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TITLE: Anosov Endomorphisms: Shift Equivalence
And Shift Equivalence Classes

NAME: ZHANG Meirong
STUDENT NO.: 18601804
DEPARTMENT: Mathematics
SPECIALITY: Pure Mathematics
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SUPERVISOR: LIAO Shantao

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Abstract

Let M be a compact manifold and f a self-diffeomorphism of M . For a hyperbolic attractor Λ of f , Williams systematically reduced the attractor $(\Lambda, f|_\Lambda)$ to the inverse limit system (Σ_g, σ_g) for some Anosov endomorphism g on a branched manifold K . Furthermore, for expanding attractors, he even reduced topological conjugacy of attractors to shift equivalence of the corresponding maps g 's. Anosov endomorphisms represent models of hyperbolic attractors in this sense. However, we show that the later reduction does not hold for general hyperbolic attractors.

In this dissertation, we will prove the following

Theorem 1 *Let a be a hyperbolic toral endomorphism. If f is sufficiently C^1 close to a , then f is shift equivalent to a if and only if f is topologically equivalent to a .*

As a corollary, we have

Theorem 2 *For general Anosov endomorphisms, inverse limit equivalence cannot imply shift equivalence.*

Recently, Lan Wen proved that an Anosov endomorphism on a branched surface with some additional assumption on surface is shift equivalent to some Anosov endomorphism on 2-torus. This leads us to consider the following question: Whether the ‘nonlinearity’ of any Anosov endomorphism on tori can be removed via shift equivalence?

This question involves shift equivalence classes. It is well known that Anosov diffeomorphisms on infranilmanifolds and expanding maps on any compact manifold are ‘linear’ under topological conjugacy.

Let $\mathcal{A}(M)$ denote the set of all C^1 Anosov endomorphisms on a manifold M . Let $\mathcal{A}^*(M) = \mathcal{A}(M) \setminus (\{\text{Anosov diffeomorphisms}\} \cup \{\text{expanding maps}\})$. Since any $f \in \mathcal{A}^*(M)$ is semi-structurally unstable, we will prove the following result.

Theorem 3 *Let M be an infranilmanifold. Then there exists a C^1 dense subset \mathcal{U} in $\mathcal{A}^*(M)$ such that every f in \mathcal{U} is not shift equivalent to any hyperbolic infranilmanifold endomorphism.*

On tori, we can obtain the following stronger result.

Theorem 4 *Let $n \geq 2$. Then there exists a residual subset \mathcal{U} in $\mathcal{A}^*(\mathbb{T}^n)$ such that every f in \mathcal{U} is not shift equivalent to any hyperbolic toral endomorphism.*

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During the Workshop on Dynamical Systems held in the International Centre for Theoretical Physics, Trieste, Italy, September—October, 1988, conversation with Professor R. F. Williams initiated my work on shift equivalence.

Dr. WEN Lan of Department of Mathematics at Peking University read carefully the manuscript of this dissertation and made many suggestions which improve the presentation of this dissertation.

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1 Introduction

It is well-known that hyperbolicity is the most important concept in differentiable dynamical systems. Hyperbolicity is a sufficient and necessary condition of locally structural stability of simpler orbits such as fixed points and periodic orbits for both continuous-time flows generated by ordinary differential equations and discrete-time flows generated by diffeomorphisms. For globally structural stability, hyperbolicity is also a fundamental condition although it is not sufficient.

At the beginning of 1960's, Peixoto characterized all structurally stable vector fields on compact orientable surfaces, which initiated the modern theory of differentiable dynamical systems. From Peixoto's consideration, one class of structurally stable systems — Morse-Smale systems — on higher dimensional manifolds has been found. Such a class of systems is just the direct generalization of Peixoto's. On the other hand, Anosov proved the structural stability for geodesic flows on manifolds of negative curvature. A remarkable difference between these two classes of systems is that the latter ones may admit infinitely many periodic orbits. This characteristic is also shared by Thom's toral hyperbolic automorphisms. Thereafter, generalization of these examples — Anosov systems — on certain manifolds has been found. Smale then found the famous 'horseshoe' which is important to characterize structural stability on higher dimensional manifolds. Based on these, Smale introduced the so-called Axiom A systems. After works of Māné, Liao and others, structurally stable systems on higher dimensional manifolds have been understood at a considerable step.

Another question for differentiable dynamical systems, which is closely related with structural stability, is the classification of systems. For Morse-Smale flows on surfaces, this has been settled down using some idea such as trees from graph theory. Due to the contribution of Franks, Newhouse and Manning, the classification of Anosov diffeomorphisms on tori can be settled down using the algebraic classification of the induced automorphisms on the fundamental groups. As a result, any Anosov diffeomorphism on tori is topologically conjugate to some 'linear' Anosov diffeomorphism.

Along with the development of differentiable dynamical systems theory, the study for iteration of noninvertible maps has attracted many attentions. Shub introduced expanding maps in late 1960's and found that these systems share many similar properties as for Anosov diffeomorphisms. Such a class of systems can only survive on infranilmanifolds. Gromov proved that all expanding maps are 'linear' in the sense of topological conjugacy.

Then there aroused the concept of Anosov endomorphisms. At the first people did think that these systems also share the same properties as for Anosov diffeomorphisms. In 1974, Māné-Pugh and Przytycki found independently some quite different properties for general Anosov endomorphisms. A remarkable result is that 'true' Anosov endomorphisms are structurally unstable. By the word 'true', we mean those Anosov endomorphisms which are neither Anosov diffeomorphisms nor expanding maps. Based on such an instability, Zhang [52] proved the existence of 'nonlinear' Anosov endomorphisms on tori in the sense of topological conjugacy. This shows that it seems impossible to give a complete classification of all Anosov endomorphisms.

