) \), where \( A_\tau(t) \equiv A(t + \tau) \).

**Example 3.1** Let \( A \) be the collection of all continuous \( n \times n \)-matrix-valued functions, i.e., \( A = C(\mathbb{R}, \mathbb{F}^{n^2}) \). Endow \( A \) with the compact-open topology on \( A \). Such a topology can be induced from the following metric on \( A \). Let \( I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots \) be a nested sequence of compact intervals of \( \mathbb{R} \) such that \( \bigcup_{k=0}^\infty I_k = \mathbb{R} \). For each \( k = 0, 1, \ldots \), define

\[
\rho_k(A, B) = \max\{ \| A(t) - B(t) \| : t \in I_k \}, \quad A, B \in A
\]

and

\[
\rho(A, B) = \sum_{k=0}^\infty 2^{-k} \frac{\rho_k(A, B)}{1 + \rho_k(A, B)}.
\]

Then \( \rho \) is a metric on \( A \) and the compact-open topology is the one induced by \( \rho \). Namely, the topology means the uniform convergence on any compact subsets.

Let now \( A \in A \) be a bounded and uniformly continuous function. Then \( Y = H(A) \) is a compact \( \sigma \)-invariant space. In particular, if \( A(t) \) is Bohr almost-periodic, then \( H(A) \) is a minimal compact invariant subset of \( \sigma \).
**Example 3.2** Let $\mathcal{A}$ be the collection of all matrix-valued functions $A(t)$ that are Bohr almost-periodic in $t \in \mathbb{R}$, and let $\mathcal{A}$ have the topology of uniform convergence on the whole $\mathbb{R}$. Then any $A \in \mathcal{A}$ gives a compact $\sigma$-minimal set $H(A)$.

**Example 3.3** Let $\mathcal{A}_p$ denote the collection of all matrix-valued functions $A(t)$ such that each component $a_{ij}(t)$ is in $L^p_{loc}(\mathbb{R})$, where $p \in [1, \infty)$ is fixed. Let $\mathcal{A}_p$ have the metric topology defined by

$$
\rho(A, B) = \sum_{k=0}^{\infty} 2^{-k} \frac{\rho_k(A, B)}{1 + \rho_k(A, B)},
$$

where

$$
\rho_k(A, B) = \max_{i,j} \left( \int_{-k}^{k} |a_{ij}(t) - b_{ij}(t)|^p dt \right)^{1/p}.
$$

If a given $A \in \mathcal{A}_p$ satisfies the additional properties:

(i) There is a $B > 0$ such that

$$
\int_0^1 |a_{ij}(t + \tau)|^p dt \leq B
$$

for all $i, j$ and for all $\tau \in \mathbb{R}$, and

(ii) For every $\varepsilon > 0$, there is a $\gamma > 0$ such that

$$
\int_{\nu}^{\nu+1} |a_{ij}(t + \tau) - a_{ij}(t)|^p dt \leq \varepsilon
$$

for all $\nu \in \mathbb{R}$ and for all $|\tau| \leq \gamma$,

then $H(A)$ is a compact $\sigma$-invariant subset of $\mathcal{A}_p$.

**Example 3.4** Let $\mathcal{A}$ denote the collection $\mathcal{A}_1$ of Example 3.3 with the following topology: $A_j \rightarrow A$ iff $\int_0^1 A_j(s)ds \rightarrow \int_0^1 A(s)ds$ uniformly on compact subsets of $\mathbb{R}$. If $A \in \mathcal{A}_1$ is bounded and $\int_0^1 A(s)ds$ is bounded, then $H(A)$ is compact.

**Example 3.5** Let $\mathcal{A}_p$ denote the collection of Example 3.3 having the following topology: For $I \subset \mathbb{R}$ and $W \subset \mathbb{R}^n$, let $C(I, W)$ be all continuous functions $F : I \rightarrow W$ with the supremum norm. A (generalized) sequence $A_j$ converges to $A$ iff for every compact interval $I \subset \mathbb{R}$, every compact $W \subset \mathbb{R}^n$, and every compact $K \subset C(I, W)$, there exists a real number $\Gamma$ and a sequence $\varepsilon_j \rightarrow 0$ such that

$$
\sup_{x \in K} \int_I |A_j(t)x(t) - A(t)x(t)|^p dt \leq \varepsilon_j^p,
$$

and for all intervals $J \subset I$ and $x \in C(I, W)$ one has

$$
\left\{ \int_J |A_j(t)x(t) - A(t)x(t)|^p dt \right\}^{1/p} \leq \Gamma \mu(J) + \varepsilon_j,
$$

where $\mu$ is the Lebesgue measure. This topology on $\mathcal{A}_p$ is not metrizable. Now suppose that $A \in \mathcal{A}_p$ satisfies (i) in Example 3.3 together with the following

(iii) For every piecewise continuous $x : \mathbb{R} \rightarrow \mathbb{R}^n$ and every $\varepsilon > 0$ there is a $\gamma > 0$ such that

$$
\int_{\nu}^{\nu+1} |A(t + \tau)x(t + \tau) - A(t)x(t)|^p dt \leq \varepsilon
$$

for all $\nu \in \mathbb{R}$ and for all $|\tau| \leq \gamma$.

Then $H(A)$ is compact.
4 Existence of Exponential Dichotomies

Consider the linear differential equation
\[ \dot{x} = A(t)x, \] (4.1)
where \( x \in X := \mathbb{F}^n \). Recall that (4.1) admits an exponential dichotomy (ED, for short) if there exists a projection \( P : X \to X \) and positive constants \( K \) and \( \alpha \) such that
\[
\| \Phi(t)P\Phi^{-1}(s) \| \leq Ke^{\alpha(s-t)}, \quad t \geq s, \tag{4.2}
\]
\[
\| \Phi(t)(I-P)\Phi^{-1}(s) \| \leq Ke^{\alpha(t-s)}, \quad t \leq s, \tag{4.3}
\]
where \( \Phi(t) \) denotes the fundamental matrix solution of (4.1) satisfying \( \Phi(0) = I \).

The existence of exponential dichotomy of (4.1) is a fundamental tool for studying the asymptotic behavior (boundedness, stability, etc.) of solutions of linear inhomogeneous equation
\[ \dot{x} = A(t)x + g(t) \] (4.4)
and nonlinear perturbations of (4.1)
\[ \dot{x} = A(t)x + g(x, t). \] (4.5)

Let us mention some known results for existence of ED.

1. If \( A(t) = A_0 \) is constant, then there exists an ED iff all of the eigenvalues of \( A_0 \) have nonzero real parts.

2. If \( A(t) \) is periodic in \( t \), then there exists an ED iff all of the Floquet multipliers lie off the unit circle.

3. The dichotomy is preserved under small perturbations of the systems. As a result, if (4.1) admits an ED, then \( \dot{x} = [A(t) + B(t)]x \) also admits one when \( B(t) \) is small in an appropriate sense.

4. An ordered pair of function spaces \((B, D)\) is termed admissible for (4.1) if for each \( g \in B \), (4.4) has at least one solution in \( D \). It is known that if an appropriate pair \((B, D)\) is admissible, then (4.1) admits an ED.

We note that when \( A(t) \) is periodic, it follows from the Floquet theory that (4.1) admits an ED iff the only bounded solution of (4.1) is the null (or trivial) solution \( x(t) \equiv 0 \). On one hand, we try to generalize in this section such a characterization of ED to more general equations including almost periodic (aperiodic, for short) ones. On the other hand, a major difference of the present method from the classical ones is that we not only study a single equation (4.1), but also imbed equation (4.1) in a family of equations of the form
\[ \dot{x} = \tilde{A}(t)x \] (4.6)
and then study all equations (4.6) as a whole. The family of equations (4.6) may take as \( \tilde{A}(t) \) in some “hull” \( H(A) = \text{cl} \{ A_r : r \in \mathbb{R} \} \) of \( A(t) \), see Example 2.2 and Section 3. Under suitable assumption on solutions of all equations (4.6), we can obtain the existence of ED for all equations (4.6). In particular, the existence of ED of (4.1) is guaranteed.
Shielding our motivated equations (4.6), we consider general linear skew-product flows. Before stating the results, we make the following

**Standing Hypotheses (SH).** $X = \mathbb{R}^n$, $Y$ is a compact Hausdorff space, $\pi$ is a linear skew-product flow on $X \times Y$ with the trivial bounded set $B = \{0\} \times Y$.

**Remark 4.1** By the Standing Hypotheses, any compact invariant set in $X \times Y$ is contained in $\{0\} \times Y$. An equivalent way to express the triviality of bounded set is that $\{0\} \times Y$ is an isolated invariant set of the flow $\pi$.

Our first result is

**Theorem 4.1** For a linear skew-product flow $\pi$ on $X \times Y$ satisfying the (SH), one has

(I) $S$ and $U$ are closed subsets of $X \times Y$.  
(II) There exist constant $K \geq 1$ and $\alpha > 0$ such that for all $(x, y) \in S$,
\[
|\varphi(x, y, t)| \leq K|x|e^{-\alpha t}, \quad t \in T^+,
\]
and for all $(x, y) \in U$,
\[
|\varphi(x, y, t)| \leq K|x|e^{\alpha t}, \quad t \in T^-.
\]

We have some preliminary results.

**Lemma 4.1** Let $K \subset X \times Y$ be compact and $(x_k, y_k) \in K$ be a sequence with limit $(x, y) \rightarrow (x, y)$.

1. If there exist $t_k \rightarrow +\infty$ such that $\pi(x_k, y_k, [0, t_k]) \subset K$ for all $k$, then $(x, y) \in S$.
2. If there exist $t'_k \rightarrow -\infty$ such that $\pi(x_k, y_k, [t'_k, 0]) \subset K$ for all $k$, then $(x, y) \in U$.
3. If both conditions (1) and (2) are met, then $(x, y) \in B$ and consequently $x = 0$.

**Proof** We need only to prove (1). In this case $\pi(x, y, T^+) \subset K$ by the compactness of $K$ and the continuity of $\pi$. As $K$ is compact, the $\omega$-limit set $\Omega(x, y)$ is nonempty and is contained in $K$. If $(x, y)$ fails to be in $S$, there exists a positive sequence $\{s_k\}$ such that $|\varphi(x, y, s_k)| \geq C_0 > 0$ for all $k$. As $\pi(x, y, T^+) \subset K$, we may assume that $\pi(x, y, s_k) \rightarrow (x^*, y^*)$. So $|x^*| \geq C_0$. Note that for any $t \in T$, we have
\[
\pi(x^*, y^*, t) = \lim_{k} \pi(x, y, t + s_k).
\]
Thus $\pi(x^*, y^*, T)$ is contained in $K$ and $(x^*, y^*) \in B$. Hence $x^* = 0$, which is a contradiction to the fact $|x^*| \geq C_0 > 0$. □

Let
\[
A = \{(x, y) \in S : |\varphi(x, y, t)| \leq 1 \text{ for all } t \in T^+\}.
\]

**Lemma 4.2** $A$ is a compact set of $X \times Y$. 
Proof As $A \subset K := \{(x, y) \in X \times Y : |x| \leq 1\}$ and $K$ is a compact set, it suffices to prove that $A$ is closed. To this end, let $(x_k, y_k) \in A$ with $(x_k, y_k) \to (x, y)$. By the continuity of $\varphi$, it is clear that $|\varphi(x, y, t)| \leq 1$ for all $t \in T^+$. As $\pi(x_k, y_k, [0, k]) \subset K$, it follows from Lemma 4.1 that $(x, y) \in S$. Hence $(x, y) \in A$.

Lemma 4.3 Let $\lambda \in (0, 1]$ be given. Then there cannot exist a sequence $(x_k, y_k) \in A$ and times $t_k \to +\infty$ such that $|\varphi(x_k, y_k, t_k)| \geq \lambda$ for all $k$.

Proof Assume this is false. Let $(\xi_k, \eta_k) = \pi(x_k, y_k, t_k) \in A$. Then $|\xi_k| \geq \lambda$. As $A$ is compact, we may assume that $(\xi_k, \eta_k) \to (\xi, \eta) \in A \subset S$. Thus $|\xi| \geq \lambda$. It is not difficult to check that $\pi(\xi_k, \eta_k, [0, t_k]) \subset A$. By Lemma 4.1, $(\xi, \eta) \in U$. As $S \cap U \subset B = \{0\} \times Y$ we have $\xi = 0$, contradicting the conclusion $|\xi| \geq \lambda$.

Lemma 4.4 (Uniform Stability) There is some $\nu \in (0, 1]$ such that if $(x, y) \in S$ and $|x| \leq \nu$, then $(x, y) \in A$.

Proof If this were false, then there exists a sequence $(x_k, y_k) \in S$ such that $|x_k| \to 0$ and $(x_k, y_k) \not\in A$. As $|\varphi(x_k, y_k, t)| \to 0$ when $t$ tends to $+\infty$, one can define $\lambda_k$ by $1/\lambda_k = \sup_{t \in T^+} |\varphi(x_k, y_k, t)| = |\varphi(x_k, y_k, \tau_k)|$, where $\tau_k \in T^+$. Then $0 < \lambda_k < 1$. By the linearity of $\varphi$, $(\lambda_k x_k, y_k) \in A$ and $|\varphi(\lambda_k x_k, y_k, \tau_k)| = 1$ for all $k$. Note that $|\lambda_k x_k| \leq |x_k| \to 0$, it follows from the continuity of $\varphi$ that $\tau_k \to +\infty$. This contradicts Lemma 4.3.

Lemma 4.5 There is a $T \in T$ with $T > 0$ such that for any $(x, y) \in S$ one has

$$|\varphi(x, y, t)| \leq (1/2)|x|$$

for all $t \geq T$.

Proof If this were false, then there exist times $t_k \to +\infty$ and $(x_k, y_k) \in S$ such that $|\varphi(x_k, y_k, t_k)| > (1/2)|x_k|$. By the linearity of $\varphi$, we may assume that $|x_k| = \nu$ for all $k$, where $\nu$ is as in Lemma 4.4. Then $(x_k, y_k) \in A$ and $|\varphi(x_k, y_k, t_k)| > \nu/2$, contradicting Lemma 4.3.

Remark 4.2 The most important step in the above proofs is the Uniform Stability in Lemma 4.4. One can express this as

$$\sup_{\substack{(x, y) \in S, |x| = 1 \\in T^+}} |\varphi(x, y, t)| < +\infty.$$  \(4.9\)

This is much like the result obtained from the uni-boundedness for linear operators in Banach spaces. More precisely, let $\Phi(y, t) : S(y) \to S(y \cdot t)$ be the linear operator defined by $\Phi(y, t)x = \varphi(x, y, t)$. By definition of $S(y)$, one has, for each $x \in S(y)$,

$$\sup_{t \in T^+} |\Phi(y, t)x| < \infty.$$  \(4.10\)

Thus it follows from the uni-boundedness result for linear operators that

$$\sup_{t \in T^+} \|\Phi(y, t)\| < \infty.$$  \(4.11\)

However, (4.11) is weaker than (4.9). It is an interesting problem how to derive Lemma 4.4 using this idea.
Now we give the proof of Theorem 4.1. For (I), we prove that $\mathcal{S}$ is closed. To this end, assume that $(x_k, y_k) \in \mathcal{S}$ have limit $(x, y)$. If $x = 0$ then it is obvious that $(x, y) \in \mathcal{S}$. Assume now that $x \neq 0$. Then for $\theta = \nu/(2|x|)$, where $\nu$ is as in Lemma 4.4, we have $(\theta x_k, y_k) \in A$ for all $k$ sufficiently large. As $A$ is closed, one has $(\theta x_k, y_k) \to (\theta x, y) \in A$. So $(x, y) \in \mathcal{S}$.

For (II) in Theorem 4.1, it can be immediately proved using Lemma 4.5. □

In the following one sees that it is the most important to study the change of stable fibers $\mathcal{S}(y)$ and unstable fibers $\mathcal{U}(y)$ with $y \in Y$. The following upper semicontinuity results on dimensions of these fibers can be derived from the fact that $\mathcal{S}$ and $\mathcal{U}$ are closed subsets of $X \times Y$.

Recall that a function $f$ from a space $Y$ to $\mathbb{R}$ is said to be upper semicontinuous if $y_k \to y$ implies that $\limsup_{k \to \infty} f(y_k) \leq f(y)$.

**Lemma 4.6** The functions $\dim \mathcal{S}(y)$ and $\dim \mathcal{U}(y)$ are upper semicontinuous functions of $y \in Y$. As a result, $\dim \mathcal{S}(z) \geq \dim \mathcal{S}(y)$ and $\dim \mathcal{U}(z) \geq \dim \mathcal{U}(y)$ for all $z \in \Omega(y)$.

**Proof** Let $y \in Y$ and $y_j \in Y$ a sequence with $y_j \to y$. Define $k = \limsup_{j \to \infty} \dim \mathcal{S}(y_j)$.

One may assume, going to a subsequence if necessary, that $\dim \mathcal{S}(y_j) = k$ for all $j$. Choose an orthogonal basis $\{e^1_j, \ldots, e^k_j\}$ of $\mathcal{S}(y_j)$. One may assume that $e^1_j \to f^1, \ldots, e^k_j \to f^k$. Of course $\{f^1, \ldots, f^k\}$ is also an orthogonal set of $X$. As $\mathcal{S}$ is closed, one sees that $\{f^1, \ldots, f^k\} \subset \mathcal{S}(y)$.

As a result, $\dim \mathcal{S}(y) \geq k$.

Before giving further results, we prove the following

**Lemma 4.7** Let $(x, y) \in X \times Y$.

(i) The limit $\lim_{t \to +\infty} |\varphi(x, y, t)|$ always exists. More precisely, the limit is $0$ if $x \in \mathcal{S}(y)$, and the limit is $+\infty$ if $x \notin \mathcal{S}(y)$.

(ii) The limit $\lim_{t \to -\infty} |\varphi(x, y, t)|$ always exists. More precisely, the limit is $0$ if $x \in \mathcal{U}(y)$, and the limit is $+\infty$ if $x \notin \mathcal{U}(y)$.

**Proof** We prove (i). Define

$$L = \limsup_{t \to +\infty} |\varphi(x, y, t)| \quad \text{and} \quad l = \liminf_{t \to +\infty} |\varphi(x, y, t)|.$$

One then has $0 \leq l \leq L \leq \infty$. Furthermore, $L = 0$ iff $(x, y) \in \mathcal{S}$. Assume that $x \notin \mathcal{S}(y)$, i.e., $L > 0$. We aim at proving $l = L = \infty$. If $L < \infty$, then $\sup_{t \in T^+} |\varphi(x, y, t)| < \infty$. Consequently, the omega limit set $\Omega(x, y)$ is a nonempty compact invariant set for $\pi$. Since $L > 0$, $\Omega(x, y) \not\subset \{0\} \times Y$. This contradicts Remark 4.1 and hence $L = \infty$.

If now $l < L = \infty$, then there exist positive sequences $s_k$, $r_k$ and $t_k$ such that $s_k < r_k < t_k$, $|\varphi(x, y, s_k)| = |\varphi(x, y, t_k)| = l + 1$ and

$$|\varphi(x, y, r_k)| = \nu_k := \max\{|\varphi(x, y, t)| : s_k \leq t \leq t_k\}$$

and $\nu_k \to \infty$ as $k \to \infty$. Define $x_k = \nu_k^{-1} \varphi(x, y, r_k)$ and $y_k = y \cdot r_k$. Then $|x_k| = 1$ and $\varphi(x, y, r_k) \leq 1$ for all $t \in [s_k - r_k, t_k - r_k]$. Note that $|\varphi(x, y, s_k - r_k)| = |\varphi(x, y, t_k - r_k)| \leq \nu_k^{-1}(l + 1) \to 0$. We see that $s_k - r_k \to -\infty$ and $t_k - r_k \to +\infty$ as $k \to \infty$. By Lemma 4.1, any accumulation point of the sequence $(x_k, y_k)$ is in $\{0\} \times Y$. This contradicts the fact that $|x_k| = 1$ for all $k$. Hence we have $l = L = \infty$. □
Remark 4.3 It is hopeful to prove that the convergence \( \lim_{t \to +\infty} |\varphi(x, y, t)| = \infty \) is uniform with respect to \((x, y) \not\in S\) in some sense. A precise statement for this shall read as:

\[
\lim_{t \to +\infty} |\varphi(x, y, t)| = \infty
\]
is uniform with respect to \((x, y) \not\in S : |x| = 1\) and \(x = x_s + x_u, x_s \in S(y), x_u \in S(y)^\perp\) and \(\angle x_s, x_u \geq \theta_0 > 0\).