In the early 1970's, in his study on hyperbolic attractors Williams systematically developed a method to reduce the attractors to inverse limit systems. For expanding attractors, he proved further that topological conjugacy for attractors can be reduced to the easily-studied shift equivalence for maps. He then obtained a satisfactory classification result for

1-dimensional expanding attractors.

In this dissertation, shift equivalence and shift equivalence classes of Anosov endomorphisms are the main objects. The following two problems will be studied:

- In what range does the reduction from inverse limit equivalence to shift equivalence hold?
- Whether the ‘nonlinearity’ of Anosov endomorphisms on tori can be removed under shift equivalence?

The second question originates from a recent work of Wen, which says that an Anosov endomorphism on a branched surface (with some additional assumption on surfaces) can be shift equivalent to some one on (non-branched) 2-torus.

The dissertation is organized as follows.

In Section 2 we have a review for the concept and main properties of Anosov endomorphisms.

Section 3 contains some metric properties for universal covering manifolds. Special emphasis is on metric properties for lifting maps.

In Section 4, we will prove that lifting systems of Anosov endomorphisms have some ‘stability’. Namely,

Theorem 4.1 *Let f and g be Anosov endomorphisms on a compact manifold M , and F and G be their liftings to universal covering manifold \bar{M} . Assume that G is C^1 close to F . Then there exists a homeomorphism $H : \bar{M} \rightarrow \bar{M}$ such that*

$$H \circ F = G \circ H.$$

Here H is C^0 close to $\text{id}_{\bar{M}}$ and both H and H^{-1} are uniformly continuous with respect to the lifting metric on \bar{M} .

Then we can obtain the following result.

Theorem 4.3 *Let f and g be an Anosov endomorphism on M . Then there exists a C^1 neighborhood \mathcal{U} of f such that for any $g \in \mathcal{U}$ one has $\text{ent}(g) = \text{ent}(f)$, where $\text{ent}(\cdot)$ stands for topological entropy.*

In Section 5 we simply explain the origin of ‘shift equivalence’ and then study shift equivalence for Anosov endomorphisms. The main result is:

Theorem 5.2 *Let $a : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a hyperbolic toral endomorphism and f be C^1 close to a . Then f is shift equivalent to a if and only if f is topologically conjugate to a .*

As a result, we have

Theorem 5.4 *For general Anosov endomorphisms f and g , f and g are shift equivalent does not necessarily imply that f and g are topologically conjugate.*

In Section 6 we study the second problem on tori and on their generalizations — infranilmanifolds. The main results are:

Theorem 6.6 *Let M be an infranilmanifold. Then there exists a dense subset \mathcal{U} in $\mathcal{A}^*(M)$ such that any $f \in \mathcal{U}$ is not shift equivalent to any ‘linear’ one.*

On tori, we can strength the above result.

Theorem 6.7 *Let $n \geq 2$. Then there exists a residual subset \mathcal{U} in $\mathcal{A}^*(\mathbb{T}^n)$ such that every f in \mathcal{U} is not shift equivalent to any hyperbolic toral endomorphism.*

Section 7 contains some notes on the possibility of global linearization of toral Anosov endomorphisms in some sense.

2 Anosov Endomorphisms

In this section we give the definition of Anosov endomorphisms which are the objects we are going to study in this dissertation. We also review the historical development of these systems and some known results.

Historically, dynamical systems theory mainly deals with flows generated by ordinary differential equations. Since 1960's, the Smale school began study discrete-time flows — iteration of diffeomorphisms. We will consider in this dissertation generalizations of discrete-time flows — iteration of maps.

For convenience, we briefly review some basic concepts in dynamical systems. For general theory of dynamical systems, one may refer to the books of Smale [40], Palis-de Melo [31] and Zhang [54].

2.1 Basic concepts from dynamical systems

As we are only interested in differentiable maps on manifolds, we do not intend to give the most general concepts for dynamical systems.

In the following, we always use X, X', Y etc. to denote metric spaces and use d, ρ etc. to denote the metrics. All maps appeared in the sequel are assumed to be continuous. We will use $C^0(X), C^0(X, Y)$ etc. to denote the sets of all continuous maps from X to itself and those from X to Y respectively. The notation $\text{Homeo}(X)$ or $\mathcal{D}^0(X)$ means the set of all self-homeomorphisms of X .

Most of concepts for iteration of noninvertible maps are similar to those for iteration of invertible maps. Some need a little modification.

The most important task for dynamical systems is to describe the structure of orbits. To this end we need not distinguish two systems in the following sense of equivalence.

Definition 2.1 Let $f \in C^0(X)$ and $g \in C^0(Y)$. We say that g is topologically conjugate (or, topologically equivalent) to f , denoted by $g \stackrel{t}{\sim} f$, if there exists a homeomorphism $h \in \text{Homeo}(X, Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

i.e., $h \circ f = g \circ h$. If h in this equality is only required to be onto, we say that g is semi-topologically conjugate to f , denoted by $g \stackrel{st}{\sim} f$. \square

For a noninvertible map $f \in C^0(X)$, the positive iterations f^n ($n \geq 0$) make sense. Fix an $x \in X$. The positive orbit starting at x is the set $\{f^n(x) : n = 0, 1, 2, \dots\}$. Since many dynamics behaviour do not only depend on positive iterations but also on 'negative' iterations, we need the following concept of orbits for noninvertible maps. Such a concept of orbits is also referred as 'inverse limit orbits' in the literature.

Definition 2.2 Let $f \in C^0(X)$ be a map which is onto. An orbit of f means a bi-infinite sequence of points $(x_n)_{-\infty}^{\infty}$ in M such that $f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Let the set of all

orbits of f be denoted by Σ_f . When X is compact, a natural metric on the produce space $\prod_{-\infty}^{\infty} X$ is

$$d((x_n), (y_n)) = \sum_{n=-\infty}^{+\infty} \frac{1}{2^{|n|}} d(x_n, y_n).$$

As a subspace of $\prod_{-\infty}^{\infty} X$, Σ_f is also compact. One then can define the shift σ_f on Σ_f by

$$\sigma_f((x_n)) = (y_n), \quad y_n = x_{n+1} \quad \text{for all } n \in \mathbb{Z}.$$

It is easy to see that $\sigma_f : \Sigma_f \rightarrow \Sigma_f$ is a homeomorphism. Moreover, if one introduces the projection $\pi_0 : \Sigma_f \rightarrow X$ by

$$\pi_0((x_n)) = x_0,$$

then one has a semi-topological conjugacy: $\pi_0 \circ \sigma_f = f \circ \pi_0$. \square

For any fixed $x_0 \in X$, the positive orbit of f starting at x_0 is unique. When f is noninvertible, the set consisting of all negative orbits of f starting at the same x_0 is in general a Cantor set.

Using the above concept of orbits, the following concept of equivalence is also of sense for noninvertible maps.

Definition 2.3 Let $f \in C^0(X)$ and $g \in C^0(Y)$. We say that g is inverse limit equivalent (or, orbit equivalent) to f , denoted by $g \overset{\sim}{\sim} f$, if their inverse limit systems are topologically conjugate, i.e., $(\Sigma_g, \sigma_g) \overset{t}{\sim} (\Sigma_f, \sigma_f)$. \square

In this dissertation we will also discuss classification of hyperbolic attractors. To this end, we need to introduce the following equivalence. One can find its background in papers of Williams [47, 48]. Some properties for such an equivalence will be discussed in Sections 5 and 6.

Definition 2.4 Let $f \in C^0(X)$ and $g \in C^0(Y)$. We say that g is shift equivalent to f , denoted by $g \overset{s}{\sim} f$, if there exist maps $r \in C^0(X, Y)$ and $s \in C^0(Y, X)$, and some integer $m \geq 0$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ r \downarrow & & \downarrow r \\ Y & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & X \\ s \uparrow & & \uparrow s \\ Y & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f^m} & X \\ r \downarrow & \begin{array}{c} \nearrow s \\ \searrow \end{array} & \downarrow r \\ Y & \xrightarrow{g^m} & Y \end{array}$$

i.e., r, s, m satisfy a set of shift equivalence equations: $r \circ f = g \circ r$, $f \circ s = s \circ g$, $s \circ r = f^m$, $r \circ s = g^m$. \square

When $m = 0$, shift equivalence is topological conjugacy. From above definitions we have the following properties.

Proposition 2.5 (i) *All of these relations are equivalence relations.*

(ii) *When f and g are homeomorphisms, all of these equivalence relations are the same.*

Proof We need only to consider shift equivalence. Suppose that $r : X \rightarrow Y$, $s : Y \rightarrow X$, and $m \geq 0$ give shift equivalence for maps f and g . Suppose further that $r' : Y \rightarrow Z$, $s' : Z \rightarrow Y$, and $m' \geq 0$ give shift equivalence for maps g and h . It is not difficult to check that the maps $r' \circ r : X \rightarrow Z$, $s \circ s' : Z \rightarrow X$ and the integer $m + m' \geq 0$ yield the desired shift equivalence for maps f and h . \square

We have the following relations for these three kinds of equivalence relations.

Proposition 2.6 (i) $g \overset{t}{\sim} f \implies g \overset{s}{\sim} f$. (ii) $g \overset{s}{\sim} f \implies g \overset{o}{\sim} f$.

Proof We need only to check (ii). Suppose r , s and m are as in Definition 2.4. Define using $r : X \rightarrow Y$ the following map $R_0 : \Sigma_f \rightarrow \Sigma_g$ by

$$R_0((x_n)) = (r(x_n)) \quad \text{for all } (x_n) \in \Sigma_f.$$

Let $R_i = R_0 \circ \sigma_f^i$, $i \in \mathbb{Z}$. Then one has $\sigma_g \circ R_i = R_i \circ \sigma_f$.

Using $s : Y \rightarrow X$, one can also define maps S_0, S_j , $j \in \mathbb{Z}$. Moreover, R_i, S_j satisfy, for any $(x_n) \in \Sigma_f$,

$$\begin{aligned} S_j \circ R_i((x_n)) &= S_j \circ R_0 \circ \sigma_f^i((x_n)) = S_j \circ R_0((x_{n+i})) \\ &= S_j((r(x_{n+i}))) = S_0 \circ \sigma_g^j((r(x_{n+i}))) = S_0((r(x_{n+i+j}))) \\ &= (s \circ r(x_{n+i+j})) = (f^m(x_{n+i+j})) = (x_{n+i+j+m}) \\ &= (x_n) \quad \text{if } i + j = -m. \end{aligned}$$

As a result, R_i and S_j give topological conjugacy for σ_f and σ_g if one takes $i, j \in \mathbb{Z}$ such that $i + j = -m$. \square

The converses of the above proposition do not hold in general. However, we will see in later sections that the converses do hold for certain special maps f and g .