Lemma 4.8 For \(k = 1, 2, \cdots\), let \(K_k\) be a linear subspace of \(X\) with \(\dim K_k \geq l\) for all \(k\). Define

\[
K = \limsup_{k \to \infty} K_k = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} K_k = \left\{ x \in X : \text{there exists a sequence } \{x_{n_j}\} \text{ such that } x_{n_j} \in K_{n_j} \text{ and } n_j \to \infty, x_{n_j} \to x \text{ as } j \to \infty \right\}.
\]

Then \(K\) contains a subspace of dimension \(l\) and the linear subspace \(\text{span}(K)\) has dimension \(\geq l\).

Proof Let \(\{e_1^k, \cdots, e_l^k\}\) be an orthogonal set of \(K_k\). One may assume that \(e_i^k \to f_i\) as \(k \to \infty\) for each \(i = 1, \cdots, l\). Since \(\{f_1^1, \cdots, f_l^1\}\) is an orthogonal set in \(K\). Meanwhile, for any set \(\{\beta_i\}\), one has \(\sum_{i=1}^{l} \beta_i e_i^k\) is in \(K_k\) and has the limit \(\sum_{i=1}^{l} \beta_i f_i\), which is also in \(K\). As a result, \(\text{span}(K)\) has dimension at least \(l\).

Now we give an important result for dimensions of stable and unstable fibers.

Lemma 4.9 (The Basic Inequalities) Let \(y \in Y, \eta \in A(y)\) and \(\eta' \in \Omega(y)\). If \(n = \dim X\), then

\[
\dim S(\eta) \geq n - \dim U(y), \quad \dim U(\eta') \geq n - \dim S(y).
\]

Proof For \(y \in Y\), let \(K(y)\) be any complementary linear subspace of \(U(y)\), i.e., \(X = K(y) \oplus U(y)\). Let \(\{t_k\}\) be any sequence in \(T^+\). Then define \(\mu_k\) by

\[
\mu_k = \min\{|\varphi(x, y, -t_k)| : x \in K(y) \text{ and } |x| = 1\}.
\]

It follows from Lemma 4.7 or more exactly, from Remark 4.3 that if \(t_k \to +\infty\) then \(\mu_k \to +\infty\). Let now \(\eta' \in A(y)\) and choose \(\{t_k\}\) so that \(t_k \to +\infty\) and \(y_k := y \cdot (-t_k) \to \eta\). Let \(K = K(y, y, -t_k)\). Then \(K_k\) is a linear subspace of \(X\) with

\[
\dim K_k = \dim K(y) = n - \dim U(y)
\]

for all \(k\). Let \(\mu_k\) be given by (4.11). Then for any \(x \in K_k\) with \(|x| \leq \mu_k\) one has \(|\varphi(x, y_k, t_k)| \leq 1\), or equivalently, for any \(x \in K_k\) with \(|x| \leq 1\) one has \(|\varphi(x, y_k, t_k)| \leq \mu_k^{-1}\).

We show now that there is an \(M < \infty\) such that

\[
\sup_{0 \leq t \leq t_k} |\varphi(x, y_k, t)| \leq M
\]

for \(k\) and all \(x \in K_k\) with \(|x| \leq 1\). Let us first note that \(\mu_k \to +\infty\) and hence \(\mu_k^{-1} \leq B\) for all \(k\) and some \(B > 0\). Assume that (4.12) were false. Then there is a sequence \(x_k \in K_k, |x_k| \leq 1\) and

\[
\beta_k := \sup_{0 \leq t \leq t_k} |\varphi(x_k, y_k, t)| = |\varphi(x_k, y_k, t_k)| \to +\infty,
\]
where \( \tau_k \in [0, t_k] \). Let
\[
(\xi_k, \eta_k) = (\beta_k^{-1} \varphi(x_k, y_k, \tau_k), y_k \cdot \tau_k) \in X \times Y.
\]
Then \( |\xi_k| = 1 \) and
\[
\pi(\xi_k, \eta_k, [-\tau_k, t_k - \tau_k]) \subset \Gamma := \{(x, y) \in X \times Y : |x| \leq 1\}.
\]
However,
\[
|\varphi(\xi_k, \eta_k, -\tau_k)| = \beta_k^{-1}|x_k| \leq \beta_k^{-1} \to 0
\]
and
\[
|\varphi(\xi_k, \eta_k, t_k - \tau_k)| = \beta_k^{-1}|\varphi(x_k, y_k, t_k)| \leq \beta_k^{-1} \mu_k^{-1} \to 0
\]
as \( k \to +\infty \). Hence \( -\tau_k \to -\infty \) and \( t_k - \tau_k \to +\infty \). Since \( \Gamma \) is compact, if \((\xi, \eta)\) is any accumulation point of \((\xi_k, \eta_k)\) then \( |\xi| = 1 \) and Lemma 4.1 implies that \((\xi, \eta) \in B = \{0\} \times Y\). This contradiction proves (4.12).

Now let \( K = \limsup_{k \to +\infty} K_k \). For any \( x \in K \) with \(|x| \leq 1\), one can choose \( x_k \in K_k \) with \(|x_k| \leq 1\) such that some subsequence of \((x_k, y_k)\) (call it again \((x_k, y_k)\)) converges to \((x, \eta)\). Inequality (4.12) means that
\[
\pi(x_k, y_k, [0, t_k]) \subset \{(x, y) \in X \times Y : |x| \leq M\}
\]
for all \( k \). So by Lemma 4.1 one has \( x \in S(\eta) \). In other words, \( K \subset S(\eta) \). Hence \( \text{span}(K) \subset S(\eta) \).

By Lemma 4.8 we have
\[
\dim S(\eta) \geq \dim K_k \equiv n - \dim U(y).
\]

The inequality \( \dim U(\eta') \geq n - \dim S(y) \) can be proved similarly. \( \Box \)

Now we reformulate the Basic Inequalities in a somewhat stronger form.

**Lemma 4.10** Let \( y \in Y \) and define \( n, k, l, k_1 \) and \( k_2 \) by: \( n = \dim X, k = \dim S(y), l = \dim U(y), k_1 = n - l \) and \( k_2 = k \). Then
\[
k_1 - k_2 = n - (\dim S(y) + \dim U(y)) \geq 0 \tag{4.13}
\]
and the following statements are valid.

(A) For all \( \eta \in A(y) \) one has
\[
\dim S(\eta) = k_1 = n - \dim U(y) \quad \text{and} \quad \dim U(\eta) = n - k_1 = \dim U(y).
\]

(B) For all \( \eta' \in \Omega(y) \) one has
\[
\dim S(\eta') = k_2 = \dim S(y) \quad \text{and} \quad \dim U(\eta') = n - k_2 = n - \dim S(y).
\]

**Proof** The relationship (4.13) is obvious because \( S(y) \cap U(y) = \{0\} \). We shall now prove (A). If \( \eta \in A(y) \), then the Basic Inequalities show that \( \dim S(\eta) \geq k_1 = n - l \). Also the upper semicontinuity of \( \dim U \) shows that \( \dim U(\eta) \geq l \). Hence
\[
n = k_1 + l \leq \dim S(\eta) + \dim U(\eta) \leq n.
\]
4. Existence of Exponential Dichotomies

As a result, \( \dim S(\eta) = k_1 \) and \( \dim U(\eta) = l = n - k_1 \).

Now we establish some relationship between ED and the dynamics of base flow for a linear skew-product flow. As usual, let \( \pi = (\varphi, \sigma) \) be a linear skew-product flow on \( X \times Y \), where \( X = \mathbb{R}^n \). For \( k = 0, 1, \cdots, n \), define

\[
Y_k = \{ y \in Y : \dim S(y) = k \quad \text{and} \quad \dim U(y) = n - k \}.
\]

We call \( k \) the index of any orbit \( \gamma(y) \), \( y \in Y_k \).

**Lemma 4.11** All \( Y_k \) are mutually disjoint invariant sets of \( \sigma \). Moreover, under the (SH) one sees that \( Y_k \) are also compact.

**Proof** The disjointness is obvious. The invariance follows from the constancy of \( \dim S \) and \( \dim U \) along orbits. The fact that \( Y_k \) is closed, therefore compact, follows from the upper semicontinuity.

Now we aim at giving the decomposition \( X = S(\eta) \oplus U(\eta) \) when \( \eta \) are in some kind of subsets of \( Y \). This will yield some ED using Theorem 4.1(II).

**Theorem 4.2** Assume that the (SH) is satisfied. Let \( y \in Y \). Then there exist integers \( k_1 \) and \( k_2 \) such that \( k_1 \geq k_2 \), \( A(y) \subset Y_{k_1} \) and \( \Omega(y) \subset Y_{k_2} \). In particular, if \( M \) is a minimal set in \( Y \) then \( M \subset Y_k \) for some \( k \).

**Proof** Lemma 4.10 shows that \( A(y) \subset Y_{k_1} \) and \( \Omega(y) \subset Y_{k_2} \), where \( k_1 = n - \dim U(y) \) and \( k_2 = \dim S(y) \). (4.13) also implies that \( k_1 \geq k_2 \). Now if \( M \) is a minimal set in \( Y \), then \( A(y) = M \subset Y_{k_1} \) and \( \Omega(y) = M \subset Y_{k_2} \). Therefore \( k_1 = k_2 = k = \dim S(y) \).

By Lemma 4.10 and Theorem 4.2, one sees that at least one of \( Y_k \) is nonempty. In what follows we distinguish between two cases:

(i) There is precisely one nonempty \( Y_k \).

(ii) There are at least two nonempty \( Y_k \).

**Lemma 4.12** Let \( y \in Y \) be such that \( A(y) \) and \( \Omega(y) \) meet a single \( Y_k \). Then \( y \) itself is in \( Y_k \) and \( H(y) \in Y_k \).

**Proof** Let \( n, k, l \), \( k_1 \) and \( k_2 \) be as before. If \( A(y) \) and \( \Omega(y) \) meet the same \( Y_k \), then it follows from Lemma 4.10 that \( k_1 = k_2 = k \). So \( y \in Y_k \) and Lemma 4.11 shows that \( H(y) \subset Y_k \).

Since every nonempty compact invariant set contains a minimal set, the next result follows from Lemma 4.11 and Theorem 4.2.

**Lemma 4.13** There exists precisely one nonempty \( Y_k \) if and only if all minimal sets in \( Y \) lie in one and the same \( Y_k \), i.e., all minimal sets have the same index.

Now we give the following result on ED.

**Theorem 4.3** Assume that the (SH) is satisfied. If there exists precisely one nonempty \( Y_k \), then the following statements are valid.
(III) $Y = Y_k$, i.e., $\dim S(y) = k$, $\dim U(y) = n - k$ and $X = S(y) \oplus U(y)$ for all $y \in Y$.

(IV) $S$ and $U$ are closed invariant subbundles of $X \times Y$.

(V) $X \times Y = S \oplus U$ is a Whitney sum.

(VI) Define $P(y) : X \to X$ to be the projection on $X$ with range $S(y)$ and null space $U(y)$. Then the mapping

$$ (x, y) \to (P(y)x, y) $$

is continuous in $X \times Y$. Furthermore, there exist positive constants $K_0$ and $\alpha$ such that

$$ \|\Phi(y, t)P(y)\Phi^{-1}(y, s)\| \leq K_0 e^{-\alpha(t-s)}, \quad s \leq t, \quad (4.15) $$

$$ \|\Phi(y, t)(I - P(y))\Phi^{-1}(y, s)\| \leq K_0 e^{-\alpha(s-t)}, \quad t \leq s, \quad (4.16) $$

i.e., $\pi$ admits an exponential dichotomy at every $y \in Y$.

**Proof**  (III) If there is a single nonempty $Y_k$, then it follows from Theorem 4.2 that for any $y \in Y$, the limit sets $A(y)$ and $\Omega(y)$ meet the same $Y_k$. Hence by Lemma 4.12 one has $y \in Y_k$. Thus $Y_k = Y$ and the other properties in (III) also hold.

(IV) By Theorem 4.1 $S$ and $U$ are closed. The constancy of $\dim S(y)$ and $\dim U(y)$ on $y \in Y$ shows that $S$ and $U$ are subbundles.

(V) This is a result of (III) and (IV).

(VI) The continuity of the mapping (4.14) follows from (V) and Proposition 1.3. The only things to be verified are inequalities (4.15) and (4.16). We verify only (4.15). Let $s$ and $t$ be in $T$ such that $s \leq t$. Choose $u \in S(y \cdot s) = \Phi(y, s)(S(y))$. Then $(u, y \cdot s) \in S$, $\Phi^{-1}(y, s)u \in S(y)$ and $P(y)\Phi^{-1}(y, s)u = \Phi^{-1}(y, s)u$. By Theorem 4.1 we get

$$ |\Phi(y, t)P(y)\Phi^{-1}(y, s)u| = |\Phi(y, t)\Phi^{-1}(y, s)u| = |\Phi(y \cdot s, t - s)u| = |\varphi(x, y \cdot s, t - s)| \leq K|u|e^{-\alpha(s-t)}. $$

Next choose $w \in U(y \cdot s) = \Phi(y, s)(U(y))$. Then $(w, y \cdot s) \in U$, $\Phi^{-1}(y, s)w \in U(y)$ and $P(y)\Phi^{-1}(y, s)w = 0$. Since any $x \in X = S(y \cdot s) \oplus U(y \cdot s)$ can be written uniquely as $x = u + w$ where $u \in S(y \cdot s)$ and $w \in U(y \cdot s)$. One then has

$$ |\Phi(y, t)P(y)\Phi^{-1}(y, s)x| = |\Phi(y, t)P(y)\Phi^{-1}(y, s)u + \Phi(y, t)P(y)\Phi^{-1}(y, s)w| \\ \leq K|u|e^{-\alpha(s-t)}. $$

Since the mapping (4.14) is continuous in $(x, y)$ and $Y$ is compact, there is a constant $k_0 > 0$ such that $|P(y)x| \leq k_0|x|$ for all $(x, y) \in X \times Y$. Therefore, in the above notation, $P(y \cdot s)x = u$ and $|u| \leq k_0|x|$. Consequently, one has

$$ |\Phi(y, t)P(y)\Phi^{-1}(y, s)x| \leq (k_0K)|x|e^{-\alpha(t-s)}, $$

which yields (4.15).

By Theorems 4.2 and 4.3, one has

**Corollary 4.1** Assume that the (SH) is satisfied. Assume, in addition, that $Y$ is minimal under $\sigma$. Then $Y = Y_k$ for some $k$ and all conclusions of Theorems 4.1 and 4.3 are valid.
In the following we prove that each compact invariant set $Y_k$ is isolated.

**Lemma 4.14** Each $Y_k$ is an isolated invariant set.

**Proof** The proof of this consists of an application of Theorem 4.3 to some linear skew-product flow formed by restricting the original flow $\pi$ to a suitable neighborhood of $Y_k$.

For each index $\alpha \in J$ and each $D \subset Y$, define the $\alpha$-neighborhood of $D$ by

$$V_{\alpha}(D) = \bigcup_{y \in D} V_{\alpha}(y).$$

Choose some $\alpha \in J$ such that the closed $\alpha$-neighborhoods $\text{cl} V_{\alpha}(Y_k)$ ($k = 0, 1, \ldots, n$) are disjoint in $Y$. Let $K$ be any invariant set in $V_{\alpha}(Y_k)$ and then let $\hat{Y} = \text{cl} (K \cup Y_k)$. Then $\hat{Y}$ is a compact invariant set in $Y$ and $\hat{Y}$ meets only the $Y_k$. Let $\hat{\pi}$ be the linear skew-product flow of $\pi$ restricted to $X \times \hat{Y}$. Correspondingly, we have $\hat{\mathcal{S}}$, $\hat{\mathcal{U}}$ and $\hat{G}_{l}$ ($l = 0, 1, \ldots, n$). It is easy to check that

$$\hat{\mathcal{B}} = \mathcal{B} \cap (X \times \hat{Y}), \quad \hat{\mathcal{S}} = \mathcal{S} \cap (X \times \hat{Y}), \quad \hat{\mathcal{U}} = \mathcal{U} \cap (X \times \hat{Y}).$$

Hence under the (SH) one has $\hat{\mathcal{B}} = \{0\} \times \hat{Y}$. Furthermore, $\hat{G}_{l} = G_{l} \cap \hat{Y}$ for each $l$. As there is only one nonempty $\hat{G}_{l}$, i.e., the $\hat{Y}_{k}$, it follows from Theorem 4.3 that $\hat{Y} = \hat{Y}_{k}$. This shows that $K \subset Y_{k}$.

In case (ii) where there are at least two nonempty $Y_k$ then one can show that the base flow $\sigma$ has a gradient-like structure.

**Theorem 4.4** Assume that the (SH) is satisfied. Assume, in addition, that there are at least two nonempty $Y_k$ and define

$$Q = \max \{ k : Y_k \text{ is nonempty} \}, \quad q = \min \{ k : Y_k \text{ is nonempty} \}.$$

Then $Y_q$ is a stable attractor of the flow $\sigma$, and $Y_Q$ is a negatively stable repeller, i.e., $Y_Q$ is a stable attractor for the reversed flow $\hat{\sigma}(y, t) = \sigma(y, -t)$. Moreover, every motion $\sigma(y, t)$ in $Y$ has its alpha (and omega) limit set in some $Y_k$.

**Proof** The last conclusion in the theorem follows directly from Theorem 4.2. We shall prove that $Y_q$ is a stable attractor. To this end, let $G_q = \{ y \in Y : \Omega(y) \subset Y_q \}$ be the region of attraction of $Y_k$. Let us first assume that $G_q$ so that $Y_q$ is an attractor. We now use Proposition 1.2 to prove the stability of $Y_q$. If $y \in G_q \setminus Y_q$, then it follows from Theorem 4.2 that $A(y) \subset Y_{k_1}$. By Lemma 4.12 it is necessary that $k_1 > q$ because $y \not\in Y_q$. Thus $A(y) \cap Y_q = \emptyset$. Consequently $Y_q$ is stable.

We shall now show that $G_q$ is open. Otherwise, there is a boundary point of $y \in G_q$, i.e., there exists a sequence $\{y_j\} \subset Y$ such that $y_j \rightarrow y$ while $\Omega(y_j)$ are not contained in $Y_q$ for all $j$. By Theorem 4.2 one has $\Omega(y_j) \cap Y_q = \emptyset$ for all $j$. By passing to a subsequence and using Theorem 4.2 one may assume that there are integers $k_1 \geq k_2 > q$ such that $A(y_j) \subset Y_{k_1}$ and $\Omega(y_j) \subset Y_{k_2}$ for all $j$. The upper semicontinuity shows that $\dim S(y) = \limsup_j \dim S(y_j)$. However, Lemma 4.10(B) shows that $\dim S(y) = \dim S(\Omega(y)) = q$ and $\dim S(y_j) = \dim S(\Omega(y_j)) = k_2$. Thus one has $q \geq k_2$, which is a contradiction. Therefore, $G_q$ is open. \qed
Remark 4.4 Let us say more about the gradient-like structure of the flow $\sigma$ on $Y$. Identifying each nonempty $Y_k$ as a point $[Y_k]$, we obtain a quotient space $\bar{Y}$. Then $\sigma$ induces a flow $\bar{\sigma}$ on $\bar{Y}$. Now each $[Y_k]$ is a fixed point of $\bar{\sigma}$ and these are only the fixed points of $\bar{\sigma}$. Theorem 4.4 implies that these are only minimal sets of $\bar{\sigma}$. Furthermore, every trajectory of $\bar{\sigma}$ in $\bar{Y}$ has its alpha (and omega) limit set at some $[Y_k]$. Moreover, this flow $\bar{\sigma}$ has the no-cycle property.

Now the following result of Selgrade is a direct consequence of Theorems 4.3 and 4.4 and Proposition 1.4.

Corollary 4.2 Assume that the (SH) is satisfied. If the flow $\sigma$ on $Y$ is chain-recurrent, then there is only one nonempty $Y_k$ and all four conclusions in Theorems 4.1 and 4.3 are valid.

Remark 4.5 As our approach is essentially local, all results in this section are valid for general linear skew-product flows in the sense of Definition 2.2 after a slight modification.