In the following we will use M, N, P etc. to denote smooth Riemannian manifolds which are also compact in most cases. For an integer $r \geq 0$, we use the following notations to denote sets of systems:

$$\begin{aligned} C^r(M) &= \{f : M \rightarrow M : f \text{ is a } C^r \text{ map}\}; \\ \mathcal{E}^r(M) &= \{f : M \rightarrow M : f \text{ is a } C^r \text{ local diffeomorphism}\}; \\ \mathcal{D}^r(M) &= \{f : M \rightarrow M : f \text{ is a } C^r \text{ diffeomorphism}\}. \end{aligned}$$

When $r = 0$, \mathcal{E}^0 and \mathcal{D}^0 denote local homeomorphisms and homeomorphisms respectively. On the space $C^0(M)$ one may introduce the metric by

$$d_0(f, g) = \sup_{x \in M} d(f(x), g(x)), \quad f, g \in C^0(M),$$

when M is compact. For the C^r -topology on the space $C^r(M)$, see Palis-de Melo [31] and Zhang [54]. We have

Proposition 2.7 For any $r \geq 1$, $\mathcal{D}^r(M) \subset_{\text{open}} \mathcal{E}^r(M) \subset_{\text{open}} C^r(M)$. \square

For the spaces $\mathcal{E}^r(M)$ and $\mathcal{D}^r(M)$ we use the C^r -topology.

Based on any equivalence one can introduce the corresponding concept of stability. In particular,

Definition 2.8 We say that $f \in C^r(M)$ ($r \geq 1$) is α -stable, if there exists a C^r -neighborhood \mathcal{U} such that for any g in \mathcal{U} one has $g \overset{\alpha}{\sim} f$, where $\alpha = t, o$ or s . We sometimes say that f is semi-structurally stable (or, st -stable) if one replaces the equivalence by $\overset{st}{\sim}$, although $\overset{st}{\sim}$ is not an equivalence relation. \square

We also refer t -stable as structurally stable and refer o -stable as orbit stability.

Of course one can derive from relations for equivalences that for the corresponding stabilities.

2.2 Anosov diffeomorphisms and expanding maps

As we described in the introduction, hyperbolicity plays a fundamental role in stability. For simple orbits such as fixed points, periodic orbits, hyperbolicity is a sufficient and necessary condition to guarantee local stability. For global stability, Smale has introduced the concept of hyperbolic (invariant) sets, Axiom A systems, etc. In this case, hyperbolicity is only a necessary condition for global stability. Some additional conditions such no-cycle, transversality conditions are also necessary, see Smale [40], Robinson [37], Liao [21, 22] and Māné [25].

Among systems having some hyperbolicity, the most important class is the set of the so-called Anosov systems. In one sentence, Anosov systems are those having hyperbolicity on the whole manifold. Let us recall the definition of hyperbolic sets.

Definition 2.9 Let $f \in \mathcal{D}^r(M)$ ($r \geq 1$). We say that a compact invariant set Λ of f (the invariance means $f(\Lambda) = \Lambda$) is hyperbolic if the restriction of tangent bundle TM to Λ , $T_\Lambda M$, has a continuous splitting $T_\Lambda M = E^s \oplus E^u$ such that

- (i) E^s and E^u are Tf -invariant, i.e., for any $x \in \Lambda$, one has $T_x f \cdot E_x^s = E_{f(x)}^s$ and $T_x f \cdot E_x^u = E_{f(x)}^u$.
- (ii) Tf is contracting and expanding on E^s and E^u respectively, i.e., there exist constants $C > 0$ and $0 < \mu < 1$ such that

$$\begin{aligned} \|T_x f^n \cdot v\| &\leq C\mu^n \|v\|, & x \in \Lambda, v \in E_x^s, n = 0, 1, 2, \dots \\ \|T_x f^n \cdot v\| &\geq C^{-1}\mu^{-n} \|v\|, & x \in \Lambda, v \in E_x^u, n = 0, 1, 2, \dots \end{aligned} \quad \square$$

Now we give the definition of Anosov diffeomorphisms.

Definition 2.10 Let M be a compact manifold. We say that $f \in \mathcal{D}^r(M)$ ($r \geq 1$) is an Anosov diffeomorphism if f has M as a hyperbolic set. The set of all C^1 Anosov diffeomorphisms on M is denoted by $\mathcal{A}_d(M)$. \square

We remark the followings.

- (i) As M is compact, the definition does not depend on the choice of Riemannian metrics on M . When M is not compact, formal generalization can be introduced. However, the concept does depend on the choice of metrics. For example, White [46] proved that the map $g(x, y) = (x+1, -y)$ is an Anosov diffeomorphism with respect to some Riemannian metric on \mathbb{R}^2 , while such a g has no any fixed point. On non-compact manifolds, a reasonable definition of Anosov diffeomorphisms remains a problem. Anosov thought in [3] that some restriction on f is necessary. We will meet in this dissertation only some special Anosov diffeomorphisms on non-compact manifolds, i.e., the liftings of Anosov endomorphisms on compact manifolds.
- (ii) The subbundles E_x^s and E_x^u , in general, only depends continuously on x , although some ‘integration’ of E_x^s and E_x^u can yield C^r stable and unstable manifolds.
- (iii) The exponential speed in contracting and expanding conditions is uniform with respect to x .
- (iv) E_x^s and E_x^u are uniquely determined by

$$E_x^{s(u)} = \{v \in T_x M : \|T_x f^n \cdot v\| \rightarrow 0 \text{ as } n \rightarrow +\infty (n \rightarrow -\infty)\}.$$

In this case the invariance holds naturally.