5 Applications of Exponential Dichotomies

Example 5.1 Let us first consider a scalar equation

$$\dot{x} = (\tan^{-1} t)x.$$  \hspace{1cm} (5.1)

Then $A(t) = \tan^{-1} t \in A$, where $A = C(\mathbb{R}, \mathbb{R})$ with the compact-open topology, cf. Example 3.1. Then the hull $H(A)$ of $A$ consists of all translations $A_\tau(t) = \tan^{-1}(t + \tau)$, $\tau \in \mathbb{R}$, and the constant functions $A_{-\infty}(t) = -\pi/2$ and $A_{+\infty}(t) = \pi/2$. It is clear that $H(A)$ is compact in $A$. Following from Example 2.2, we have a linear skew-product flow on $X \times Y$, where $X = \mathbb{R}$ and $Y = H(A)$. For this flow, one has

$$S(y) = \begin{cases} \{0\}, & \text{if } y \in Y \backslash \{A_{-\infty}\}, \\ X, & \text{if } y = A_{-\infty}, \end{cases} \quad \text{and} \quad U(y) = \begin{cases} \{0\}, & \text{if } y \in Y \backslash \{A_{+\infty}\}, \\ X, & \text{if } y = A_{+\infty}. \end{cases}$$

Thus $B = \{0\} \times Y$. Clearly, this flow has properties (I) and (II) in Theorem 4.1, while we have no ED as in Theorem 4.3. As for the base flow $\sigma$ on $Y$, it has gradient-like structure as in Theorem 4.4. In this example, $\sigma$ has two fixed points $A_{\pm\infty}$. Moreover $Y_0 = \{A_{+\infty}\}$ and $Y_1 = \{A_{-\infty}\}$. These are the only compact invariant sets in $Y$. For every $y \in Y \backslash \{A_{+\infty}, A_{-\infty}\}$, one has $A(y) = Y_1$ and $\Omega(y) = Y_0$.

Example 5.2 (Stability Theory) For Eq. (4.1), the null solution is asymptotically stable if, in the language of the associated LSPF given by Example 2.2, $S(A) = X$, i.e., for all $x \in X$, $\varphi(x, A, t) \to 0$ as $t \to \infty$. The null solution of (4.1) is uniformly stable if there is a $\nu > 0$ such that if $\tau \geq 0$ and $x \in X$ satisfying $|\varphi(x, A, \tau)| \leq \nu$, then $|\varphi(x, A, \tau + t)| \leq 1$ for all $t \geq 0$. It is not difficult to show that if $Y = H(A)$ is compact minimal and for one $\tilde{A} \in Y$ the null solution of (4.6) is uniformly stable, then for every $\tilde{A} \in Y$ the null solution of (4.6) is uniformly stable. Moreover, for any $\tilde{A} \in Y$, one has

$$|\varphi(x, \tilde{A}, t)| \leq \nu^{-1} |x|$$  \hspace{1cm} (5.2)

for all $t \geq 0$. Clearly, (5.2) implies the null solution of (4.6) is uniformly stable. (Compare (5.2) with Lemma 4.5.)
In 1962, W. Hahn posed the problem of whether asymptotic stability implies uniform stability for linear equations with almost-periodic coefficients. C. C. Conley and R. K. Miller gave a negative answer to this by constructing a scalar equation $\dot{x} = a(t)x$ with the property that every solution $\varphi(t) \to 0$ as $t \to \infty$, while the null solution is not uniformly stable. For such an example, we note that $Y = H(a)$ is a compact minimal set (in the topology given by Example 3.2) and $S(a) = X := \mathbb{R}$. If conclusion (II) of Theorem 4.1 were valid with $S = X \times Y$, then the null solution would be uniformly stable. Thus, the hypothesis that $B = \{0\} \times Y$ in Theorem 4.1 must fail. Hence, there exists $\tilde{a} \in Y$ such that equation $\dot{x} = \tilde{a}(t)x$ has a nontrivial bounded solution $\varphi(x_0, \tilde{a}, t)$. In response of Hahn’s problem, if $B = \{0\} \times Y$, then asymptotic stability for any one equation (4.6) does imply not only uniform stability but also asymptotic stability for all the equations since $S = X \times H(A)$ by Corollary 4.1. Furthermore statement (VI) implies an exponential decay in $S$, which means that the null solution is exponentially asymptotically stable.

**Example 5.3** Let $A$ be as in Example 3.1. Consider the $2 \times 2$ matrix-valued function

$$A(t) = \begin{pmatrix} \frac{2}{2 + f(t)} & \frac{3}{2} + f(t) \\ 1 & 1 \end{pmatrix}, \quad t \in \mathbb{R},$$

where $f(t) = (1/\pi) \tan^{-1} t$. Then $A(t)$ is bounded and uniformly continuous in $t \in \mathbb{R}$, and the hull $H(A)$ is $A$ a compact set and consists of all translations $A_\tau$, $\tau \in \mathbb{R}$, and the two limiting matrices

$$A_{-\infty} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_{+\infty} = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix},$$

where $A_{\pm\infty} = \lim_{\tau \to \pm\infty} A_\tau$. As in Example 5.1, one has $\{A_{+\infty}\} =$ the alpha limit set of $A$ and $\{A_{-\infty}\} =$ the omega limit set of $A$. It is easy to see that $\dim S(A_{+\infty}) = 0$ and $\dim S(A_{-\infty}) = 1$. Since $\dim S(y)$ increases as $y$ ‘runs’ from the alpha limit set to the omega limit set, it follows from Theorem 4.2 that the induced LSPF on $\mathbb{R}^2 \times H(A)$ fails to satisfy the (SH). Namely, there exists some $\tilde{A} \in H(A)$ such that (4.6) has a nontrivial bounded solution. When $\tilde{A} = A_{\pm\infty}$, (4.6) has only trivial bounded solution. Thus the equation $\dot{x} = A_\tau(t)x$ for some $\tau \in \mathbb{R}$, and therefore the equation $\dot{x} = A(t)x$ itself admits some nontrivial bounded solutions.

This phenomenon admits many generalizations. One of these is

**Theorem 5.1** Let $A(t)$ be an $n \times n$ matrix-valued function with continuous coefficients and the two limits

$$A_{\pm\infty} = \lim_{t \to \pm\infty} A(t)$$

exist. Assume that the eigenvalues of $A_{\pm\infty}$ have nonzero real parts. Let $\dim S(A_{\pm\infty})$ be the dimensions of stable sets of $\dot{x} = A_{\pm\infty}x$ respectively. If $\dim S(A_{-\infty}) < \dim U(A_{+\infty})$, then the bounded solutions $B(A)$ of the equation $\dot{x} = A(t)x$ has dimension at least $k = \dim S(A_{+\infty}) - \dim S(A_{-\infty}) \geq 1$.

The fact that $\dim B(A) \geq 1$ can be proved as in Example 5.3. As for $\dim B(A) \geq k$, we will prove it later.
Now we turn to consider LSPFs in the sense of Definition 2.2. Let $M^n$ be an $n$-dimensional compact smooth manifold. Let $F : M^n \to M^n$ be a diffeomorphism of class $C^1$. Then $F$ induces a discrete flow $\sigma$ on $M^n$ by

$$\sigma(y, t) = F^t(y), \quad y \in M^n, \quad t \in \mathbb{Z}.$$ 

Furthermore, $\sigma$ induces a tangent discrete LSPF on the tangent bundle $TM^n$ by

$$\pi(x, y, t) = DF^t(y)x, \quad x \in T_yM^n, \quad y \in M^n, \quad t \in \mathbb{Z}.$$ 

(Recall that the points on $TM^n$ are denoted by $(x, y)$, where $y \in M^n$ and $x \in T_yM^n = $ tangent space of $M^n$ at $y$.) We call $F$ an \textit{quasi-Anosov diffeomorphism} if the bounded set $B$ of $\pi$ is trivial, i.e., $B = (TM^n)_0$ = zero section of $TM^n$. We call $F$ an \textit{Anosov diffeomorphism} if (i) $TM^n = \mathcal{S} \oplus \mathcal{U}$ is a Whitney sum, and (ii) the rate of decay in $\mathcal{S}$ and $\mathcal{U}$ are exponential.

Now the following theorem is a direct result of Corollaries 4.1 and 4.2.

**Theorem 5.2** Let $F : M^n \to M^n$ be a diffeomorphism such that $B = (TM^n)_0$. If one of the following holds:

(i) The flow $\sigma$ on $M^n$ is chain-recurrent, or

(ii) $\dim \mathcal{S}(y)$ is the same over every minimal set in $M^n$,

then $F$ is an Anosov diffeomorphism.

Franks and Robinson have constructed a quasi-Anosov diffeomorphism on a compact 3-dimensional manifold which is not an Anosov diffeomorphism. However, when $n = \dim M^n \leq 2$, we have

**Corollary 5.1** A quasi-Anosov diffeomorphism on a compact manifold of dimension $\leq 2$ is necessarily an Anosov diffeomorphism.

**Proof** In Theorem 4.4, if there are at least two nonempty $Y_k := (M^n)_k$. Then $q < Q$. If $q = 0$ then at points $y \in Y_q$ the mapping is expanding thus contradicting the fact that $Y_q$ is a stable attractor. Therefore $q > 0$. Similarly, $Q < n$. Hence $0 < q < Q < n$ implies that $n \geq 3$. Now if $n \leq 2$ we can apply Theorem 4.3 to obtain that $F$ is an Anosov diffeomorphism. \hfill \qed

### 6 Existence of Trichotomies

Let us see something that cannot be solved using the results on exponential dichotomy in Section 4. Consider a differentiable flow $\sigma : M^n \times \mathbb{R} \to M^n$ generated by a vector $V$ of $M^n$. Then the tangent flow $\pi$ of $\sigma$ on $TM^n$ is a LSPF in the sense of Definition 2.2. If $\sigma$ has a periodic trajectory $\gamma = \gamma(y), \quad y \in M^n$, then $\pi$ has always a nontrivial bounded solution $(V(\sigma(y, t)), \sigma(y, t))$. Thus $B \neq (TM^n)_0$ and the theory of exponential dichotomy in Section 4 is not applicable to this case. In fact, from Theorem 5.2 one sees that theory of ED has possible applications to dynamical systems with discrete time. In this section we need to develop a new theory, i.e., that for trichotomy, so that we can deal with dynamical systems with continuous time.
Let $\pi = (\varphi, \sigma)$ be a LSPF on $X \times Y$, where $X = \mathbb{F}^n$ and $Y$ is always a compact Hausdorff space. So we have $\mathcal{B}, \mathcal{S}$ and $\mathcal{U}$. In this section we are interested in establishing the Whitney decomposition

$$X \times Y = \mathcal{S} \oplus \mathcal{B} \oplus \mathcal{U}.$$  

(6.1)

In order to motivate our theory, we consider the following example.

**Example 6.1** Consider the LSPF generated by a linear differential equation

$$\dot{x} = Ax, \quad x \in \mathbb{C}^2,$$  

(6.2)

with constant coefficients. Without loss of generality, we assume that $A$ takes one of the following forms:

$$\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{C}$ and $\lambda \neq \mu$. In the first two cases, the eigenvectors span the space $\mathbb{C}^2$. These eigenvectors lie in $\mathcal{S}, \mathcal{B}, \mathcal{U}$ depending on whether the real part of the associated eigenvalues is negative, zero, or positive. In any case, one always has $\mathbb{C}^2 = \mathcal{S} \oplus \mathcal{B} \oplus \mathcal{U}$. In the third case, $\mathcal{B}$ is nontrivial iff $\text{Re}\lambda = 0$. Furthermore, $\mathcal{B}$ is 1-dimensional and $\mathcal{S} = \mathcal{U} = \{0\}$. Therefore, one does not have $\mathbb{C}^2 = \mathcal{S} \oplus \mathcal{B} \oplus \mathcal{U}$. Such a phenomenon is called “shearing phenomenon”. In this section we will impose some hypothesis to eliminate this phenomenon for general LSPFs so that one has the decomposition (6.1).

This follows from the following observation. For equation (6.2), the bounded set $\mathcal{B}$ consists of almost periodic functions of $t$. If $\mathcal{B}$ is not trivial, there will be positive constants $C_0$ and $C_1$ such that

$$C_0|\varphi(0)| \leq |\varphi(t)| \leq C_1|\varphi(0)|, \quad t \in \mathbb{R}.$$  

(6.3)

This property will constitute part of our hypothesis.

Let $\mathcal{W} \subset X \times Y$. We say that $\mathcal{W}$ satisfies Hypothesis (BB) if

(i) $\mathcal{W}$ is an invariant subbundle for the flow $\pi$, and

(ii) there exists positive constants $C_0, C_1$ and $C_2$ such that

$$C_0|x| \leq |\Phi(y, t)x| = |\varphi(x, y, t)| \leq C_1|x|$$  

(6.4)

for all $y \in Y$, $t \in T$ and $x \in \mathcal{W}(y)$, and

$$|\Phi(y, t)\Phi^{-1}(y, s)x| \leq C_2|x|$$  

(6.5)

for all $y \in Y$, $s, t \in T$ and $x \in \mathcal{W}(y \cdot s)$.

We remark that two inequalities (6.4) and (6.5) are equivalent statements. Note also that if $\mathcal{W}$ satisfies (BB) then $\mathcal{W} \subset \mathcal{B}$. If $\mathcal{B}$ is trivial then $\mathcal{B}$ itself satisfies (BB). Hypothesis (BB) is actually the uniform stability of $\pi$ restricted to $\mathcal{W}$ when $t$ tends both to $+\infty$ and to $-\infty$.

Let $\mathcal{W}$ be a subbundle of $X \times Y$. We define the **orthogonal complement** $\mathcal{W}^\perp$ by

$$\mathcal{W}^\perp = \{(x, y) \in X \times Y : x \perp \mathcal{W}(y)\}.$$
Then $W^\perp$ is also a subbundle and that $X \times Y = W \oplus W^\perp$ is a Whitney decomposition. Define $\hat{P} : X \times Y \to X \to Y$ by $\hat{P}(x, y) = (P(y)x, y)$ where $P(y)$ is the projection on $X$ with range $W^\perp(y)$ and null $W(y)$. Then $\hat{P}$ is continuous jointly in $x$ and $y$.

When $W$ is an invariant subbundle under the flow $\pi$, the orthogonal subbundle (or, the conjugate subbundle in Liao’s terminology) $W^\perp$ may not be invariant. However, one can construct the following flow $\hat{\pi}$ on $W^\perp$.

Lemma 6.1 Let $W$ be an invariant subbundle of $X \times Y$ and define $\hat{\pi}$ by

$$\hat{\pi}(x, y, t) = \hat{P}(\pi(x, y, t)) = (P(y \cdot t)\varphi(x, y, t), y \cdot t)$$

for $(x, y) \in W^\perp$ and $t \in T$. Then $\hat{\pi}$ is a LSPF on $W^\perp$.

Proof It is a simple computation. □

The main result in this section is

Theorem 6.1 Assume that $W \subset X \times Y$ is an invariant subbundle and satisfies $(BB)$. If $\hat{\mathcal{B}}$, the associated bounded set for the induced flow $\hat{\pi}$ on $W^\perp$, is trivial. Assume further that $Y$ is compact and has one of the following properties:

(i) The flow $\sigma$ on $Y$ is chain-recurrent, or
(ii) there exists precisely one integer $k$ such that every minimal set of $Y$ lies in $\hat{\mathcal{Y}}_k = \{ y \in Y : \dim \hat{\mathcal{S}}(y) = k \}$.

Then the following conclusions are valid.

(VII) $W = \mathcal{B}$,
(VIII) $\mathcal{S}$ and $\mathcal{U}$ are subbundles of $X \times Y$. Furthermore

$$\mathcal{S} = \lim \sup_{t \to -\infty} \pi(\hat{\mathcal{S}}, t), \quad \mathcal{U} = \lim \sup_{t \to +\infty} \pi(\hat{\mathcal{U}}, t),$$

where $\hat{\mathcal{S}}$ and $\hat{\mathcal{U}}$ are the stable and unstable sets for the flow $\hat{\pi}$,

(IX) $\dim \mathcal{S}(y) = \dim \hat{\mathcal{S}}(y)$ and $\dim \mathcal{U}(y) = \dim \hat{\mathcal{U}}(y)$ for all $y \in Y$.

(X) $X \times Y = \mathcal{S} \otimes \mathcal{B} \otimes \mathcal{U}$ is a Whitney decomposition.

(XI) Let $P_\omega(y)$ be the projection of $X$ with range $\omega(y)$ and the null space spanned by the remaining two subspaces, where $\omega$ stands for $\mathcal{S}$, $\mathcal{B}$ or $\mathcal{U}$. Moreover, there exists positive constants $K$ and $\beta$ such that

$$\| \Phi(y, t)P_s(y)\Phi^{-1}(y, s) \| \leq Ke^{-\beta(t-s)}, \quad s \leq t,$$

$$\| \Phi(y, t)P_u(y)\Phi^{-1}(y, s) \| \leq Ke^{-\beta(s-t)}, \quad t \leq s,$$

$$\| \Phi(y, t)P_b(y)\Phi^{-1}(y, s) \| \leq K, \quad \forall s, t \in T.$$

Remark 6.1 Under the assumptions in Theorem 6.1, the assumption (i) does imply (ii) using the theory in Section 4.
Let \((x, y) \in X \times Y\). Define \(v = P(y)x\) and \(u = x - P(y)x\). Then \((u, x) \in W\) and \((v, y) \in W^\perp\).

The solutions \(\varphi(u, y, t)\) and \(\varphi(v, y, t)\) have also decompositions. As \(W\) is \(\pi\)-invariant, one has \((\varphi(u, y, t), y \cdot t) \in W\), i.e.,

\[
\varphi(u, y, t) = A(y, t)u,
\]

where \(A(y, t) : W(y) \to W(y \cdot t)\) is a linear transformation. One also has

\[
\varphi(v, y, t) = B(y, t)v + D(y, t)v,
\]

where \(B(y, t) : W^\perp(y) \to W^\perp(y \cdot t)\) and \(D(y, t) : W^\perp(y) \to W^\perp(y \cdot t)\) are linear transformations.

If one writes \(x\) as in the vector form \(x = \begin{pmatrix} u \\ v \end{pmatrix}\), then

\[
\varphi(x, y, t) = \Phi(y, t)x = \Phi(y, t) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A(y, t)u + B(y, t)v \\ D(y, t)v \end{pmatrix} = \begin{pmatrix} A(y, t) \\ 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\]

By the group property of the flow \(\pi\), one has

\[
A(y \cdot t, s)A(y, t) = A(y, t + s), \quad D(y \cdot t, s)D(y, t) = D(y, t + s),
\]

for all \(y \in Y\) and for all \(t, s \in T\). In particular, by setting \(s = -t\) one sees that

\[
A^{-1}(y, t) = A(y \cdot t, -t), \quad D^{-1}(y, t) = D(y \cdot t, -t).
\]

Define \(u_t\) and \(v_t\) by

\[
u_t = A(y, t)u + B(y, t)v, \quad v_t = D(y, t)v.
\]

Then \(u_t\) and \(v_t\) denote the \(t\)-evaluation of the \(u\) and \(v\) coordinates under the flow \(\pi\). In these coordinates, the induced flow \(\hat{\pi}\) on \(W^\perp\) is given by

\[
\hat{\pi}(v, y, t) = (D(y, t)v, y \cdot t),
\]

that is \(\hat{\Phi}(y, t) = D(y, t)\). Note also that the mapping

\[
(u, y, t) \to (A(y, t)u, y \cdot t)
\]

is simply the restriction of the original flow \(\pi\) to the invariant subbundle \(W\).

The next lemma gives an important variation-of-constants formula for the coordinate \(u_t\).