Anosov systems originate from the study of Anosov on geodesic flows of negatively curved manifolds [1, 2]. One may also refer to Chapter 3 of Arnold [4] and Manning [28] for such geodesic flows. Later it was found that Thom’s examples also share such a hyperbolicity. Projecting the linear transformations such as $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ of \mathbb{R}^2 onto $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ one can obtain Anosov diffeomorphisms on \mathbb{T}^2 . Those ‘linear’ examples are essentially all Anosov diffeomorphisms so far we have found.

The most remarkable property for Anosov diffeomorphisms is the following stability result.

Theorem 2.11 (Anosov) *An Anosov diffeomorphism on a compact manifold is ε -structurally stable under C^1 perturbations. Namely, if $f \in \mathcal{A}_d(M)$, then for any C^1 diffeomorphism g which is C^1 close to f , there exists an $h \in \text{Homeo}(M)$, which is close to id_M , such that $h \circ f = g \circ h$. \square*

An analytic proof of J. Mather for above theorem was given in Smale [40]. Such an idea was later expanded to solve nonlinear conjugating equations. For the proof using pseudo orbit shadowing technique, see the work of Japan school, e.g., Morimoto [29].

It was found that those manifolds having Anosov diffeomorphisms have some restriction. From Thom’s examples, one has constructed many ‘linear’ Anosov diffeomorphisms on tori, nilmanifolds, and infranilmanifolds. We will use the terminology of hyperbolic automorphisms for those examples, see Shub [38] and Franks [15]. The detailed construction of these examples will be given in the next section. Under topological conjugacy, those are all examples of Anosov diffeomorphisms so far one has found. In fact there is a famous conjecture in this connection.

Conjecture 2.12 *Let M be a compact manifold. Then any Anosov diffeomorphism on M is topologically conjugate to some hyperbolic automorphism on an infranilmanifold. In particular, M is itself homeomorphic to an infranilmanifold. \square*

In attacking this conjecture, Franks [15] developed the concept of π_1 property. Some results on this are also given in Zhang [51]. After work of Manning, we know that

Theorem 2.13 (Franks [15] and Manning [27]) *Let M an infranilmanifold. Then any Anosov diffeomorphism on M is topologically conjugate to some hyperbolic automorphism.* \square

It remains unknown whether other manifolds except those algebraic ones can support also Anosov diffeomorphisms. See Newhouse [30] and Brin-Manning [10]. However, Franks [17] used some foliation theory developed by Haefliger [17] to prove that the manifold M has necessarily some topological properties such as $\pi_i(M) = 0$, $i \geq 2$, if M can support an Anosov diffeomorphism. See Franks [15] and Anosov [3]. For the related work on Anosov flows, the situation is more complicated, see Tomter [42, 43].

For noninvertible systems, M. Shub defined based on the above algebraic examples the following concept of expanding maps.

Definition 2.14 (Shub [38]) Let M be a compact manifold. We say that $f \in C^r(M)$ ($r \geq 1$) is an expanding map if there exist constants $C > 0$ and $\lambda > 1$ such that

$$\|T_x f^n \cdot v\| \geq C\lambda^n \|v\|, \quad x \in M, v \in T_x M, n = 0, 1, 2, \dots$$

The collection of all C^1 expanding maps on M is denoted by $\mathcal{E}_p(M)$. \square

Those maps share some similar properties with Anosov diffeomorphisms.

Theorem 2.15 (Shub [38]) *All expanding maps on compact manifolds are C^1 structurally stable.* \square

For expanding maps, we know more.

Theorem 2.16 (Gromov [16]) *If f is an expanding map on a compact manifold M , then f is necessarily topologically conjugate to some expanding endomorphism on some infranilmanifold. In particular, M is itself homeomorphic to an infranilmanifold.* \square

Similar to hyperbolic sets, Zhang [53] studied expanding (invariant) sets of noninvertible maps. See also Przytycki [35].

It is worth mentioning the work of R. F. Williams on hyperbolic attractors in late 1960's. He systematically developed the idea of reducing along stable foliations and then expressed the complicated attractors to the inverse limit systems of some hyperbolic systems on branched manifolds. In particular, he reduced further the equivalence for expanding attractors to the so-called shift equivalence for the corresponding expanding maps. As a result, he has obtained a satisfactory classification for 1-dimensional expanding attractors. See Williams [47, 48]. For further results on expanding attractors, see also Bothe [8], Farrell-Jones [13], Jones [18] and Plykin [34].

For ergodicity of Anosov diffeomorphisms and expanding maps, one can refer to Bowen [9], Krzyżewski [20] and Māné [24].

2.3 Anosov endomorphisms

As Anosov diffeomorphisms and expanding maps share many similar properties, a natural idea to expand and to unite them as Anosov endomorphisms. Here are some definitions for Anosov endomorphisms.

Definition 2.17 ([15, 38]) Let $f \in \mathcal{E}^r(M)$ ($r \geq 1$). We say that f is an ‘Anosov endomorphism’ if TM has a TF -invariant continuous splitting $TM = E^s \oplus E^u$ such that Tf contracts E^s and expands E^u respectively. \square

Such a concept of Anosov endomorphisms contains well-understood Anosov diffeomorphisms and expanding maps. It was suspected in a relatively not so short period that those systems are also structurally stable. Later it was found that such a class of systems is no longer structurally stable. In fact such a definition is not satisfactory and now has been neglected. Let us call those systems in Definition 2.17 ‘hyperbolic systems’ and see something on them.

For a noninvertible map $f \in C^r(M)$, we have for any fixed $x_0 \in M$ a uniquely determined positive orbit. So $E_{x_0}^s$ can be uniquely determined by

$$E_{x_0}^s = \{v \in T_{x_0}M : \lim_{n \rightarrow +\infty} \|Tf_{x_0}^n \cdot v\| = 0\}.$$

Firstly, if f is not expanding, then any subspace $\tilde{E}_{x_0}^u$ complement to $E_{x_0}^s$ in $T_{x_0}M$ is also an expanding subspace for $Tf_{x_0}^n$ on which the expanding speed depends on the angle between $E_{x_0}^s$ and $\tilde{E}_{x_0}^u$. As the negative orbits are not uniquely determined, we have no a natural choice for $E_{x_0}^u$. As a result, the expanding property for Tf^n on E^u has imposed some restriction on f .

Secondly, according to Definition 2.2, for any $x_0 \in M$, one may take an orbit $(x_n) \in \Sigma_f$. Then for any $n = -1, -2, \dots$, we have a sequence of linear operators

$$Tf_{x_n} : T_{x_n}M \rightarrow T_{x_{n+1}}M, \quad n = -1, -2, \dots$$

As f is a local diffeomorphism, we have the inverses

$$(Tf_{x_n})^{-1} : T_{x_{n+1}}M \rightarrow T_{x_n}M, \quad n = -1, -2, \dots$$

The composition of the first $|n|$ operators yields a sequence of linear operators, for which we denote also by $Tf_{x_0}^n$,

$$Tf_{x_0}^n : T_{x_0}M \rightarrow T_{x_n}M, \quad n = -1, -2, \dots$$

Along with (x_n) we can define reasonably

$$Tf_{x_m}^n : T_{x_m}M \rightarrow T_{x_{n+m}}M, \quad n, m \in \mathbb{Z}.$$

Now we find that $E_{x_0}^u$ in Definition 2.17 should be

$$E_{x_0}^u = \{v \in T_{x_0}M : \lim_{n \rightarrow +\infty} \|Tf_{x_0}^{-n} \cdot v\| = 0\}.$$

It is obvious that the right-hand side does not only depend on x_0 but also on the whole negative orbit $(x_n)_{-\infty}^{-1}$. For this reason, we use the notation $E_{x_0:(x_n)}^u$ to denote the right-hand

of the above equality. When f has some C^1 perturbation $g \in \mathcal{E}^r(M)$, then for any orbit (y_n) of g which is close to (x_n) , one has the corresponding $E_{y_0:(y_n)}^u$. As $E_{y_0:(y_n)}^u$ can depend on (y_n) , g may no longer be a ‘hyperbolic’ map. As a result, the set of all hyperbolic maps as in Definition 2.17 is not open in the space of $C^1(M)$. In fact M̄an̄e–Pugh [26] proved that the interior of above hyperbolic maps is just the set consisting of all Anosov diffeomorphisms and expanding maps.

Following above analysis, one has the following reasonable definition for Anosov endomorphisms due to the important contribution of M̄an̄e–Pugh [26] and Przytycki [35]. We first give the definition of M̄an̄e–Pugh.

Definition 2.18 (M̄an̄e–Pugh [26]) *Let $f \in \mathcal{E}^r(M)$ ($r \geq 1$). We say that f is an Anosov endomorphism if there exists a Tf -invariant continuous subbundle $E^s \subset TM$ such that Tf is contracting on E^s and the induced map $\bar{T}f$ of Tf on quotient bundle TM/E^s is expanding.* \square

The metric on the quotient bundle TM/E^s is given by

$$\|[v]\| = \inf\{\|w\| : w \in [v]\}, \quad v \in TM.$$

Form the following proposition one can see what an Anosov endomorphism does mean.

Proposition 2.19 (M̄an̄e–Pugh [26]) *Let $f \in \mathcal{E}^r(M)$. Then f is an Anosov endomorphism \iff the lifting map of f to the universal covering manifold is an Anosov diffeomorphism.* \square

We will briefly review some on universal covering manifolds and some metric properties for liftings in the next section.

We use $\mathcal{A}(M)$ to denote the set of all C^1 Anosov endomorphisms as in Definition 2.18. Some properties for Anosov endomorphisms are similar to that for Anosov diffeomorphisms. See [26, 35]. However, for ‘true’ Anosov endomorphisms in $\mathcal{A}^*(M) = \mathcal{A}(M) \setminus (\mathcal{A}_d(M) \cup \mathcal{E}_p(M))$ one has a different stability from Anosov diffeomorphisms.

Theorem 2.20 (M̄an̄e–Pugh [26]) *Let $f \in \mathcal{A}(M)$. Then f is ε -structurally stable if and only if $f \in \mathcal{A}(M) \setminus \mathcal{A}^*(M)$.* \square

From above theorem, one sees that Anosov diffeomorphisms and expanding maps are of some speciality among Anosov endomorphisms. The introduction of general Anosov endomorphisms does not provide more examples of stable systems. However, Anosov endomorphisms have the following stability.

Theorem 2.21 (M̄an̄e–Pugh [26]) *Any $f \in \mathcal{A}^*(M)$ is o -stable under C^1 perturbations.* \square

M̄an̄e–Pugh’s proof for Theorem 2.20 is of geometrical nature, from which one finds that the instability of Anosov endomorphisms originates from some ‘independence’ for negative orbits starting at the same initial point.