**Lemma 6.2** Let \(W\) and \(u_t\) and \(v_t\) be as above. For any \(t \in T\) and all integers \(n\) one has

\[
u_{nt} = A(y, nt)u + A(y, nt) \sum_{i=1}^{n} A^{-1}(y, it)B(y \cdot (i - 1)t, t)D(y, (i - 1)t)v.
\]
When \( n = 1 \), (6.10) holds because \( D(y, 0) = I \). Assume now that (6.10) holds for \( n \). It follows (6.7) that

\[
\begin{align*}
u_{(n+1)t} &= A(y \cdot nt, t)u_{nt} + B(y \cdot nt, t)v_{nt} \\
&= A(y \cdot nt, t) \left[A(y, nt)u + A(y, nt) \sum_{i=1}^{n} \Gamma_i \right] + B(y \cdot nt, t)v_{nt},
\end{align*}
\]

where \( \Gamma_i = A^{-1}(y, it)B(y \cdot (i - 1)t, t)D(y, (i - 1)t)v \). Using the group properties of \( A(y, t) \) and \( D(y, t) \) and the fact that \( v_{nt} = D(y, nt)v \) we get

\[
\begin{align*}
u_{(n+1)t} &= A(y, (n + 1)t)u + A(y, (n + 1)t) \sum_{i=1}^{n+1} \Gamma_i \\
&= A(y, (n + 1)t)u + A(y, (n + 1)t) \sum_{i=1}^{n+1} \Gamma_i,
\end{align*}
\]

where \( \Gamma_{n+1} = A^{-1}(y, (n + 1)t)B(y \cdot nt, t)D(y, nt)v \). Thus the lemma is proved. \( \square \)

Remark 6.2 When \( T = \mathbb{R} \) and \( B(y, t) \) is continuously differentiable in \( t \) at \( t = 0 \),

\[
b(y) = \frac{\partial}{\partial t} B(y, t)|_{t=0},
\]

one has

\[
u_t = A(y, t) \left[u + \int_0^t A^{-1}(y, s)b(y \cdot s)D(y, s)vds \right],
\]

which coincides with the usual variation-of-constants formula for ODEs.

Observing that the mapping \((u, y, t) \rightarrow (A(y, t)u, y \cdot t)\) is precisely the restriction of the flow \( \pi \) to \( \mathcal{W} \), one has an equivalent statement for (BB) in terms of the local coordinates \( u \).

Lemma 6.3 Let \( \mathcal{W} \) be an invariant subbundle in \( X \times Y \) and let \( A(y, t), B(y, t) \) and \( D(y, t) \) be as above. If \( \mathcal{W} \) satisfies (BB), then one has

\[
C_0 |u| \leq |A(y, t)u| \leq C_1 |u| \tag{6.11}
\]

for all \((u, y) \in \mathcal{W} \) and \( t \in T \), and

\[
|A(y, t)A^{-1}(y, s)u| \leq C_2 |u| \tag{6.12}
\]

for all \( s, t \in T \), where \( C_j \) are same as in (BB).

Now we give the proof of Theorem 6.1. First we note that the theory in Section 4 for ED applies to the flow \( \hat{\pi} \) on \( \mathcal{W}^\perp \) although there is a little bit difference between the definitions of the sets \( \hat{Y}_k \). However, it is easy to check that under the assumption (i) or (ii) in the statement of Theorem 6.1, there will be only one nonempty set \( Y_k = \{ y \in Y : \dim \hat{S}(y) = k \} \) and \( \dim \hat{U}(y) = \) the fiber dimension of \( \mathcal{W}^\perp \) minus \( k \). As a result, Theorem 4.3 is applicable to \( \hat{\pi} \) on \( \mathcal{W}^\perp \). Namely, one can write \( \mathcal{W}^\perp = \hat{S} \oplus \hat{U} \) as a Whitney sum. Moreover, let \( \hat{Q} : \mathcal{W}^\perp \to \mathcal{W}^\perp \) be defined by \( \hat{Q}(x, y) = (Q(y)x, y) \), where \( Q(y) \) is the projection on \( \mathcal{W}^\perp(y) \) with range \( \hat{S}(y) \) and null \( \hat{U}(y) \).
Then, by recalling that $\hat{\Phi}(y, t) = D(y, t)$ for all $y \in Y$, there are some positive constants $\alpha$ and $K_0$ such that
\[
\|D(y, t)Q(y)D^{-1}(y, s)\| \leq K_0 e^{-\alpha(t-s)}, \quad s \leq t,
\]
\[
\|D(y, t)[I - Q(y)]D^{-1}(y, s)\| \leq K_0 e^{-\alpha(s-t)}, \quad t \leq s.
\]
(6.13)

The subbundles $\hat{S}$ and $\hat{U}$ are $\hat{\pi}$-invariant but are in general not $\pi$-invariant. From these subbundles and $W$ one can construct subbundles $\mathcal{E}_s = \hat{S} \oplus W$ and $\mathcal{E}_u = \hat{U} \oplus W$, which may be called the stable-center and unstable-center subbundles of $\pi$ respectively. Our first aim is to prove

**Proposition 6.1** The subbundles $\mathcal{E}_s$ and $\mathcal{E}_u$ are $\pi$-invariant.

**Proof** As before, let $P(y)$ denote the projection on $X$ with range $W^\perp(y)$ and null $W(y)$. If $(x, y) \in \hat{S}$, by the invariance of $\hat{S}$ under $\hat{\pi}$, one has
\[
\varphi(x, y, t) = P(y)\varphi(x, y, t) + [I - P(y\cdot t)]\varphi(x, y, t) = \hat{\Phi}(x, y, t) + [I - P(y\cdot t)]\varphi(x, y, t) \in \hat{S}(y\cdot t) + W(y\cdot t) = \mathcal{E}_s(y\cdot t).
\]
Thus $\pi(x, y, t) = (\varphi(x, y, t), y\cdot t) \in \mathcal{E}_s$. This proves that $\pi(\hat{S}, t) \subset \mathcal{E}_s$ for all $t$. Since $\pi(W, t) \subset W \subset \mathcal{E}_s$, we prove that
\[
\pi(\mathcal{E}_s, t) = \pi(\hat{S} \oplus W, t) = \pi(\hat{S}, t) + \pi(W, t) \subset \mathcal{E}_s + \mathcal{E}_s = \mathcal{E}_s.
\]
This proves that $\mathcal{E}_s$ is $\pi$-invariant. Likewise, $\mathcal{E}_u$ is also $\pi$-invariant. \hfill \Box

Now let us restrict our attention to the invariant subbundles $\hat{S}$ and $\mathcal{E}_s$. Define $\hat{S}_t = \pi(\hat{S}, t)$, i.e., $\hat{S}_t$ is the time evolution of $\hat{S}$ under the flow $\pi$. It is easy to see that each $\hat{S}_t$ is a subbundle of $\mathcal{E}_s$ and $\dim \hat{S}_t(y) = k$ for all $y \in Y$. Define
\[
\mathcal{S}' = \limsup_{t \to -\infty} \hat{S}_t = \bigcap_{t \in T} \bigcup_{t \leq \tau} \hat{S}_t,
\]
cf. Lemma 4.8. At a first glimpse, $\mathcal{S}'$ may not be a subbundle. However, after a series of steps we will prove that $\mathcal{S}'$ has some desired properties. Denote by $\mathcal{S}'(y)$ the fibers of $\mathcal{S}'$ as usual. Similar to Lemma 4.8 one knows that the fiber $\mathcal{S}'(y)$ contains a linear subspace of $X$ of dimension $k$ for each $y \in Y$.

As $\mathcal{E}_s$ is a closed set invariant under $\pi$, one sees that $\mathcal{S}'$ is a closed set and is contained in $\mathcal{S}_s$. We will prove that $\mathcal{S}' \subset \mathcal{E}_s$. In fact a further analysis shows that the equality described as in Theorem 6.1 holds also. To this end we need to analyze the growth of $\varphi(x, y, t)$ for $(x, y) \in \mathcal{S}'$. This will be done in the following propositions.

Let $(x, y) = (u + v, y) \in \mathcal{E}_s$, where $(u, y) \in W$ and $(v, y) \in \hat{S} \subset W^\perp$. Then the solution $\varphi(x, y, t)$ can be expressed as the sum $u_t + v_t$.

**Proposition 6.2** There exist positive constants $\alpha$, $C_3 (\ll 1)$ and $C_4 (\gg 1)$ with the property that if $(u, y) \in W$ and $(v, y) \in \hat{S}$ satisfy
\[
|v| \leq C_3|u|,
\]
(6.14)
then
\[
|v_t| \leq C_4 e^{-\alpha t}|u_t|, \quad t \in T^+.
\]
(6.15)
Moreover, if the inequality (6.14) is strict, then inequality (6.15) is also strict.
Proof We need not consider the case $u = 0$ because this implies that $v = 0$ and $u_t = 0$ and $v_t = 0$ for all $t$.

At first we consider the case $T = \mathbb{Z}$. We apply (6.10) with $t = 1$. Then one has

$$|u_n| = |A(y, n)u + \sum_{i=1}^{n} A(y, n)A^{-1}(y, i)B(y \cdot (i - 1), 1)D(y, i - 1)v|$$

$$\geq |A(y, n)u| - \sum_{i=1}^{n} |A(y, n)A^{-1}(y, i)B(y \cdot (i - 1), 1)D(y, i - 1)v|$$

$$\geq C_0|u| - C_2 \sum_{i=1}^{n} \|B(y \cdot (i - 1), 1)\| \|D(y, i - 1)v|,$$

where (6.11) and (6.12) in (BB) are used. Note that $B(y, 1)$ is continuous in compact space $y \in Y$. One has

$$M_1 := \sup\{\|B(y, 1)\| : y \in Y\} < +\infty.$$

Note that $(v, y) \in \hat{S}$, the stable set of $\pi$. One then has

$$|D(y, i - 1)v| = |\hat{\Phi}(y, i - 1)v| \leq K_0e^{-\alpha(i-1)|v|}$$

for all $i = 1, 2, \ldots$, where (6.13) is used. These imply that

$$|u_n| \geq C_0|u| - C_2M_1K_0\left(\sum_{i=1}^{n} e^{-\alpha(i-1)}\right) |v| \geq C_0|u| - C_2M_1K_0M_2|v|,$$

where $M_2 = (1 - e^{-\alpha})^{-1}$. Thus, if

$$|v| \leq \bar{C}_3|u|$$

for $\bar{C}_3 = C_0(2C_2M_1K_0M_2)^{-1}$, then

$$|u_n| \geq \frac{1}{2}C_0|u|, \quad n = 0, 1, 2, \ldots$$

We remark here that $|u_n| = O(|u|)$ in this case.

When (6.16) is satisfied, we have by noticing that $|v_n| \leq K_0e^{-\alpha n}|v|$ for all $n \geq 0,

$$\frac{|v_n|}{|u_n|} \leq \frac{K_0e^{-\alpha n}|v|}{(C_0/2)|u|} \leq \bar{C}_4 e^{-\alpha n}, \quad n = 0, 1, 2, \ldots$$

where $\bar{C}_4 = 2K_0C_0^{-1}\bar{C}_3$. This establishes (6.15) for the case $T = \mathbb{Z}$ if one sets $C_3 = \bar{C}_3$ and $C_4 = \bar{C}_4$.

For the case $T = \mathbb{R}$, (6.18) also holds for nonnegative integers. We use the continuity of $\pi$ to find a constant $\bar{C}_3 > 0$ such that if $|v| \leq \bar{C}_3|u|$ then

$$|v_{\tau}| \leq \bar{C}_3|u_{\tau}|, \quad 0 \leq \tau \leq 1.$$

For general $t \in \mathbb{R}^+$, write $t = n + \tau$, where $n \in \mathbb{Z}^+$ and $\tau \in [0, 1)$. Consequently, $u_t = (u_{\tau})_n$ and $v_t = (v_{\tau})_n$. By (6.18) we have

$$\frac{|v_t|}{|u_t|} = \frac{|(v_{\tau})_n|}{|(u_{\tau})_n|} \leq 2K_0C_0^{-1}e^{-\alpha n}|v_{\tau}| \leq C_4 e^{-\alpha \tau},$$

where $C_4 = \bar{C}_4 e^{\alpha}$. This proves (6.15).

The strict inequality (6.15) can be proved by noticing that (6.17) is strict if the strict inequality (6.14) is satisfied. \qed
Proposition 6.3 Let \( \tau \geq 0 \) be such that \( C_4 e^{-\alpha \tau} = C_3 \). If \( |v| \geq C_3|u| \) then \( |v_t| \geq C_3|u_t| \) for all \( t < -\tau \).

**Proof** If it is false, there would be some \( s < -\tau \) such that \( |v_s| < C_3|u_s| \) and \( |v| \geq C_3|u| \). Note that \( u_s \in \mathcal{W}(y \cdot s) \). As \( \mathcal{E}_s \) is \( \pi \)-invariant, one has \( v_s \in \mathcal{W}(y \cdot s) \cap \mathcal{E}_s(y \cdot s) \subset \hat{S}(y \cdot s) \). Now one can apply the strict inequality \( (6.15) \) to \( u' = u_s, v' = v_s \) and \( t = -s \). Note that \( (v_s)_{-s} = u \) and \( (v_s)_{-s} = v \). The inequality \( (6.15) \) will yield that \( s \geq -\tau \), which contradicts our assumption. \( \square \)

Proposition 6.4 Let \( C_3 \) and \( \tau \) be as above. Let \( a \in T \) with \( a \leq -\tau \). If \( (x, y) \in \pi(\hat{S}, a) \) has the decomposition \( (x, y) = (u, y) + (v, y) \in \mathcal{W} \oplus \hat{S} \), then \( |v_t| \geq C_3|u_t| \) for all \( 0 \leq t < -a - \tau \).

**Proof** Let \( (x_0, y_0) = \pi(x, y, -a) \in \hat{S} \). Now one can apply Proposition 6.3 to the decomposition \( (x_0, y_0) = (0, y_0) + (x_0, y_0) =: (u_0, y_0) + (v_0, y_0) \) and \( t < -\tau \). Then one gets
\[
|v_{t-a}| = |(v_0)_t| \geq C_3|(u_0)_t| = C_3|u_{t-a}|
\]
for all \( t < -\tau \). Thus the proposition is proved. \( \square \)

Now we can characterize the growth of \( \varphi(x, y, t) \) for \( (x, y) \in S' \).

Proposition 6.5 Let \( (x, y) = (u + v, y) \in S' \), where \( (u, y) \in \mathcal{W} \) and \( (v, y) \in \hat{S} \). Then one has
\[
|v_t| \geq C_3|u_t|, \quad t \geq 0.
\]

**Proof** \( (x, y) \in \mathcal{E}' \) means that \( (x, y) = \lim_n (x^n, y^n) \), where \( (x^n, y^n) \in \pi(\hat{S}, a_n) \) with \( a_n \to -\infty \). Let \( (x^n, y^n) = (u^n, y^n) + (v^n, y^n) \in \mathcal{W} \oplus \hat{S} \). Without loss of generality, one may assume that \( (u^n, y^n) \to (u, y) \in \mathcal{W} \) and \( (v^n, y^n) \to (v, y) \in \hat{S} \). By Proposition 6.4 one has
\[
|v^n_t| \geq C_3|u^n_t|, \quad 0 \leq t < -a_n - \tau.
\]
By the continuity, the limit gives \( (6.19) \) because \( -a_n - \tau \to \infty \).

We remark here that \( (6.19) \) actually holds for all \( t \in T \). \( \square \)

Proposition 6.6 \( S' \subset \mathcal{S} \).

**Proof** Let \( (x, y) \in \mathcal{S}' \). We need to show that \( |\varphi(x, y, t)| \to 0 \) as \( t \to \infty \). Let \( (x, y) = (u, y) + (v, y) \in \mathcal{W} \oplus \hat{S} \) and \( \varphi(x, y, t) = u_t + v_t \). Since \( (v_t, y \cdot t) = (D(y, t)v, y \cdot t) = \hat{\pi}(v, y, t) \), one has \( |v_t| \to 0 \) as \( t \to \infty \) because \( (v, y) \in \hat{S} \). By \( (6.19) \) \( |u_t| \to 0 \) as \( t \to \infty \). Therefore \( |\varphi(x, y, t)| \to 0 \) as \( t \to \infty \). \( \square \)

Proposition 6.7 \( S' \) is a subbundle of \( \mathcal{E}_s \) and \( S' = S \cap \mathcal{E}_s \). Furthermore \( \dim S'(y) = \dim \hat{S}(y) \) for all \( y \in Y \).

**Proof** Note that \( k = \dim \hat{S}(y) \) and \( l = \dim \mathcal{W}(y) \) are constant for all \( y \in Y \). Thus \( \dim \mathcal{E}_s(y) = k + l \).

As \( S' \subset \mathcal{E}_s \), by Proposition 6.6 \( S' \subset \mathcal{E}_s \cap \mathcal{S} \). From the definition of \( (BB) \), it is easy to see that \( S(y) \cap \mathcal{W}(y) = \{0\} \) for all \( y \). Thus \( (\mathcal{E}_s(y) \cap S(y)) \cap \mathcal{W}(y) = \{0\} \). This shows that \( \dim [\mathcal{E}_s(y) \cap S(y)] \leq k \). On the other hand, \( \mathcal{E}_s(y) \cap S(y) \) contains \( S'(y) \) which contains at least a subspace of dimension \( k \). Therefore, \( \mathcal{E}_s(y) \cap S(y) = S'(y) \). This shows that \( S' = S \cap \mathcal{E}_s \), which is a subbundle of \( \mathcal{E}_s \) because \( S' \) is closed. \( \square \)
Proposition 6.8 $S'$ is $\pi$-invariant and $E_s = S' \oplus W$ is a Whitney decomposition.

Proof Since both $E_s$ and $S$ are $\pi$-invariant, $S' = E_s \cap S$ is also $\pi$-invariant.

First one has $S'(y) \cap W(y) = E_s(y) \cap S(y) \cap W(y) = \{0\}$ for all $y \in Y$. Now the fact $E_s(y)$ is spanned by $S'(y)$ and $W(y)$ follows from $\dim S(y) = k$, $\dim W(y) = l$ and from $\dim E_s(y) = k+l$. 
\[\square\]

Analogously, let $E_u = W \oplus \hat{U}$ and

$$U' = \limsup_{t \to +\infty} \pi(\hat{U}, t).$$

Then

Proposition 6.9 $U'$ is a subbundle of $E_u$ with $U' \subset U$ and $U' = E_u \cap U$. Furthermore, $U'$ is $\pi$-invariant and $E_u = U' \oplus W$ is a Whitney sum, and $\dim U(y) = \dim \hat{U}(y)$ for all $y \in Y$.

Proposition 6.10 $X \times Y = W \oplus S' \oplus U'$ is a Whitney sum.

Proof

$$X \times Y = W \oplus W^\perp = W \oplus (\hat{S} \oplus \hat{U})$$

$$= (W \oplus \hat{S}) \oplus \hat{U} = E_s \oplus \hat{U}$$

$$= (S' \oplus \hat{W}) \oplus \hat{U} = S' \oplus (W \oplus \hat{U})$$

$$= S' \oplus E_u = S' \oplus W \oplus U'.$$

Proposition 6.11 $W = \mathcal{B}$, $S' = \mathcal{S}$ and $U' = \mathcal{U}$. Furthermore $X \times Y = \mathcal{B} \oplus \mathcal{S} \oplus \mathcal{U}$ is a Whitney sum.

Proof By (BB) one has simply $W \subset \mathcal{B}$. For the opposite inclusion, let $(x, y) \in \mathcal{B}$. Write $(x, y) = (u, y) + (v, y) \in W \oplus W^\perp$. Let $P(y)$ be the projection on $X$ with range $W^\perp(y)$ and null $W(y)$. Then $\|P(y)\| \leq K$ for some $K$ and for all $y \in Y$ because $Y$ is compact. Now

$$|D(y, t)v| = |\varphi(v, y, t)| = |P(y \cdot t)\varphi(v, y, t)| \leq K|\varphi(v, y, t)|$$

$$= K|\varphi(x, y, t) - \varphi(u, y, t)| \leq K|\varphi(x, y, t)| + K|\varphi(u, y, t)|$$

$$\leq K|\varphi(x, y, t)| + KC_1|u|,$$

where (BB) is used. This shows that $\sup_t |D(y, t)v| < \infty$ because $(x, y) \in \mathcal{B}$. Namely, $(v, y) \in \hat{\mathcal{B}}$. Thus $v = 0$ and $(x, y) \in \mathcal{W}$. This proves that $\mathcal{B} = \mathcal{W}$.

For the conclusion $S' = \mathcal{S}$, it suffices to prove that $\mathcal{S} \subset E_s$ by noticing Proposition 6.7. As $X \times Y = E_s \oplus \hat{U}'$, any $(x, y) \in \mathcal{S}$ can be written as $(x, y) = (u, y) + (v, y) \in E_s \oplus \hat{U}'$. As $|\varphi(x, y, t)| \to 0$ as $t \to +\infty$, and $(u, y) \in E_s$ shows that $|\varphi(u, y, t)|$ is bounded in $t \in [0, \infty)$, we have

$$\varphi(v, y, t) = \varphi(x, y, t) - \varphi(u, y, t)$$

is bounded in $t \in [0, \infty)$. On the other hand, $|\varphi(v, y, t)| \to 0$ as $t \to -\infty$ because $(v, y) \in \hat{U} \subset \mathcal{U}$. Therefore $|\varphi(v, y, t)|$ is bounded in $t \in T$ and $(v, y) \in \mathcal{B} \cap \hat{U}' = \{0\} \times Y$. Consequently $v = 0$ and $(x, y) \in E_s$. 
\[\square\]
Now we end the proof of Theorem 6.1 by proving (XI). To this end, let $P_b(y)$ be the projection on $X$ with range $\mathcal{B}(y)$ and null $\mathcal{S}(y) \oplus \mathcal{U}(y)$. For $(x, y) \in X \times Y$, let $(x, y) = (P_b(y)x, y) + ((I - P_b(y))x, y) = (u, y) + (v, y)$. As $\mathcal{B}$ and $\mathcal{S} \oplus \mathcal{U}$ are $\pi$-invariant, one has

$$P_b(y)\Phi^{-1}(y, s) = P_b(y)\Phi^{-1}(y, s)P_b(y \cdot s) = \Phi^{-1}(y, s)P_b(y \cdot s).$$

If $\tilde{u} = P_b(y \cdot s)x$, then

$$P_b(y)\Phi^{-1}(y, s)x = \Phi^{-1}(y, s)P_b(y \cdot s)x = \Phi^{-1}(y, s)\tilde{u} = A^{-1}(y, s)\tilde{u},$$

because $A(y, t)$ denotes the restriction of $\Phi(y, t)$ to $\mathcal{B}$. Thus

$$\Phi(y, t)P_b(y)\Phi^{-1}(y, s)x = \Phi(y, t)A^{-1}(y, s)\tilde{u} = A(y, t)A^{-1}(y, s)\tilde{u}.$$ 

Now Lemma 6.3 shows that

$$|\Phi(y, t)P_b(y)\Phi^{-1}(y, s)x| \leq |A(y, t)A^{-1}(y, s)\tilde{u}| \leq C_2|P_b(y \cdot s)x| \leq C_2K|x|.$$

For the exponential estimates in (XI) one may apply the theory in Section 4 to the restricted flow of $\pi$ to the invariant subbundle $\mathcal{S} \oplus \mathcal{U}$. This completes the proof of Theorem 6.1. \hfill $\square$

Let us compare (BB) with Favard’s conditions concerning the existence of almost periodic solutions to

$$\dot{x} = A(t)x + f(t), \quad (6.20)$$

where $A(t)$ and $f(t)$ are aperiodic in $t$. Favard’s conditions are:

(i) The inhomogeneous equation (6.20) has at least one bounded solution on $t$; and

(ii) for every $\tilde{A} \in \mathcal{A} = H(A)$, the homogeneous equation $\dot{x} = \tilde{A}(t)x$ has the property that its every nontrivial bounded solution $\varphi(t)$ satisfies $\inf\{||\varphi(t)|| : t \in \mathbb{R}\} > 0$.

From condition (ii) we introduce the following concept.

**Definition 6.1** For a LSPF $\pi = (\varphi, \sigma)$, we say that $\pi$ or its bounded set $\mathcal{B}$ has the **Favard property** if every $(x, y) \in \mathcal{B}$ with $x \neq 0$ satisfies

$$\alpha(x, y) := \inf\{|\varphi(x, y, t)| : t \in T\} > 0. \quad (6.21)$$

It is obvious that if $\mathcal{B}$ satisfies (BB) then $\mathcal{B}$ has Favard property. Conversely, we have

**Proposition 6.12** If a LSPF $\pi = (\varphi, \sigma)$ has a minimal base flow $\sigma$, then Favard property implies also that $\mathcal{B}$ satisfies (BB).

**Remark 6.3** There are examples showing that the implication from Favard property to (BB) will fail if the base flow $\sigma$ on $Y$ is only assumed to satisfy one of the following properties:

(i) $\sigma$ is chain-recurrent in $Y$;

(ii) $Y$ is compact, connected and contains a collection of minimal subsets whose union is dense; or, even

(iii) $Y$ is compact, connected and equal to union of its minimal subsets.
7 Dynamical Spectrum

In this section we present some general spectral theory for cocycles or linear skew-product flows from the view of point of dynamics. Such a dynamical spectrum theory will be compared with the Lyapunov exponents and the measure spectrum in the next section.

In order that the theory to be developed is applicable to invariant sets of differentiable dynamical systems, we generalize the concepts of bundles and LSPFs in Sections 1 and 2 a little bit.

**Definition 7.1** A triple \((E, Y, p)\) is called a bundle with base \(Y\) and projection \(p\), if

(i) \(E\) and \(Y\) are Hausdorff spaces and \(p : E \to Y\) is a continuous mapping of \(E\) onto \(Y\),

(ii) for each \(y \in Y\), \(X_y := p^{-1}(y)\) is a vector space, and

(iii) for each \(y \in Y\), there exists an open set \(G \subset Y\) with \(y \in G\), a finite-dimensional vector space \(X\), and a homeomorphism \(\tau : p^{-1}(G) \to X \times G\) which can be represented in the form \(\tau(x, \eta) = (\tau_y x, \eta)\), where \(\tau_y\) is a linear isomorphism from the fiber \(X_y\) onto \(X\).

For the trivial bundle \(E = X \times Y\), where \(X\) is a linear space, then \(p\) is the natural projection onto \(Y\). Also \(X_y = p^{-1}(y) = X \times \{y\}\) and the mapping \(\tau\) is the identity.

In the definition of bundles, it is possible for the dimension of the fiber \(X_y\) to vary with \(y \in Y\). However, it is necessary that \(\text{dim} X_y\) is constant for \(y\) in each connected component of \(Y\). We always assume that \(\max \text{dim} X_y\) is finite and the dimension of \(E\) is defined to be this number. As \(X = \mathbb{R}^n\), each fiber admits an inner product \(\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_y\) as well as the associated norm \(|\cdot| = |\cdot|_y\). When (and we always assume that) \(Y\) is metrizable or compact, then the inner product and the norm can be chosen to vary continuously on \(Y\).

For the points in \(E\), we use the notation \((x, y)\) and \((u, y) + (v, y) = (u + v, y)\) for \(u, v \in X_y\). For a subset \(V\) the fiber \(V(y)\) is

\[V(y) = \{x \in X_y : (x, y) \in V\} = \{(x, y) \in E : (x, y) \in V\}.\]

More generally, if \(M \subset Y\) we define

\[V(M) = \cup_{y \in M} V(y) = \{(x, y) \in V : y \in M\}.\]

Thus \(E(M)\) is the restriction of \(E\) to \(M\). In this case, \(E(M)\) is also a bundle with base \(M\) and the projection \(p|E(M)\). The zero section of \(E\) is \(E_0 = \{(x, y) : y \in Y, x = 0_y\}\).

**Definition 7.2** A subset \(V\) of \(E\) is called a subbundle if \(V\) is a closed set of \(E\) such that \((i) \ V(y)\) is a linear subspace of \(X_y\) for each \(y \in Y\), and \((ii) \ \text{dim} V(y)\) is constant on each component of \(Y\). In this case, \(V\) is itself a bundle and the linear subspaces \(V(y)\) vary continuously on \(Y\).

**Definition 7.3** A flow \(\pi\) on \(E\) is said to be a linear skew-product flow if \(\pi\) can be represented in the form

\[\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t)) = (\Phi(y, t)x, y \cdot t),\]

where \(\sigma\) is a flow on \(Y\), and the mapping from \(X_y\) onto \(X_{y, t}\) given by \(x \to \varphi(x, y, t)\) is linear in \(x\).
The linear mappings \( \Phi(y, t) \) have the cocycle property
\[
\Phi(y \cdot t, s) \Phi(y, t) = \Phi(y, t + s).
\] (7.1)
It is obvious that \( \dim X_{y \cdot t} = \dim X_y \) for each \( y \in Y \) and all \( t \in T \).

Besides the important examples 2.2 and 2.3 of LSPFs in Section 2, we present another class of LSPFs.

**Example 7.1** Let \( \sigma(y, t) = y \cdot t \) be a flow on a Hausdorff space \( Y \). For any given continuous matrix-valued function \( A : Y \to \text{gl}(\mathbb{R}^n) \), one can obtain a LSPF as follows. Let
\[
\pi(x, y, t) = (\phi(x, y, t), y \cdot t),
\]
where \( \phi(x, y, t) \) is the solution of the following problem at time \( t \):
\[
\dot{x} = A(y \cdot t)x, \quad x(0) = x.
\]
Then \( \pi \) is a LSPF on \( \mathbb{R}^n \times Y \). This example is a generalization of Example 2.2 in some sense. Such LSPFs are the objective of some extensive works.

**Definition 7.4** Let \( M \) be a compact \( \sigma \)-invariant subset of \( Y \). We say that \( M \) is dynamically connected (or equivalently, invariantly connected) if \( M \) cannot be written as two disjoint nonempty compact invariant sets.

In the case \( T = \mathbb{R} \), dynamically connected is simply connected in topology.

For a LSPF \( \pi \), we are interested in finding the decomposition of \( E \) into invariant subbundles under \( \pi \). For this, let \( M \subset Y \) and \( V_1(M), \ldots, V_l(M) \) be a collection of subbundles of \( E(M) \). If

(i) \( V_i(y) \cap V_j(y) = \{0\} \) for \( i \neq j \) and \( y \in M \),
(ii) \( X_y = V_1(y) \oplus \cdots \oplus V_l(y) \) for all \( y \in M \),

then we express this as
\[
E(M) = V_1(M) \oplus \cdots \oplus V_l(M) \quad \text{(Whitney sum)}.
\]

Let \( \pi = (\phi, \sigma) \) be a LSPF on \( E \). For any \( \lambda \in \mathbb{R} \), define \( \pi_\lambda \) by
\[
\pi_\lambda(x, y, t) = (e^{-\lambda t}\phi(x, y, t), y \cdot t) = (\Phi_\lambda(y, t)x, y \cdot t).
\]
It is easy to verify that \( \pi_\lambda \) is also a LSPF on \( E \). Furthermore, \( \Phi_\lambda(y, t) = e^{-\lambda t}\Phi(y, t) \). Define \( B_\lambda, S_\lambda \) and \( U_\lambda \) to be the bounded, stable and unstable sets of \( \pi_\lambda \) respectively. Then their fibers are always subspaces. Moreover, \( S_\lambda(y) \) is increasing in \( \lambda \) and \( U_\lambda(y) \) is decreasing in \( \lambda \).

**Definition 7.5** Let \( M \subset Y \) be a subset. We say that \( \pi_\lambda \) admits an exponential dichotomy over \( M \) if there is a bundle projector \( P : E(M) \to E(M) \) and positive constants \( \alpha \) and \( K \) such that
\[
\|\Phi_\lambda(y, t)P(y)\Phi_\lambda^{-1}(y, s)\| \leq Ke^{-\alpha(t-s)}, \quad s \leq t,
\]
\[
\|\Phi_\lambda(y, t)[I - P(y)]\Phi_\lambda^{-1}(y, s)\| \leq Ke^{-\alpha(s-t)}, \quad t \leq s,
\]
for all \( y \in M \). Note that when \( M = \{y\} \) is a single point, the ED over \( M \) corresponds to the usual concept of dichotomy at \( y \).
Notice that ED over $M$ has two ingredients. The first one is that the projection $P(y)$ varies continuously over $M$. The second is that the constants $\alpha$ and $K$ are independent of $y \in M$.

In the sequel, we assume that $Y$ is a compact Hausdorff space and $\pi$ a LSPF on $E$, a bundle with base $Y$. Now we introduce the dynamical spectrum for $\pi$. For a point $y \in Y$, define the resolvent $\rho(y)$ by

$$\rho(y) = \{ \lambda \in \mathbb{R} : \pi_\lambda \text{ admits an exponential dichotomy over } \{y\}\}.$$ 

The spectrum $\Sigma(y)$ is the complement $\Sigma(y) = \rho(y)^c = \mathbb{R} \setminus \rho(y)$. For an arbitrary subset $M \subset Y$, we define the spectrum of $(E, \pi)$ over $M$, or simply, the spectrum by

$$\Sigma(M) = \bigcup_{y \in M} \Sigma(y),$$

and the resolvent of $(E, \pi)$ over $M$, or simply, the resolvent by

$$\rho(M) = \bigcap_{y \in M} \rho(y) = \Sigma(M)^c.$$ 

Such a spectrum is also called dynamical spectrum.

In the case of a LSPF generated by a family of ODEs (cf. Example 2.2), we see that $\lambda \in \rho(A)$ iff the shifted system

$$\dot{x} = (A(t) - \lambda I)x$$

admits an ED.

Note that if $\pi_\lambda$ admits ED over a point $y$, then $\pi_\lambda$ admits ED over the hull $H(y)$. Thus

**Lemma 7.1** $\Sigma(y) = \Sigma(H(y))$ for all $y \in Y$. In particular, if $M$ is a compact minimal set in $Y$, then $\Sigma(y) = \Sigma(M)$ for all $y \in M$.

When $M$ is a compact $\sigma$-invariant set, it follows from Section 4 that (i) $\lambda \in \rho(M) \iff \pi_\lambda$ admits ED over $M$; and (ii) $\lambda \in \rho(M) \implies B_\lambda(M) = E_0(M)$, the zero section of $E(M)$. However, if $M$ is a compact minimal set, or more generally, a compact chain-recurrent set, one can say more.

**Lemma 7.2** Let $M$ be a compact chain-recurrent set or a compact minimal set. Then $\lambda \in \rho(M) \iff \pi_\lambda$ admits an ED over $M \iff B_\lambda(M) = E_0(M)$.

**Lemma 7.3** Let $M$ be a compact invariant set in $Y$ and $\lambda \in \mathbb{R}$. Then

(A) If $\|\Phi_\lambda(y, t)\| \to 0$ as $t \to +\infty$ for each $y \in M$, then $\Sigma(M) \subseteq (-\infty, \lambda)$ and $S_\mu(M) = E(M)$ for all $\mu \geq \lambda$.

(B) If $\|\Phi_\lambda(y, t)\| \to 0$ as $t \to -\infty$ for each $y \in M$, then $\Sigma(M) \subseteq (\lambda, +\infty)$ and $U_\mu(M) = E(M)$ for all $\mu \leq \lambda$.

**Proof** We will prove (A) because (B) is similar. For each $y \in Y$, there is $T(y) > 0$ such that $\|\Phi_\lambda(y, t)\| < 1/2$ for all $t \geq T(y)$. In particular, $\|\Phi_\lambda(y, T(y))\| < 1/2$. By the continuity of $\Phi_\lambda$, there is a neighborhood $N(y)$ of $y$ such that $\|\Phi_\lambda(\eta, T(y))\| < 1/2$ for all $\eta \in N(y)$. As $M$ is compact, we can find a finite number of points $\{y_1, y_2, \cdots, y_m\}$ such that $\bigcup_{j=1}^m N(y_j) = M$. Let
Lemma 7.4 Let $M$ be a compact invariant set in $Y$. Then the resolvent $\rho(M)$ is open in $\mathbb{R}$.

**Proof** This essentially follows from the robustness of ED. Let $\lambda \in \rho(M)$. Then there is a projector $P : E(M) \to E(M)$ and positive constants $K$ and $\alpha$ such that

\[
\|\Phi_\lambda(y, t) P(y) \Phi_\lambda^{-1}(y, s)\| \leq Ke^{-\alpha(t-s)}, \quad s \leq t,
\]

\[
\|\Phi_\lambda(y, t) |I - P(y)| \Phi_\lambda^{-1}(y, s)\| \leq Ke^{-\alpha(s-t)}, \quad t \leq s,
\]

for all $y \in M$. Let $\beta = \alpha/2$ and assume that $|\lambda - \mu| < \beta$. Then

\[
\|\Phi_\mu(y, t) P(y) \Phi_\mu^{-1}(y, s)\| \leq Ke^{-\beta(t-s)}, \quad s \leq t,
\]

\[
\|\Phi_\mu(y, t) |I - P(y)| \Phi_\mu^{-1}(y, s)\| \leq Ke^{-\beta(s-t)}, \quad t \leq s,
\]

for all $y \in M$. Hence $\mu \in \rho(M)$, i.e., $\rho(M)$ is open. Moreover, since the ED for $\pi_\lambda$ and $\pi_\mu$ have the same projector $P$ we see that $S_\lambda(M) = S_\mu(M)$ and $U_\lambda(M) = U_\mu(M)$ whenever $|\lambda - \mu| < \beta$. \hfill $\Box$

**Lemma 7.5** Let $M$ be a compact invariant set in $Y$. Then the spectrum $\Sigma(M)$ is compact in $\mathbb{R}$. More specifically, there exists an $a > 0$ such that $S_\lambda(M) = E(M)$ if $\lambda > a$, and $U_\lambda(M) = E(M)$ if $\lambda < -a$. 

$T_j = T(y_j)$. Without loss of generality, we assume that $T_1 \leq T_j \leq T_m$ for all $1 \leq j \leq m$. Define $K$ by

\[
K = \max\{\|\Phi_\lambda(y, t)\| : y \in M, \ 0 \leq t \leq T_m\} < \infty.
\]

Fix now $t \geq 0$ and $y \in Y$. Then $y \in \mathcal{N}(y_{j_1})$ for some $j_1$ and $\|\Phi_\lambda(y, T_{j_1})\| < 1/2$. Furthermore, $y \cdot T_{j_1} \in \mathcal{N}(y_{j_2})$ for some $j_2$ and

\[
\|\Phi_\lambda(y, T_{j_1} + T_{j_2})\| = \|\Phi_\lambda(y \cdot T_{j_1}, T_{j_2}) \Phi_\lambda(y, T_{j_1})\| < (1/2)^2.
\]

Repeat this process until one has

\[
\tau = T_{j_1} + T_{j_2} + \cdots + T_{j_l} \leq t < \tau + T_{j_{l+1}}
\]

and $\|\Phi_\lambda(y, \tau)\| < (1/2)^l$. Note that $0 \leq t - \tau \leq T_m$ and $l > t/T_m - 1$. Thus

\[
\|\Phi_\lambda(y, t)\| = \|\Phi_\lambda(y \cdot \tau, t - \tau) \Phi_\lambda(y, \tau)\| \leq K(1/2)^l \leq K(1/2)^l/T_m^{-1} = K' e^{-\alpha t},
\]

where $K' = 2K$ and $\alpha = -T_m^{-1} \log \frac{1}{2} > 0$.

Since the late estimate is uniform in $y$, one has, for $t \geq s$,

\[
\|\Phi_\lambda(y, t) \Phi_\lambda^{-1}(y, s)\| = \|\Phi_\lambda(y \cdot t, t - s)\| \leq K' e^{-\alpha(t-s)};
\]

i.e., $\pi_\lambda$ admits ED over $M$ with the stable bundle $S_\lambda(M) = E(M)$. Thus $\lambda \in \rho(M)$. when $\mu \geq \lambda$, one has

\[
\|\Phi_\mu(y, t)\| = e^{(\lambda - \mu)t} \|\Phi_\lambda(y, t)\| \to 0 \quad \text{as} \quad t \to +\infty
\]

for all $y$. By what we have just proved, $\mu \in \rho(M)$. This proves that $\Sigma(M) \subseteq (-\infty, \lambda)$. \hfill $\Box$
**Proof** By Lemma 7.4 we need only prove that $\Sigma(M)$ is bounded. To this end we will prove that there are positive constants $K$ and $a$ such that

$$\|\Phi(y,t)\| \leq Ke^{at}$$

(7.2)

for all $y \in Y$ and all $t \in T$. The proof is similar to that for Lemma 7.3. Define $K$ by

$$K = \max\{\|\Phi_\lambda(y,t)\| : y \in M, ~|t| \leq 1\} < \infty.$$ 

Fix now $t \in T$ and $y \in Y$. Assume that $t \geq 0$. Let $m$ be an integer such that $m \leq t < m + 1$. Then

$$\|\Phi(y,t)\| = \|\Phi(y_m, t - m)\Phi(y_{m-1}, 1) \cdots \Phi(y_0, 1)\| \leq K^{m+1} \leq K \cdot K^t = Ke^{at},$$

where $y_0 = y$, $y_j = \sigma(y_{j-1}, 1)$, $j = 1, \cdots, m$, and $a = \log K$. \hfill \Box

**Lemma 7.6** Let $M$ be a nonempty set in $Y$ and assume that $\dim E = n \geq 1$. Then the spectrum $\Sigma(M)$ is nonempty.