Now we give another definition for Anosov endomorphisms which is given by Przytycki basing on idea in Definition 2.2. Such a definition is more intrinsic, namely an Anosov endomorphism means Tf is of uniform hyperbolicity along all orbit $(x_n) \in \Sigma_f$. More precisely,

Definition 2.22 (Przytycki [35]) Let $f \in \mathcal{E}^r(M)$ ($r \geq 1$). We say that f is an Anosov endomorphism if there exist constants $C > 0$ and $0 < \mu < 1$ such that for each $(x_n) \in \Sigma_f$, one has a decomposition

$$\bigcup_{n \in \mathbb{Z}} T_{x_n} M = E^s \oplus E^u = \bigcup_{n \in \mathbb{Z}} E_{x_n}^s \oplus E_{x_n}^u,$$

which is invariant under Tf and

$$\begin{aligned} \|Tf^n \cdot v\| &\leq C\mu^n \|v\|, & v \in E^s, n = 0, 1, 2, \dots \\ \|Tf^n \cdot v\| &\geq C^{-1}\mu^{-n} \|v\|, & v \in E_x^u, n = 0, 1, 2, \dots \end{aligned} \quad \square$$

This does not mean that TM has a hyperbolic splitting as a whole. For any $(x_n) \in \Sigma_f$, $E_{x_0:(x_n)}^u$ depends on the whole orbit (x_n) while $E_{x_0:(x_n)}^s = E_{x_0}^s$ depends only on the initial point x_0 . The difference between two definitions is just the Mâné-Pugh's one uses the homeomorphisms of lifting systems and Przytycki's uses the homeomorphisms of inverse limit systems. A relation for them is the latter one implies the former one.

Proposition 2.23 *Definition 2.22 implies Definition 2.18.*

Proof Let f be an Anosov endomorphism as in Definition 2.22. From [35] one knows that for any $x \in M$, E_x^s is uniquely determined and E^s is a continuous subbundle. We need to check that $\bar{T}f$ is expanding on TM/E^s .

By [35], the angle between $E_{x_0}^s$ and $E_{x_0:(x_n)}^u$ continuously depends on $(x_n) \in \Sigma_f$. As Σ_f is compact, there exist $0 < \alpha < \beta < \pi$ such that

$$0 < \alpha \leq \angle E_{x_0}^s, E_{x_0:(x_n)}^u \leq \beta < \pi \quad \text{for all } (x_n) \in \Sigma_f.$$

As a result, there exist $0 < \rho_1 < \rho_2$ such that

$$\rho_1(\|v_s\| + \|v_u\|) \leq \|v_s + v_u\| \leq \rho_2(\|v_s\| + \|v_u\|) \quad (2.1)$$

for all $v_s \in E_{x_0}^s$, $v_u \in E_{x_0:(x_n)}^u$ and all $(x_n) \in \Sigma_f$.

Let now v^\perp be in $E_{x_0}^{s\perp}$ (the orthogonal complement of $E_{x_0}^s$). We have a decomposition, according to $T_{x_0}M = E_{x_0}^s \oplus E_{x_0}^u$,

$$v^\perp = v_s + v_u.$$

It then follows from $\langle v^\perp, v^\perp \rangle = \langle v_s + v_u, v^\perp \rangle = \langle v_u, v^\perp \rangle$ that

$$\|v^\perp\| = \|v_u\| \cos \angle v_u, v^\perp.$$

Hence there exist $0 < c_1 < c_2$ such that

$$c_1 \|v^\perp\| \leq \|v_u\| \leq c_2 \|v^\perp\|, \quad v^\perp \in E_{x_0}^{s\perp}. \quad (2.2)$$

From $0 = \langle v^\perp, v_s \rangle = \langle v_s, v_s \rangle + \langle v_u, v_s \rangle$, we have $\|v_s\| = -\|v_u\| \cos \angle v_s, v_u$. This, together with (2.2), shows that there exist $0 < c_3 < c_4$ such that

$$c_3 \|v^\perp\| \leq \|v_s\| \leq c_4 \|v^\perp\|, \quad v^\perp \in E_{x_0}^{s\perp}. \quad (2.3)$$

For any $n \geq 1$,

$$Tf^n \cdot v^\perp = Tf^n \cdot v_s + Tf^n \cdot v_u \in E_{x_n}^s + E_{x_n: \sigma_f^n((x_m))}^u.$$

As

$$\|Tf^n \cdot v_s\| \leq C\mu^n \|v_s\| \leq c_4 C\mu^n \|v^\perp\| \quad (2.4)$$

and

$$\|Tf^n \cdot v_u\| \geq C^{-1}\mu^{-n} \|v_u\| \geq c_3 C^{-1}\mu^{-n} \|v^\perp\|, \quad (2.5)$$

we have for $n \gg 1$ (independent of v^\perp !) that $\angle Tf^n \cdot v^\perp, E_{x_n}^s$ is close to $\angle E_{x_n: (x_m)}^u, E_{x_n}^s$. Hence there exist $N \gg 1$ and $C' > 0$ such that

$$\begin{aligned} \|Tf^n \cdot [v^\perp]\| &= \|Tf^n \cdot v^\perp\| \sin \angle Tf^n \cdot v^\perp, E_{x_n}^s \\ &\geq C' \|Tf^n \cdot v^\perp\|, \quad v^\perp \in E_{x_0}^{s\perp}, \quad n \geq N. \end{aligned} \quad (2.6)$$

Following inequalities (2.1) and (2.3)—(2.6), we have

$$\begin{aligned} \|Tf^n \cdot [v^\perp]\| &\geq C' \rho_1 (\|Tf^n \cdot v_s\| + \|Tf^n \cdot v_u\|) \\ &\geq C' \rho_1 c_3 C^{-1} \mu^{-n} \|v^\perp\| \\ &= C' \rho_1 c_3 C^{-1} \mu^{-n} \|[v^\perp]\| \end{aligned}$$

for all $v^\perp \in E_{x_0}^{s\perp}$ and $n \geq N$. This proves that $\bar{T}f$ is expanding on TM/E^s . \square

It is reasonable to think that those two definitions are coincident although we could not prove this at the moment, see Anosov [3]. As we need both the technique of liftings and the instability of Anosov endomorphisms, we are considering Anosov endomorphisms under Przytycki's definition.¹⁾

For Anosov endomorphisms, Przytycki has some analytic discussion. In fact he has studied in [35] hyperbolic invariant sets for maps. The following sharp results were proved by him.