**Proof** Pick $y_0 \in M$ and let $M_0 = H(y_0) \subset M$. Then $M_0$ is a compact invariant set and $\Sigma(M_0) = \Sigma(y_0) \subset \Sigma(M)$. We will prove that $\Sigma(M_0)$ is nonempty.

Let $K$ and $a$ be as in (7.2). If $\lambda > a$ then $\lambda \in \rho(M_0)$ and $S_\lambda(M_0) = E(M_0)$, $U_\lambda(M_0) = E_0(M_0)$. Similarly, if $\lambda < -a$ then $\lambda \in \rho(M_0)$ and $S_\lambda(M_0) = E_0(M_0)$, $U_\lambda(M_0) = E(M_0)$. Define $\lambda_0 = \inf\{\lambda \in \rho(M_0) : S_\lambda(M_0) = E(M_0)\}$. Then $-\alpha \leq \lambda_0 \leq a$. Assume, by contrary, that $\lambda_0 \in \rho(M_0)$. There are two cases to consider: (i) $S_{\lambda_0}(M_0) = E(M_0)$, or (ii) $S_{\lambda_0}(M_0) \neq E(M_0)$. In case (i), one has $S_{\lambda_0}(M_0) = E(M_0)$ for $\lambda$ in a neighborhood of $\lambda_0$, which contradicts the definition of $\lambda_0$. In case (ii) one must have $U_{\lambda_0}(M_0) \neq E_0(M_0)$ since $S_{\lambda_0}(y) + U_{\lambda_0}(y) = X_0$ for all $y \in M_0$. By Lemma 7.4, one has $U_\lambda(M_0) \neq E_0(M_0)$ for $\lambda$ in a neighborhood of $\lambda_0$. However, for $\lambda > \lambda_0$ we see from the definition of $\lambda_0$ that $U_\lambda(M_0) = E_0(M_0)$. Such a contradiction shows that $\lambda_0 \in \Sigma(M_0)$. \hfill \Box

**Lemma 7.7** Let $M$ be a compact invariant set in $Y$. Let $\lambda_1$, $\lambda_2 \in \rho(M)$ with $\lambda_1 < \lambda_2$. If $S_{\lambda_1}(M) = S_{\lambda_2}(M)$ and $U_{\lambda_1}(M) = U_{\lambda_2}(M)$, then $[\lambda_1, \lambda_2] \subset \rho(M)$.

**Proof** Since $\lambda_1$, $\lambda_2 \in \rho(M)$ and $S_{\lambda_1}(M) = S_{\lambda_2}(M)$, $U_{\lambda_1}(M) = U_{\lambda_2}(M)$, the projectors in ED corresponding to $\lambda_1$ and $\lambda_2$ must coincide. Let $P$ be the common projector. Then there are positive constants $a$ and $K$ such that

$$\|\Phi_\lambda(y,t)P(y)\Phi_\lambda^{-1}(y,s)\| \leq Ke^{-\alpha(t-s)},$$

$$s \leq t,$$

$$\|\Phi_\lambda(y,t)[I - P(y)]\Phi_\lambda^{-1}(y,s)\| \leq Ke^{-\alpha(s-t)},$$

$$t \leq s,$$

for all $y \in M$, where $\lambda = \lambda_i$, $i = 1, 2$. These inequalities are equivalent to

$$e^{-\lambda(t-s)}\|\Phi(y,t)P(y)\Phi^{-1}(y,s)\| \leq Ke^{-\alpha(t-s)},$$

$$s \leq t,$$

$$e^{-\lambda(t-s)}\|\Phi(y,t)[I - P(y)]\Phi^{-1}(y,s)\| \leq Ke^{-\alpha(s-t)},$$

$$t \leq s,$$

where $\lambda = \lambda_i$, $i = 1, 2$. Thus the latter inequalities hold also for all $\lambda \in [\lambda_1, \lambda_2]$. Now the lemma follows. \hfill \Box
Lemma 7.8 Let $M$ be a compact invariant set in $Y$. Let $\lambda_1, \lambda_2 \in \rho(M)$ with $\lambda_1 < \lambda_2$. Then the following statements are equivalent:

(A) $(\lambda_1, \lambda_2) \cap \Sigma(M) \neq \emptyset$.
(B) $\mathcal{U}_{\lambda_1}(M) \cap S_{\lambda_2}(M) \neq E_0(M)$.
(C) $\dim S_{\lambda_1}(M) < \dim S_{\lambda_2}(M)$.
(D) $\dim \mathcal{U}_{\lambda_1}(M) > \dim \mathcal{U}_{\lambda_2}(M)$.

Furthermore, $\mathcal{F} = \mathcal{U}_{\lambda_1}(M) \cap S_{\lambda_2}(M)$ is a nonempty invariant subbundle of $E$.

**Proof** (C) $\iff$ (D). Since $\lambda_i \in \rho(M)$ we have $\dim S_{\lambda_i}(y) + \dim \mathcal{U}_{\lambda_i}(y) = n$ for all $y \in M$ and $i = 1, 2$. Now the equivalence of (C) and (D) is immediate.

(A) $\Rightarrow$ (C) and (D). If there is $\mu \in (\lambda_1, \lambda_2) \cap \Sigma(M)$, by the last lemma either $S_{\lambda_1}(M) \neq S_{\lambda_2}(M)$ or $\mathcal{U}_{\lambda_1}(M) \neq \mathcal{U}_{\lambda_2}(M)$. So (C) and (D) follows from the monotonicity of $S_{\lambda}$ and $\mathcal{U}_{\lambda}$.

(C) and (D) $\Rightarrow$ (B). For every $y \in M$,

$$\dim \mathcal{U}_{\lambda_1}(y) \cap S_{\lambda_2}(y) \geq \dim \mathcal{U}_{\lambda_1}(y) + \dim S_{\lambda_2}(y) - n > \dim \mathcal{U}_{\lambda_1}(y) + \dim S_{\lambda_1}(y) - n = 0.$$

(B) $\Rightarrow$ (A). Define $\mu = \inf \{ \lambda \in \rho(M) : S_{\lambda}(M) = S_{\lambda_2}(M) \}$. Arguing as in Lemma 7.6, we get $\lambda_1 < \mu < \lambda_2$ and $\mu \in \Sigma(M)$.

Finally, since both $\mathcal{U}_{\lambda_1}(M)$ and $S_{\lambda_2}(M)$ are invariant subbundles of $E(M)$, it follows that the intersection $\mathcal{F} = \mathcal{U}_{\lambda_1}(M) \cap S_{\lambda_2}(M)$ is also an invariant subbundle of $E(M)$.

Lemma 7.9 Let $M$ be a compact invariant set in $Y$. Let $\lambda_1, \lambda_2 \in \rho(M)$ with $\lambda_1 < \lambda_2$. Assume that $(\lambda_1, \lambda_2) \cap \Sigma(M) \neq \emptyset$. Let $\mathcal{F} = \mathcal{U}_{\lambda_1}(M) \cap S_{\lambda_2}(M)$ and $\hat{\pi}$ be the restriction of $\pi$ to $\mathcal{F}$. If $\hat{\Sigma}(M)$ is the spectrum of $(\mathcal{F}, \hat{\pi})$, then $\hat{\Sigma}(M) = \Sigma(M) \cap (\lambda_1, \lambda_2)$.

**Proof** We will make use of the following equality

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

frequently, where $A$, $B$ and $C$ are subspaces of $X$ with $B \cap C = \{0\}$ and $C \subseteq A$.

The first step is to prove that $\hat{\Sigma}(M) \subseteq \Sigma(M)$, or, equivalently $\rho(M) \subseteq \hat{\rho}(M)$. Let $\lambda \in \rho(M)$ and define

$$\mathcal{F}_s = \mathcal{F} \cap S_{\lambda}(M), \quad \mathcal{F}_u = \mathcal{F} \cap \mathcal{U}_{\lambda}(M).$$

Clearly $\mathcal{F}_s$ and $\mathcal{F}_u$ are invariant subbundles. We claim that $\mathcal{F} = \mathcal{F}_s \oplus \mathcal{F}_u$. If $\lambda < \lambda_1$, one has $\mathcal{F}_s = \mathcal{F}_0$, the zero section, and $\mathcal{F}_u = \mathcal{F}$ from which the assertion follows. This is also true when $\lambda > \lambda_2$. When $\lambda \in [\lambda_1, \lambda_2]$,

$$\mathcal{F}_s = \mathcal{F} \cap S_{\lambda} = \mathcal{U}_{\lambda_1} \cap S_{\lambda_2} \cap S_{\lambda} = \mathcal{U}_{\lambda_1} \cap S_{\lambda}$$

and

$$\mathcal{F}_u = S_{\lambda_2} \cap \mathcal{U}_{\lambda}.$$

Now one has

$$\mathcal{U}_{\lambda_1} = \mathcal{U}_{\lambda_1} \cap (\mathcal{U}_{\lambda_2} \oplus S_{\lambda_2}) = \mathcal{U}_{\lambda_2} \oplus (\mathcal{U}_{\lambda_1} \cap S_{\lambda_2}) = \mathcal{U}_{\lambda_2} \oplus \mathcal{F}. \quad (7.3)$$
Similarly, 
\[ \mathcal{U}_{\lambda_1} = \mathcal{U}_{\lambda_1} \cap (\mathcal{U}_\lambda \oplus \mathcal{S}_\lambda) = \mathcal{U}_\lambda \oplus \mathcal{F}_s \]
and 
\[ \mathcal{U}_\lambda = \mathcal{U}_\lambda \cap (\mathcal{U}_{\lambda_2} \oplus \mathcal{S}_{\lambda_2}) = \mathcal{U}_{\lambda_2} \oplus \mathcal{F}_u. \]

From the latter two equalities, 
\[ \mathcal{U}_{\lambda_1} \mathcal{U}_{\lambda_2} \oplus \mathcal{F}_s \oplus \mathcal{F}_u, \]
which, together with (7.3), implies that \( \mathcal{F}_s \oplus \mathcal{F}_u \subseteq \mathcal{F} \). So the claim follows. As \( \lambda \in \rho(M) \), there is a bundle projector \( P : E(M) \to E(M) \) with range \( \mathcal{S}_\lambda(M) \) and null space \( \mathcal{U}_\lambda(M) \), and positive constants \( \alpha \) and \( K \) such that 
\[ \| \Phi_\lambda(y, t) P(y) \Phi_\lambda^{-1}(y, s) \| \leq Ke^{-\alpha(t-s)}, \quad s \leq t, \]
\[ \| \Phi_\lambda(y, t) [I - P(y)] \Phi_\lambda^{-1}(y, s) \| \leq Ke^{-\alpha(s-t)}, \quad t \leq s, \]
for all \((x, y) \in E(M)\). Let \( \hat{P} \) be the restriction of \( P \) to \( \mathcal{F} \). Then \( \hat{P} \) has range \( \mathcal{F}_s \) and null space \( \mathcal{F}_u \). By restricting the above inequalities to all \((x, y) \in \mathcal{F}\) one then has the ED of \( \hat{\pi}_\lambda \) over \( M \). This proves that \( \lambda \in \hat{\rho}(M) \).

Moreover, as \( \mathcal{F} \subseteq \mathcal{S}_{\lambda_2}(M) \) it follows that \( |\hat{\Phi}_{\lambda_2}(y, t)x| \to 0 \) as \( t \to \infty \) for all \((x, y) \in \mathcal{F}\). By Lemma 7.3 one has \( \hat{\Sigma}(M) \subseteq (\lambda_1, \infty) \). As \( \mathcal{F} \subseteq \mathcal{U}_{\lambda_1}(M) \), one has \( \hat{\Sigma}(M) \subseteq (-\infty, \lambda_2) \) by again Lemma 7.3. As a result, \( \hat{\Sigma}(M) \subseteq \Sigma(M) \cap (\lambda_1, \lambda_2) \).

Now we prove the opposite inclusion. To this end, let \( \lambda \in \hat{\rho}(M) \cap (\lambda_1, \lambda_2) \). Then there is a bundle projector \( Q : \mathcal{F} \to \mathcal{F} \) and positive constants \( \alpha \) and \( K \) such that 
\[ \| \hat{\Phi}_\lambda(y, t) Q(y) \hat{\Phi}_\lambda^{-1}(y, s) \| \leq Ke^{-\alpha(t-s)}, \quad s \leq t, \]
\[ \| \hat{\Phi}_\lambda(y, t) [I - Q(y)] \hat{\Phi}_\lambda^{-1}(y, s) \| \leq Ke^{-\alpha(s-t)}, \quad t \leq s, \]
for all \( y \in M \). Let \( P_1 \) be the projector on \( E(M) \) corresponding to \( \lambda_1 \in \rho(M) \). Let \( P \) be the projector on \( E(M) \) with range \( \mathcal{F} \) and null space \( \mathcal{S}_{\lambda_1}(M) + \mathcal{U}_{\lambda_2}(M) \). Then \( P_\lambda = P_1 + QP \) is a projector on \( E(M) \) with range \( \mathcal{S}_{\lambda_1}(M) + \hat{\mathcal{S}}_\lambda(M) \) and null space \( \hat{\mathcal{U}}_\lambda(M) + \mathcal{U}_{\lambda_2}(M) \), where \( \hat{\mathcal{S}}_\lambda(M) \) and \( \hat{\mathcal{U}}_\lambda(M) \) denote, respectively, the range and null space of \( Q \) in \( \mathcal{F} \). By using (7.4) and the fact that \( \lambda_1, \lambda_2 \in \rho(M) \) and \( \lambda \in (\lambda_1, \lambda_2) \), it is easy to derive the ED of \( \pi_\lambda \) with the projector \( P_\lambda \). Thus \( \lambda \in \rho(M) \).

Now we can state the main result for spectrum and invariant splittings for LSPFs on compact invariant sets in \( Y \). Let \( M \) be a compact invariant set in \( Y \). Then the spectrum \( \Sigma(M) \) is a closed set lying in some bounded interval \([-a, a]\). Choose now points \( \lambda_0, \ldots, \lambda_m \) in the resolvent \( \rho(M) \) such that \( \lambda_0 < \lambda_1 < \cdots < \lambda_m \) and \( \lambda_0 < -a \) and \( \lambda_m > a \). In addition let us require that the spectrum \( \Sigma(M) \) meets each interval \((\lambda_{i-1}, \lambda_i), i = 1, 2, \ldots, m \). Then each set 
\[ \mathcal{V}_i = \mathcal{V}_i(M) = \mathcal{U}_{\lambda_{i-1}}(M) \cap \mathcal{S}_{\lambda_i}(M), \quad i = 1, 2, \ldots, m, \]
is a nonempty invariant subbundle of \( E(M) \). Let \( \pi^i \) be the restriction of \( \pi \) to \( \mathcal{V}_i \) and \( \Sigma_i(M) \) the spectrum of \( (\mathcal{V}_i, \pi^i) \) over \( M \). Then one has

**Theorem 7.1** Assume that \( \dim E(M) \geq 1 \). Then the following statements are valid:

(A) \( \Sigma_i(M) = \Sigma(M) \cap (\lambda_{i-1}, \lambda_i) \).
(B) $\Sigma(M) = \cap_{i=1}^{m} \Sigma_i(M)$.
(C) $\dim \mathcal{V}_i(M) \geq 1$.
(D) $\mathcal{V}_i(M) \cap \mathcal{V}_j(M) = E_0(M)$ when $i \neq j$.
(E) $E(M) = \mathcal{V}_1(M) \oplus \cdots \oplus \mathcal{V}_m(M)$ (Whitney sum).

**Proof** Statement (A) follows from Lemma 7.9. For (B), note that $\Sigma(M) \subseteq [-a, a]$ and $\lambda_0 < -a$, $\lambda_m > a$. (C) follows from Lemma 7.8(B). For (D), let us assume that $i + 1 \leq j$. As $\mathcal{V}_i \subset S_{\lambda_i}(M)$ and $\mathcal{V}_j \subset U_{\lambda_j}(M)$, (D) then follows because $S_{\lambda_i}(M) \cap U_{\lambda_j}(M) = E_0(M)$. (E) can be proved as follows.

$$E(M) = U_{\lambda_0}(M) = U_{\lambda_0}(M) \cap (S_{\lambda_1}(M) \oplus U_{\lambda_1}(M))$$
$$= U_{\lambda_0}(M) \cap S_{\lambda_1}(M) \oplus U_{\lambda_1}(M)$$
$$= \mathcal{V}_1(M) \oplus U_{\lambda_2}(M)$$
$$= \mathcal{V}_1(M) + \mathcal{V}_2(M) \oplus U_{\lambda_2}(M)$$

Repeating this, one has the desired decomposition in (E). \hfill \Box

When the compact invariant set $M$ is dynamically connected, one has more detailed information on spectrum.

**Theorem 7.2** Let $M$ be a compact, dynamically connected, invariant set in $Y$. Let $n = \dim E(M) \geq 1$. Then the spectrum $\Sigma(M)$ is the union of $k$ compact intervals

$$\Sigma(M) = [a_1, b_1] \cup \cdots \cup [a_k, b_k],$$

where $k \leq n$ and $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. Furthermore, if $\lambda_0, \lambda_1, \ldots, \lambda_k$ are chosen in the resolvent $\rho(M)$ so that

$$\lambda_0 < a_1 \leq b_1 < \lambda_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k < \lambda_k,$$

then for $1 \leq i \leq k$,

$$\mathcal{V}_i = \mathcal{V}_i(M) = U_{\lambda_{i-1}}(M) \cap S_{\lambda_i}(M)$$

is an invariant subbundle of $E(M)$ with $\dim \mathcal{V}_i(y) = n_i$ for all $y \in M$, where $n_i \geq 1$ and $n_1 + \cdots + n_k = n$. Moreover, $\mathcal{V}_i(M) \cap \mathcal{V}_j(M) = E_0(M)$ if $i \neq j$ and $E(y) = \mathcal{V}_1(y) \oplus \cdots \oplus \mathcal{V}_m(y)$ for all $y \in Y$. Finally, the spectrum $\Sigma_i(M)$ of $(\mathcal{V}_i, \pi^i)$, where $\pi^i$ is the restriction of $\pi$ to $\mathcal{V}_i$, is simply the interval $[a_i, b_i]$.

**Proof** We need only to prove that $\rho(M)$ consists of $(k + 1)$ open intervals where $k \leq n$. Otherwise, if $\rho(M)$ contains $(n + 2)$ components, then one can choose points $\lambda_0$, $\lambda_1$, $\ldots$, $\lambda_{n+1}$ in $\rho(M)$ such that $\Sigma(M)$ meets each interval $(\lambda_{i-1}, \lambda_i)$, $i = 1, \ldots, n + 1$. Now the invariant subbundle $\mathcal{V}_i(M) = S_{\lambda_i}(M) \cap U_{\lambda_i}(M)$ has dimension $\geq 1$ for each $i$. Since $M$ is invariantly connected, $\dim \mathcal{V}_i(y) = n_i \geq 1$ is constant for all $y \in M$. Now we have a contradiction

$$n = \dim E(y) \geq \sum_{i=1}^{n+1} n_i \geq n + 1.$$
The remainder of the theorem now follows from Theorem 7.1.

The decomposition of spectrum

\[ \Sigma(M) = \bigcup_{i=1}^{k} [a_i, b_i] \]

and the corresponding decomposition of bundle

\[ E(M) = V_1(M) \oplus \cdots \oplus V_k(M) \]

shall be called the spectral decomposition of \((E, \pi)\). The invariant subbundle \(V_i(M)\) will be referred to as the \(i\)th spectral subbundle, or sometimes as the spectral subbundle associated with the \(i\)th spectral interval \([a_i, b_i]\).

**Example 7.2 AUTONOMOUS EQUATIONS.** Consider the linear autonomous differential equation \(\dot{x} = Ax\), where \(A\) is a constant matrix and \(x \in X = \mathbb{F}^n\). Let \(\Lambda(A)\) be the collection of eigenvalues of \(A\). Then \(\Sigma(A) = \text{Re} \, \Lambda(A)\), when the associated LSPF \(\pi(x, A, t) = (\exp(tA)x, A)\) on the space \(X \times \{A\}\) is considered. In this case the spectral subbundle associated with \(\mu \in \Sigma(A)\) can be described in terms of the generalized eigenspaces of \(A\), i.e.,

\[ V_{\mu} = \bigoplus_{\lambda \in \Lambda(A), \text{Re} \lambda = \mu} E_{\lambda}, \quad \text{where} \quad E_{\lambda} = \{x \in X : (A - \lambda I)^n x = 0\}. \]

**Example 7.3 PERIODIC EQUATIONS.** Consider the linear periodic differential equation \(\dot{x} = A(t)x\), where \(A(t + \omega) \equiv A(t)\) is \(\omega\)-periodic. This equation induces a LSPF \(\pi\) on the space \(X \times \gamma(A)\), where \(\gamma(A) = \{A_{\tau} : \tau \in \mathbb{R}\}\) which is homeomorphic to a circle \(S^1\). By the Floquet theory, there is a change of variable \(y = P(t)x\), where \(P(t)\) is nonsingular and periodic, such that the equation is reduced to \(\dot{y}' = By\). Now the spectrum \(\Sigma(\gamma(A))\) of \(\pi\) over \(X \times \gamma(A)\) is precisely that of \(\dot{y}' = By\), i.e., \(\Sigma(\gamma(A)) = \Sigma(B)\). The spectral subbundle associated with \(\mu \in \Sigma(\gamma(A))\) can be described using the generalized eigenspace of \(B\) and the matrix function \(P(t)\).

**Example 7.4** The stronger conclusions of Theorem 7.2 are not valid if \(M\) is not dynamically connected. A counterexample is the collection of scalar equations \(x' = \frac{1}{n} x, \, n \in \mathbb{N}, \, x' = 0 \cdot x\). Then this LSPF has the spectrum \(\Sigma(M) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\).

**Example 7.5 ANOSOV Diffeomorphisms.** Consider, for example, a linear Anosov diffeomorphism \(F\) on 2-torus \(T^2\) induced by the \(2 \times 2\) matrix

\[ A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}. \]

The eigenvalues of \(A\) are \(\lambda_1 = 3 \pm \sqrt{8}\). Then the linearized flow \(\pi\) of \(F\) on the tangent bundle \(TT^2\) is a LSPF. So \(TT^2\) has a Whitney sum \(TT^2 = S \oplus U\). Note that periodic points of \(F\) are dense in \(T^2\). One can easily prove that \(\Sigma(T^2) = \{\log \lambda_1, \log \lambda_2\} = \{\log(3 + \sqrt{8}), \log(3 - \sqrt{8})\}\). Furthermore the spectral bundle associated with \(\log \lambda_2 = \log(3 - \sqrt{8})\) (respectively, \(\log \lambda_1 = \log(3 + \sqrt{8})\)) is just \(S\) (respectively, \(U\)).
Example 7.6 Tangent Flows on Manifolds. Let $M$ be a compact differentiable manifold and $\sigma$ be the flow induced by a $C^1$ vector field $F$ on $M$. Then $\sigma$ induces a LSPF $\pi$ on the tangent bundle $TM$. The one always has $0 \in \Sigma(M)$. In fact, if the vector field vanishes everywhere, then $\Sigma(M) = \{0\}$. Then spectral subbundle associated with $\lambda = 0$ is $TM$. On the other hand, there is $y_0 \in M$ such that $F(y_0) \neq 0$. Then $\pi$ has a nonzero orbit

$$\pi(F(y_0), y_0, t) = (F(\sigma(y_0, t)), \sigma(y_0, t)),$$

which is bounded because of the compactness of $M$ and of the continuity of $F$. So $B_0(M) \neq \{0\}$ and $0 \in \Sigma(M)$.

Example 7.7 Anosov Flows. The Anosov diffeomorphism $F$ in Example 7.5 can be suspended to give an Anosov flow on $M = T^2 \times S^1$. Then its induced LSPF has the spectrum

$$\Sigma = \{\log(3 + \sqrt{8}), 0, \log(3 - \sqrt{8})\}.$$

The spectral bundles associated with $\lambda_i$ are perpendicular to the $S^1$-direction, while that associated with $\lambda = 0$ consists of all tangent vectors pointing in the $S^1$-direction.

8 Perturbation of Dynamical Spectrum

9 Dynamical Spectrum—Another Approach

10 Lyapunov Characteristic Exponents

In this section we aim at giving some connection between dynamical spectrum and Lyapunov characteristic exponents. A more precise result will be given in the next section based on the Multiplicative Ergodic Theorem.

In the following we always assume that $\sigma$ is flow on $Y$ which is a compact, dynamically connected, (invariant) space. Now for any LSPF $\pi$ on $E$, a bundle over $Y$, has the spectral decomposition described in Theorem 7.2.

Given a point $(x, y) \in E$. We always assume that $x \neq 0$. Then one can define four Lyapunov characteristic exponents of $(x, y)$ by

$$\lambda^+_s(x, y) = \limsup_{t \to \pm \infty} \frac{1}{t} \log |\varphi(x, y, t)|,$$

$$\lambda^+_i(x, y) = \liminf_{t \to \pm \infty} \frac{1}{t} \log |\varphi(x, y, t)|.$$

If it happens that the following two limits exist and are equal

$$\lim_{t \to +\infty} \frac{1}{t} \log |\varphi(x, y, t)| = \lim_{t \to -\infty} \frac{1}{t} \log |\varphi(x, y, t)|,$$

then we shall denote the common value as $\lambda(x, y)$. In this case we say that $(x, y)$ has a Lyapunov exponent $\lambda(x, y)$. When we write the symbol $\lambda(x, y)$ this always means that the limits in (10.3) exist and $\lambda(x, y)$ is the common value.
Theorem 10.1  If \((x, y)\) is in the \(i\)th spectral subbundle \(\mathcal{V}_i\) with \(x \neq 0\), then the four Lyapunov characteristic exponents lie in the \(i\)th spectrum interval \([a_i, b_i]\). In particular, if \(a_i = b_i\), then all \((x, y) \in \mathcal{V}_i, x \neq 0, \) have a Lyapunov exponent \(\lambda(x, y) = a_i = b_i\).

Proof  Let \((x, y) \in \mathcal{V}_i\) with \(x \neq 0\). We will prove that \(\lambda^+_s(x, y) \leq b_i\) and \(\lambda^+_i(x, y) \geq a_i\). These will imply that \(\lambda^+_s(x, y), \lambda^+_i(x, y) \in [a_i, b_i]\).

Fix any \(\lambda \in (b_i, a_{i+1})\). Then \(\lambda \in \rho(Y)\) and there is a projector \(P : E \to E\) and positive constants \(K\) and \(\alpha\) such that

\[
\|\Phi \lambda(y, t)P(y)\Phi_{\lambda}^{-1}(y, s)\| \leq Ke^{-\alpha(t-s)}, \quad s \leq t.
\]

Thus

\[
\|\Phi(y, t)P(y)\Phi_{\lambda}^{-1}(y, 0)\| \leq Ke^{(\lambda-\alpha)(t-s)}, \quad s \leq t.
\]

When \((x, y) \in \mathcal{V}_i\) then \(P(y)x = x\). Now we use the last inequality to obtain

\[
|\varphi(x, y, t)| = |\Phi(y, t)P(y)\Phi_{\lambda}^{-1}(y, 0)x| \leq K|x|e^{(\lambda-\alpha)t}, \quad t \geq 0.
\]

Therefore, \(\lambda^+_s(x, y) \leq \lambda - \alpha < \lambda\). Since \(\lambda \in (b_i, a_{i+1})\) is arbitrary one has \(\lambda^+_s(x, y) \leq b_i\).

The inequality \(\lambda^+_i(x, y) \geq a_i\) can be proved similarly by taking \(\lambda \in (b_{i-1}, a_i)\) and considering the second estimate in the ED corresponding to \(\lambda\).

Let \(P_i : E \to E\) denote the projector on \(E\) with range \(\mathcal{V}_i\) and null the sum of the remaining \(\mathcal{V}_j\).

Theorem 10.2  For any \((x, y) \in E \setminus E_0\), the four Lyapunov characteristic exponents lie in the spectrum \(\Sigma\). More precisely, define \(p = p(x, y)\) and \(q = q(x, y)\) by

\[
p = \max\{i : P_i(y)x \neq 0\}, \quad q = \min\{i : P_i(y)x \neq 0\}.
\]

Then one has

\[
a_p \leq \lambda^+_i(x, y) \leq \lambda^+_s(x, y) \leq b_p, \quad \lambda^-_s(x, y) \leq \lambda^-_i(x, y) \leq b_q. \tag{10.4}
\]

Proof  Let us prove (10.4). So \((x, y)\) can be written as

\[
(x, y) = (x_p, y) + (x_p', y),
\]

where \((x_p, y) \in \mathcal{V}_p\) with \(x_p \neq 0\), and \((x_p', y) \in \bigoplus_{j=1}^{p-1} \mathcal{V}_j\). Note that \(\lambda^+_i(x_p, y), \lambda^+_s(x_p, y) \in [a_p, b_p]\) and \(\lambda^+_i(x_p', y) \leq \lambda^+_s(x_p', y) \leq b_{p-1} < a_p\). The results follow from the following facts:

\[
\lambda^+_i(x, y) = \lambda^+_i(x_p, y), \quad \lambda^+_s(x, y) = \lambda^+_s(x_p, y). \tag{10.5}
\]

Now we fix a \(y \in Y\). Define two sets

\[
\Lambda^+_i(y) = \{\lambda^+_i(x, y) : x \in E_y \setminus \{0\}\}, \quad \Lambda^-_i(y) = \{\lambda^-_i(x, y) : x \in E_y \setminus \{0\}\}.
\]

It is not difficult to prove the following conclusion.
Lemma 10.1 For every \( y \in Y \), each of the sets \( \Lambda^+_s(y) \) and \( \Lambda^-_s(y) \) contains no more than \( n \) elements, where \( n = \dim E \).

Our next main result is that all endpoints in the dynamical spectrum can be realized using Lyapunov characteristic exponents of “many” points in \( Y \), provided that \( Y \) is a minimal set with respect to the flow \( \sigma \). This is described in the terms of topology. It will be an interesting problem to describe such points in terms of invariant measures.

Before giving the precise statement, let us prove a preliminary result.

Lemma 10.2 Assume that the spectrum \( \Sigma \) contains only a single spectral interval \([a, b]\). Then there exist \((x_i, y_i) \in E\) such that \( x_i \neq 0 \), \( i = 1, 2, \) and \( \lambda^+_x(x_1, y_1) = b \) and \( \lambda^-_x(x_2, y_2) = a \).

Proof Let us prove the first conclusion. Otherwise, for any \( y \in Y \), there is \( \varepsilon = \varepsilon(y) > 0 \) such that \( \lambda^+_x(x, y) < b - \varepsilon \) for all \( x \in X_y \) with \(|x| = 1\). Recall that \( \Lambda^+_x(y) \) attains only finitely many different values. Thus

\[
\|\Phi_b(y, t)\| = \sup_{|x|=1} |\Phi_b(y, t)x| = \sup_{|x|=1} e^{-bt} |\varphi(x, y, t)| 
\leq \sup_{|x|=1} e^{-(b-\varepsilon)t} |\varphi(x, y, t)| \to 0 \quad \text{as } t \to \infty
\]

for all \( y \in Y \). Thus \( b \in \rho(Y) \), a contradiction. \( \Box \)

Theorem 10.3 Assume that \( Y \) is a minimal set with respect to the flow \( \sigma \). Let \([a, b]\) be any spectral interval and \( \mathcal{V} \) the associated spectral subbundle. Define \( G^+ \) and \( G^- \) by

\[
G^+(G^-) = \{ y \in Y : \text{there exists an } x \in \mathcal{V}(y) \text{ such that } \lambda^+_x(x, y) = b \ (\lambda^-_x(x, y) = a) \}.
\]

Then \( G^+ \), \( G^- \) and \( G^+ \cap G^- \) are all residual \( G_\delta \)-subsets of \( Y \).

Proof We will prove that \( G^+ \) is a residual \( G_\delta \)-subset of \( Y \). Note that \( \lambda = \lambda^+_x(x, y) \) is characterized by: (i) for any \( \varepsilon > 0 \) there is a sequence \( t_n \to +\infty \) such that

\[
e^{-(\lambda-\varepsilon)t_n} |\varphi(x, y, t_n)| \to \infty,
\]

and (ii) for any \( \varepsilon > 0 \)

\[
e^{-(\lambda+\varepsilon)t} |\varphi(x, y, t)| \to 0 \quad \text{as } t \to +\infty.
\]

Let \( E^+ = \{ y \in Y : \lambda^+_x(x, y) < b \text{ for all } x \in \mathcal{V}(y) \setminus \{0\} \} \). The \( G^+ = Y \setminus E^+ \). Thus we need only to prove that \( E^+ \) is a countable union of closed nowhere dense sets.

For any \( k, l \in \mathbb{N} \), define

\[
E_{kl} = \{ y \in Y : |\varphi(x, y, t)| \leq k|x|e^{(b-1/|l|)t} \quad \text{for all } t \geq 0, \ x \in \mathcal{V}(y) \}.
\]

Note that \( \Lambda^+_x(y) \) contains only finitely many values. Then

\[
E^+ = \bigcup_{k,l=1}^{\infty} E_{kl}.
\]
Obviously, each $E_{kl}$ is closed. Now we prove that $E_{kl}$ is nowhere dense. Otherwise, $E_{kl}$ contains a nonempty open subset $U$. As $Y$ is minimal, there exist $\{t_1, \cdots, t_m\}$ with $t_i \geq 0$ and

$$Y = U \cdot t_1 \cup \cdots \cup U \cdot t_m. \quad (10.6)$$

Let

$$K_0 = \sup \{|\varphi(x, y \cdot \tau, -\tau)| : y \in Y, x \in V(y), |x| = 1, \tau = t_i, 1 \leq i \leq m\} < \infty.$$ 

Given $\tau \in \{t_1, \cdots, t_m\}$ and $y \in E_{kl}$. One has

$$|\varphi(x, y \cdot \tau, t)| = |\varphi(\varphi(x, y \cdot \tau, -\tau), y, \tau + t)|$$
$$\leq K_0 e^{(b-1/l)(\tau + t)}$$
$$\leq K_0 e^{(b-1/l)\tau} e^{(b-1/l)t}$$

for all $t \geq 0$ and all $x \in V(y)$ with $|x| = 1$. By (10.6), one has $Y = E_{kl}$ if $K \in \mathbb{N}$ with $K \geq K_0 e^{(b-1/l)\tau}$. This implies that $\lambda^+_k(x, y) \leq b - 1/l$ for all $(x, y) \in E$, a contradiction to the above lemma. \(\square\)

**Corollary 10.1** Assume that $Y$ is a minimal set with respect to the flow $\sigma$. Let $\pi$ be a LSPF on $E$ with the spectral decomposition in Theorem 7.2. Then the set $G$ of all $y \in Y$ such that there exist $x_{-1}, \cdots, x_k^-, x_1^+, \cdots, x_k^+$ in $E_y$ with

$$\lambda^+_k(x_j^+, y) = b_j \quad \text{and} \quad \lambda^-_i(x_j^-, y) = a_j \quad (j = 1, \cdots, k)$$

is a residual $G_\delta$-subset of $Y$.

11 Multiplicative Ergodic Theorem for Cocycles

Let $Y$ be a compact Hausdorff space and $\sigma(y, t) = y \cdot t$ a flow on $Y$. Here we use $y$ denote points in $Y$ and the time $t$ is in $T$. For an integer $n \geq 1$, let $\text{Gl}(n)$ denote the group of all isomorphisms on $\mathbb{R}^n$. A cocycle on $Y$ (over $y \cdot t$) is a continuous mapping $\Phi : Y \times T \rightarrow \text{Gl}(n)$ satisfying

$$\Phi(y, t + s) = \Phi(y \cdot t, s)\Phi(y, t)$$

for all $y \in Y$ and $t, s \in T$. Note that $\Phi$ is a cocycle on $Y$ if and only if

$$\pi(x, y, t) = (\Phi(y, t)x, y \cdot t)$$

is a LSPF on $\mathbb{R}^n \times Y$. When $T = \mathbb{R}$ we say that the flow $\pi$ is smooth if the mapping

$$A : y \rightarrow \frac{d}{dt}\Phi(y, t)|_{t=0}$$

exists and is continuous. In this case the cocycle $\Phi(y, t)$ is simply the fundamental matrix solution of

$$\dot{x} = A(y \cdot t)x, \quad x \in \mathbb{R}^n$$

satisfying $\Phi(y, 0) = I$. This is the prototypical example of a cocycle.
11. Multiplicative Ergodic Theorem for Cocycles

As in the previous section, for \( x \in \mathbb{R}^n \) with \( x \neq 0 \), and \( y \in Y \), the four Lyapunov characteristic exponents are denoted by \( \lambda_\pm^x(x, y) \) and \( \lambda_\pm^y(x, y) \). If the following two limits exist and equal

\[
\lim_{t \to +\infty} \frac{1}{t} \log |\Phi(x, y, t)| = \lim_{t \to -\infty} \frac{1}{t} \log |\Phi(x, y, t)|,
\]

then we shall denote the common value as \( \lambda(x, y) \). In the following, the symbol \( \lambda(x, y) \) means the limits in (11.1) exist and are equal.

Two cocycles \( \Phi \) and \( \Psi \) (over the same base flow \( y \cdot t \)) is called \textit{cohomologous} if there is a continuous mapping \( V : Y \to \text{Gl}(n) \) such that

\[
\Phi(y, t) = V(y \cdot t) \Psi(y, t) V(y)^{-1}
\]

for all \( y \in Y \) and all \( t \in T \), where \(-1\) denotes the matrix inverse. Note that in the terms of LSPFs, if one introduces a coordinate change \((x, y) \to (V(y)x, y)\) on \( \mathbb{R}^m \times Y \), then (11.2) means that the flow \( \pi = \pi_\Psi \) can be changed into the flow \( \pi = \pi_\Phi \) under the change \((x, y) \to (V(y)x, y)\). It is easy to see that cohomologous cocycles have the same Lyapunov (characteristic) exponents in the following sense:

\[
\lambda_\pm^y(\tilde{x}, y; \Psi) = \lambda_\pm^x(x, y; \Phi), \quad \lambda_\pm^y(\tilde{x}, y; \Psi) = \lambda_\pm^x(x, y; \Phi), \quad \lambda(\tilde{x}, y; \Psi) = \lambda(x, y; \Phi),
\]

where \( \tilde{x} = V(y)x \).

For \( 0 \leq k \leq n \) let \( G(n, k) \) be the Grassman manifold of \( k \)-planes in \( \mathbb{R}^n \), and let \( G(n) = \bigcup_{k=0}^n G(n, k) \) be the disjoint union of these compact manifolds. We should introduce some index sets. For any \( k \in \{1, \ldots, n\} \) let \( N(k) \) be the set of those vectors \( \vec{n} = (n_1, \ldots, n_k) \) such that \( n_i \geq 1 \) for all \( 1 \leq i \leq k \) and \( n_1 + \cdots + n_k = n \). Finally, \( P \) denotes the collection of all pairs \( p = (k, \vec{n}) \), where \( 1 \leq k \leq n \) and \( \vec{n} \in N(k) \). The following two theorems are the statements of the Multiplicative Ergodic Theorem.

At first we recall that from the Krylov-Bogoliubov theorem one knows that for a flow \( y \cdot t \) on \( Y \) there is an invariant probability measure \( \mu \) on \( Y \). This means that \( \mu(A \cdot t) = \mu(A) \) for all Borel set \( A \subset Y \) and all \( t \in T \). The invariant measure \( \mu \) is \textit{ergodic} if \( \mu(A \Delta A \cdot t) = 0 \) for all \( t \in T \) implies that either \( \mu(A) = 0 \) or \( \mu(A) = 1 \). Recall that \( A \Delta B = (A \setminus B) \cup (B \setminus A) \) is the symmetric difference.

11.1 Statement of Main Results

**Theorem 11.1** Let \( Y \) be a compact Hausdorff space with a flow \( y \cdot t \) and let \( \mu \) be an invariant probability measure on \( Y \). Let \( \Phi \) be a cocycle on \( Y \) over \( y \cdot t \). Then there exist:

(I) an invariant set \( Y_\mu \subseteq Y \) with \( \mu(Y_\mu) = 1 \);

(II) a measurable decomposition \( Y_\mu = \cup_{p \in P} Y_\mu(p) \), where \( Y_\mu(p) \) is invariant;

(III) measurable mappings \( \lambda_1, \ldots, \lambda_k : Y_\mu(p) \to \mathbb{R} \) with

\[
\lambda_1(y) < \cdots < \lambda_k(y), \quad y \in Y_\mu(p);
\]

(IV) measurable mappings \( W_i : Y_\mu(p) \to G(n, n_i) \), \( 1 \leq i \leq k \), where \( p = (k, \vec{n}) \) and \( \vec{n} = (n_1, \ldots, n_k) \);
such that for any \( y \in Y_\mu(p) \) one has

(V) \( W_1(y), \ldots, W_k(y) \) are independent linear subspaces of \( \mathbb{R}^n \);

(VI) \( \mathbb{R}^n = W_1(y) \oplus \cdots \oplus W_k(y) \);

(VII) if \( x \in W_i(y) \) with \( x \neq 0 \), \( 1 \leq i \leq k \), then \( \lambda(x, y) = \lambda_i(y) \).

If, in addition, \( \mu \) is an ergodic measure, then precisely one \( Y_\mu(p) \) has positive measure (consequently, has measure 1), and the mappings \( \lambda_i : Y_\mu \to \mathbb{R} \) are constants, \( 1 \leq i \leq k \).

**Theorem 11.2** Let \( E \) be a vector bundle over compact Hausdorff space \( Y \) and let \( \pi \) be a LSPF on \( E \). Let \( \mu \) be an invariant probability measure on \( Y \). Then the conclusions of Theorem 11.1 remain valid where \( W_i \) now assume values in the appropriate Grassman bundles over \( Y \). When \( \mu \) is ergodic, the measurable spectrum \( \Sigma^m(\mu) \) is defined to be the collection \( \{\lambda_1, \cdots, \lambda_k\} \), the numbers \( n_1, \cdots, n_k \) are the multiplicities of the spectral values \( \lambda_1, \cdots, \lambda_k \). When \( \mu \) is not ergodic, then the measurable spectrum is \( \Sigma^m(\mu, y) = \{\lambda_1(y), \cdots, \lambda_k(y)\} \) and the multiplicities \( n_1, \cdots, n_k \) depend on \( y \in Y_\mu(p) \). For an ergodic measure \( \mu \), the measurable bundle associated with a spectral value \( \lambda_i \), \( 1 \leq i \leq k \), is

\[
W_i = \{(x, y) : x \in W_i(y), y \in Y_\mu\}.
\]

If \( \mu \) is not ergodic, then the measurable bundles are defined similarly on each of the invariant sets \( Y_\mu(p) \).

In order to emphasize the dynamics aspect, we use \( \Sigma^d \) to denote the dynamical spectrum introduced in Section 7. When \( Y \) is dynamically connected then \( \Sigma^d = \bigcup_{i=1}^k [a_i, b_i] \). In this case the boundary of dynamical spectra is the set \( \partial \Sigma^d = \{a_1, \cdots, a_k, b_1, \cdots, b_k\} \). However we reserve the notation \( \Sigma^d \) the (continuous) invariant spectral subbundles in the Whitney decomposition in Section 7 so that we have \( \mathbb{R}^n \times Y = \bigcup_{i=1}^k V_i \).

Now the main result in this section is the following connection between the measurable and the dynamical spectra.

**Theorem 11.3** Let \( \pi \) be a LSPF on \( \mathbb{R}^n \times Y \) where \( Y \) is compact and dynamically connected. Then one has

\[
\partial \Sigma^d \subseteq \bigcup_{\mu} \Sigma^m(\mu) \subseteq \Sigma^d,
\]  

(11.3)

where the union is either over all invariant probability measures \( \mu \) on \( Y \) or over all ergodic measures on \( Y \). Let \( \mu \) be an invariant probability measure on \( Y \) and let \( \Sigma^m(\mu, y) = \{\lambda_1(y), \cdots, \lambda_k(y)\} \) be the measurable spectrum for \( y \in Y_\mu(p) \). Then for each \( j \) there is precisely one spectral interval \( [a_j, b_j] \) such that \( \lambda_j(y) \in [a_j, b_j] \) for all \( y \in Y_\mu(p) \). Also the associated measurable bundle \( W_j(y) \) satisfies \( W_j(y) \subseteq V_i(y) \) for all \( y \in Y_\mu(p) \). Finally, one has \( V_i(y) = \sum W_j(y) \) for all \( y \in Y_\mu(p) \), where the summation is over all \( j \) such that \( \lambda_j(y) \in [a_j, b_j] \).

The above theorem states roughly that for any given invariant measure \( \mu \), the Lyapunov exponents are always in the dynamical spectrum, and when \( y \in Y_\mu \), the measurable subbundles \( W_j(y) \) are the refinements of the spectral subbundles \( V_i(y) \). Conversely, any boundary point of the dynamical spectrum must be the Lyapunov exponent with respect to some invariant measure.
11. Multiplicative Ergodic Theorem for Cocycles

µ. A more delicate problem is to describe the content of spectral intervals \([a_i, b_i]\) using Lyapunov exponents, i.e., whether the second inclusion in (11.3) is an equality. Another problem is the size of the Lyapunov exponents \(\lambda_j(y)\) in the spectral intervals \([a_i, b_i]\).

The next theorem is concerned directly with the problem of computing the measurable spectrum \(\Sigma^m(\mu, y)\). To this end, we need the notion of wedge product. For \(1 \leq k \leq n\), let \(\Lambda^k\mathbb{R}^n\) denote the vector space generated by all \(k\)-fold wedge products \(x_1 \wedge \cdots \wedge x_k, \ x_i \in \mathbb{R}^n\). Recall that the wedge product is linear in each factor and antisymmetric. If \(L : \mathbb{R}^n \to \mathbb{R}^n\) is a linear mapping, then it induces a linear mapping \(\Lambda^kL : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n\) by

\[
\Lambda^kL(x_1 \wedge \cdots \wedge x_k) := (Lx_1) \wedge \cdots \wedge (Lx_k).
\]

Since \(\Lambda^k(LM) = (\Lambda^kL)(\Lambda^kM)\), one has that if \(\Phi(y, t)\) is a cocycle on \(Y\) then \(\Lambda^k\Phi(y, t)\) is also a cocycle.

Let us write the Lyapunov exponents, for \(y \in Y_\mu(p)\),

\[
\lambda_1(y) < \cdots < \lambda_k(y)
\]

with multiplicities \(n_1, \cdots, n_k\) as in the form

\[
\gamma_1(y) \leq \gamma_2(y) \leq \cdots \leq \gamma_n(y)
\]

(11.4)

where \(\gamma_i(y)\) is repeated \(n_i\)-times in (11.4), \(1 \leq i \leq k\).

**Theorem 11.4** Under the assumption of Theorem 11.1, then for any \(y \in Y_\mu(p)\), one has

(i) \(\lim_{t \to +\infty}(1/t) \log \|\Phi(y, t)\| = \gamma_n(y)\),

(ii) \(\lim_{t \to +\infty}(1/t) \log \|\Lambda^k\Phi(y, t)\| = \gamma_n(y) + \gamma_{n-1}(y) + \cdots + \gamma_{n-k}(y)\) for \(2 \leq k \leq n\),

(iii) \(\lim_{t \to -\infty}(1/t) \log \|\Phi(y, t)\| = \gamma_1(y)\),

(iv) \(\lim_{t \to -\infty}(1/t) \log \|\Lambda^k\Phi(y, t)\| = \gamma_1(y) + \gamma_2(y) + \cdots + \gamma_k(y)\) for \(2 \leq k \leq n\).

**Remark 11.1** The theorems are valid for cocycles with values in \(\text{Gl}(n, \mathbb{C})\) and can be extended to cocycles on general bundles with compact Hausdorff bases.

11.2 Lemmas

As for the dynamical and measurable spectra and the corresponding bundle decompositions of cocycles, a natural isomorphism is as follows, which is wider than the cohomology.

Let \(Y_i\) be compact Hausdorff spaces and \(\sigma_i\) flows on \(Y_i, i = 1, 2\). We say that \(h : Y_1 \to Y_2\) is an epimorphism if \(h\) is continuous, onto and satisfies \(h(\sigma_1(y_1, t)) = \sigma_2(h(y_1), t)\). Let now \(\Phi_i\) be cocycles on \(Y_i\) over \(\sigma_i, i = 1, 2\). We say that \(\Phi_1\) is homomorphic to \(\Phi_2\) if there is an epimorphism \(h : Y_1 \to Y_2\) and a continuous \(H : Y_1 \to \text{Gl}(n)\) such that \(H(\sigma_1(y_1, t))\Phi_1(y_1, t) = \Phi_2(h(y_1), t)H(y_1)\).

Our step is to replace a compact Hausdorff space with a compact metric space and a cocycle with a homomorphic one.

**Lemma 11.1** Let \(\Phi_1\) be a cocycle on a compact Hausdorff space \(Y_1\) with a flow \(\sigma_1(y_1, t) = y_1 \cdot t\). Then there exist a compact metric space \(Y_2\) with a flow \(\sigma_2\), an epimorphism \(h : Y_1 \to Y_2\), and a cocycle \(\Phi_2\) on \(Y_2\) such that \(\Phi_1(y_1, t) = \Phi_2(h(y_1), t)\).
Proof} If \( T = \mathbb{Z} \) let \( T' = \mathbb{Z} \). If \( T = \mathbb{R} \) let \( T' \) be any countable dense set of \( \mathbb{R} \). Let \( A \) be the closed subalgebra of \( C(Y_1, \mathbb{R}) \) generated by all functions \( \{ y \to \phi_{ij}(y, t') : t' \in T' \} \), where \( \Phi_1(y, t) = (\phi_{ij}(y, t))_{n \times n} \). Then \( A \) is a separable subalgebra of \( C(Y_1, \mathbb{R}) \). By the cocycle equality, one sees that \( A \) is invariant in the sense that if \( g \in A \) and \( \tau \in T \) then \( g \cdot \tau = g(\cdot + \tau) \in T \). Now the Stone theorem says that \( A = C(Y_2, \mathbb{R}) \), where \( Y_2 \) is the maximal ideal space of \( A \). As \( A \) is separable, \( Y_2 \) is a compact metric space.

The space \( Y_2 \) can be realized as follows. Define an equivalence \( \sim \) on \( Y_1 \) by

\[
y_1 \sim y_1 \iff \Phi(y_1, \tau) = \Phi_1(y_1, \tau) \quad \text{for all } \tau \in T'.
\]

We assert that

\[
y_1 \sim y_1 \implies y_1 \cdot t \sim y_1 \cdot t \quad \text{for all } t \in T.
\]

In fact, let us assume that \( y_1 \sim y_1 \). For any given \( t \in T \) and \( \tau \in T' \), there exists a sequence \( \{ \tau_n \} \subset T' \) such that \( \tau_n \to t + \tau \) as \( n \to \infty \) because \( T' \) is dense in \( T \). By the continuity, one has \( \Phi_1(y_1, \tau_n) \to \Phi_1(y_1, t + \tau) \) and \( \Phi_1(y_1, \tau_n) \to \Phi_1(y_1, t + \tau) \) as \( n \to \infty \). Similarly there is a sequence \( \{ \tau'_n \} \subset T' \) such that \( \tau'_n \to t \) and \( \Phi_1(y_1, \tau'_n) \to \Phi_1(y_1, t) \) and \( \Phi_1(y_1, \tau'_n) \to \Phi_1(y_1, t) \) as \( n \to \infty \). Now it follows from the cocycle property that

\[
\Phi_1(y_1 \cdot t, \tau) = \Phi_1(y_1, t + \tau) \Phi_1^{-1}(y_1, t) = \lim_{n \to \infty} \Phi_1(y_1, \tau_n) \Phi_1^{-1}(y_1, \tau'_n) = \lim_{n \to \infty} \Phi_1(y_1, \tau_n) \Phi_1^{-1}(y_1, \tau'_n) = \Phi_1(y_1, t + \tau) \Phi_1^{-1}(y_1, t) = \Phi_1(y_1 \cdot t, \tau).
\]

This proves the assertion.

Let now \( Y_2 = Y_1/\sim \) be the quotient space and \( h : Y_1 \to Y_2 \) be the natural projection. Then \( Y_2 \) is a compact metrizable space and \( h \) is continuous and onto. Define the flow \( \sigma_2 \) on \( Y_2 \) as \( \sigma_2([y_1], t) = [\sigma_1(y_1), t] \). It is easy to see that for any \( t \in T \) the flow \( \Phi_1(y_1, t) \) depends only on the equivalence class \([y_1] \). We can define a mapping \( \Phi_2([y_1], t) = \Phi_1(y_1, t) \). Now it can be checked that \( \Phi_2 \) is a cocycle on \( Y_2 \). Moreover one has \( \Phi_1(y_1, t) = \Phi_2([y_1], t) \equiv \Phi_2(h(y_1), t) \).

The next step is to replace a continuous cocycle with a smooth one when \( T = \mathbb{R} \).

**Lemma 11.2** Let \( \Phi \) be a cocycle over a compact Hausdorff space \( Y \) with \( T = \mathbb{R} \). Then \( \Phi \) is cohomologous to a smooth cocycle \( \Psi \) over \( Y \), i.e., \( \Phi(y, t) \) is the fundamental matrix solution to \( \dot{x} = A(y \cdot t)x \) where \( A \) will be given explicitly later.

**Proof** Let \( V \subset \text{Gl}(n) \) be a compact convex neighborhood of the identity \( I \). Choose \( r > 0 \) such that \( \Phi(y, t) \in V \) for all \( y \in Y \) and all \( 0 \leq t \leq r \). Define \( H(y) = (1/r) \int_0^r \Phi(y, s)ds \). Then \( H(y) \) is invertible, and it is easily verified that the cocycle

\[
\Psi(y, t) = H(y \cdot t) \Phi(y, t) H(y)^{-1} = \frac{1}{r} \int_t^{t+r} \Phi(y, s)ds H(y)^{-1},
\]

which is cohomologous to \( \Phi \), is the fundamental matrix solution to \( \dot{x} = A(y \cdot t)x \) where

\[
A(y) = \frac{1}{r} \left[(\Phi(y, r) - I)H(y)^{-1}.
\]
11.3 Triangularization of Cocycles

We consider the theory of Gram-Schmidt factorization of isomorphisms on \( \mathbb{R}^n \), where \( \mathbb{R}^n \) has the Euclidean inner product \( \langle \cdot, \cdot \rangle \). For \( L \in \text{Gl}(n) \), we identify \( L \) with the \( n \times n \) matrix with columns \( Le_i \), where \( \{e_1, \cdots, e_n\} \) is an orthogonal basis in \( \mathbb{R}^n \). Let \( \mathcal{O}(n) \) be the orthogonal group of \( \mathbb{R}^n \) and \( T_+(n) \) the group of all upper triangular linear matrices \( L \in \text{Gl}(n) \) with positive entries on the main diagonal. Then \( \mathcal{O}(n) \cap T_+(n) = \{I\} \).

From the Gram-Schmidt orthogonalization process one knows that for any \( A \in \text{Gl}(n) \) there are uniquely \( G(A) \in \mathcal{O}(n) \) and \( T(A) \in T_+(n) \) such that

\[
G(A) = AT(A).
\]

It is easy to see that both \( T(A) \) and \( G(A) \) are smooth functions of \( A \). One has the equalities

\[
G(AB) = G(AG(B)), \quad T(AB) = T(B)T(ABT(B)). \tag{11.5}
\]

In fact, let \( U, V \in \mathcal{O}(n) \) be

\[
U = ABT(AB), \quad V = G(AG(B)) = ABT(B)T(ABT(B)).
\]

Then

\[
U^{-1}V = T(AB)^{-1}T(B)T(ABT(B)) \in \mathcal{O}(n) \cap T_+(n).
\]

Hence \( U = V \), which proves (11.5).

Next let \( \Phi : Y \to \text{Gl}(n) \) be a cocycle on \( Y \). One has the factorization

\[
G(\Phi(y,t)U) = \Phi(y,t)UT(\Phi(y,t)U) \tag{11.6}
\]

for every \( (y,U) \in Y \times \mathcal{O}(n) =: \mathcal{Y} \) and every \( t \in T \). This permits us to define a flow on \( \mathcal{Y} \) as follows: For every \( \phi = (y,U) \in \mathcal{Y} \) and \( t \in T \),

\[
\phi \cdot t = (y,t,G(\Phi(y,t)U)). \tag{11.7}
\]

One can check that (11.7) actually defines a flow on \( \mathcal{Y} \).

A cocycle \( \Phi \) on \( Y \) defines a LSPF \( \pi \) on \( \mathbb{R}^n \times Y \):

\[
\pi(x,y,t) = (\Phi(y,t)x, y \cdot t).
\]

This flow can be lifted to a new flow \( \tilde{\pi} \) on \( \mathbb{R}^n \times \mathcal{Y} \) by

\[
\tilde{\pi}(x,\phi,t) = (\Phi(y,t)x, \phi \cdot t), \quad \phi = (y,U) \in \mathcal{Y}. \tag{11.8}
\]

In fact, let \( q : Y \times \text{Gl}(n) \to \text{Gl}(n) \) and \( r : Y \times \text{Gl}(n) \to \text{Gl}(n) \), (or, \( r : \mathcal{Y} \to Y \)) be the natural projections. Define \( \Psi(\phi,t) \) by

\[
\Psi(\phi,t) = q(\phi \cdot t)^{-1}\Phi(y,t)q(\phi) = G(\Phi(y,t)U)^{-1}\Phi(y,t)U, \tag{11.9}
\]

where \( (-1)^{-1} \) denotes the matrix inverse. Since \( \phi \cdot t \) is a flow on \( \mathcal{Y} \), it follows that \( \Psi \) is a cocycle on \( \mathcal{Y} \), and

\[
\tilde{\pi}(x,\phi,t) = (\Psi(\phi,t)x, \phi \cdot t) \tag{11.10}
\]

is a LSPF on \( \mathbb{R}^n \times \mathcal{Y} \) which is cohomologous to \( \tilde{\pi} \).

We summarize these as the following lemma.
Lemma 11.3 Let $\Phi$ be a cocycle on $Y$. Then (i) (11.7) defines a flow on $\mathcal{Y}$; (ii) (11.9) defines a cocycle on $\mathcal{Y}$; and (iii) the induced LSPF $\tilde{\pi}$ (11.10) of $\Psi$ is cohomologous to the lifted LSPF $\hat{\pi}$ (11.8) of $\pi$. Moreover, one has

$$\Psi(\phi, t) = T(\Phi(y, t)U)^{-1} \in T_n$$ (11.11)

for all $\phi = (y, t) \in \mathcal{Y}$ and all $t \in T$.

Remark 11.2 When $T = \mathbb{R}$ and $\Phi(y, t)$ is a smooth cocycle generated by a differential equation

$$\dot{x} = A(y \cdot t)x, \quad x \in \mathbb{R}^n, \ y \in Y;$$ (11.12)

then $\Psi(\phi, t)$ is also a smooth cocycle which is generated by the equation

$$\tilde{x}' = B(\phi \cdot t)\tilde{x}, \quad \tilde{x} \in \mathbb{R}^n, \ \phi \in \mathcal{Y},$$ (11.13)

where $\phi = (y, t)$ and

$$B = G^{-1}(AG - G'), \quad G = G(\Phi(y, t)U), \quad G' = \frac{d}{dt}G.$$

The change of variables which maps solutions of (11.13) onto those of (11.12) is

$$x = G(\Phi(y, t)U)\tilde{x}.$$

Also since the fundamental matrix solution of (11.2) is $\Psi$, a upper triangular matrix, we see that $B$ is also upper triangular.

This triangularization technique has been frequently used since Lyapunov, Perron, Diliberto, Liao, Oseledec.

11.4 Relationship Between Invariant Measures

Between the flow on $Y$ and the extended flow on $\mathcal{Y} = Y \times O(n)$, there is an epimorphism $r : \mathcal{Y} \to Y$, i.e., $r(\phi \cdot t) = r(\phi) \cdot t$. Because of Lemma 11.1, one may assume that $Y$ (and therefore $\mathcal{Y}$) is a compact metric space so that the Riesz Representation Theorem applies.

References

11. Multiplicative Ergodic Theorem for Cocycles