Theorem 2.24 (Przytycki [35]) *Let $f \in \mathcal{A}^*(M)$. Then f is not structurally stable.* \square

Theorem 2.25 (Przytycki [35]) *Let $f \in \mathcal{A}^*(M)$. Then for any C^1 neighborhood \mathcal{U} of f , there exists some $g \in \mathcal{U}$ such that there is no any map $\varphi : M \rightarrow M$ which is onto and the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \varphi \downarrow & & \downarrow \varphi \\ M & \xrightarrow{g} & M \end{array}$$

In one sentence, f is not semi-structurally stable. \square

Since we use in many times some special Anosov endomorphisms which are generalization of above 'linear' ones, we give their definitions. Some general properties for them will be introduced later. See also Franks [15].

¹⁾ After we finish this dissertation, we found that these two definitions are actually coincident. For a proof of this, see K. Sakai, Anosov maps on closed topological manifolds, *J. Math. Soc. Japan*, **39** (1978), 505–519. As a result, we need not to distinguish these two definitions.

Example 2.26 (Hyperbolic Nilmanifold Endomorphism (HNM, for short)) Thom's examples on tori have two classes of generalizations. The first one is to noninvertible maps and the second one is to more general manifolds. Let G be a connected and simply connected Lie group. Suppose that G is nilpotent. Let Γ be a Lie subgroup of G . Then Γ acts on G and one has the orbit space G/Γ . Suppose that Γ is a uniform (i.e., G/Γ is compact) discrete (in topology) subgroup. Then G/Γ is a compact manifold of the same dimension with G . Such a manifold G/Γ is called a nilmanifold. If $A : G \rightarrow G$ is a Lie homomorphism such that $A(\Gamma) \subset \Gamma$, then A induces a C^∞ map a on $M = G/\Gamma$ by

$$a(\Gamma x) \stackrel{\text{def}}{=} \Gamma A(x), \quad x \in G.$$

If the tangent map $T_e A : T_e G =: g \rightarrow g$ is hyperbolic, then $a : M \rightarrow M$ is an Anosov endomorphism which is called a Hyperbolic Nilmanifold Endomorphism (HNE). In particular, if $A(\Gamma) = \Gamma$ then a is an Anosov diffeomorphism and a is called a Hyperbolic Nilmanifold Automorphism (HNA). If $T_e A : g \rightarrow g$ is expanding, then a is an expanding map and is called an Expanding Nilmanifold Endomorphism (ENE). \square

A more generalization of above examples is to infranilmanifolds.

Example 2.27 (Hyperbolic Infranilmanifold Endomorphism (HIE)) Let G be a connected and simply connected Lie group and $K \subset \text{Aut}(G)$ be a finite subgroup of automorphism group $\text{Aut}(G)$. One then can define the semi-direct product $K \cdot G$ as follows. Let $(k_i, g_i) \in K \cdot G$, where $k_i \in K$ and $g_i \in G$, $i = 1, 2$. The semi-direct product is defined by

$$(k_1, g_1) \cdot (k_2, g_2) \stackrel{\text{def}}{=} (k_1 k_2, g_1 k_2(g_2)).$$

Then $K \cdot G$ is also a Lie group. Let $\Gamma \subset K \cdot G$ be a uniform, discrete Lie subgroup. Then Γ can act on G in the following way:

$$(k, h) \cdot g \stackrel{\text{def}}{=} h k(g), \quad (k, h) \in \Gamma, \quad g \in G.$$

In this case the orbit space $M = G/\Gamma$ is a compact manifold which is called an infranilmanifold.

According to Auslander [6], if one embeds G into $K \cdot G$, then $\Gamma \cap G$ is a uniform subgroup of G and the index of $\Gamma \cap G$ in Γ is finite. Hence $M = G/\Gamma$ has a covering manifold $M' = G/\Gamma \cap G$ which is a nilmanifold.

Let now $A : K \cdot G \rightarrow K \cdot G$ be a Lie automorphism. Suppose that $A(G) \subset G$ and $A(\Gamma) \subset \Gamma$. Then A induces a C^∞ map $a : M = G/\Gamma \rightarrow M$. If $T_e A : T_e G =: g \rightarrow g$ is hyperbolic, then $a : M \rightarrow M$ is an Anosov endomorphism, which is called a Hyperbolic Infranilmanifold Endomorphism (HIE). \square

In this dissertation we use $\mathcal{H}_l(M)$ to denote the collection of all HIEs on an infranilmanifold M , while $\mathcal{H}_l^*(M) = \mathcal{H}_l(M) \cap \mathcal{A}^*(M)$ the collection of those HIEs which are neither hyperbolic infranilmanifold automorphisms nor expanding infranilmanifold endomorphisms.

As mentioned before, these linear ones are all known Anosov diffeomorphisms and expanding maps under topological conjugacy (see Theorems 2.13 and 2.16). However, the situation for $\mathcal{A}^*(M)$ is quite different. In fact we have applied the results of Przytycki to prove in [52] the following result.

Theorem 2.28 (Zhang [52]) *Let $n \geq 2$. Then there exists a residual subset \mathcal{U} in $\mathcal{A}^*(\mathbb{T}^n)$ such that any $f \in \mathcal{U}$ is topologically conjugate to any hyperbolic toral endomorphism. \square*

In Section 6 we will use the further instability result for systems in $\mathcal{A}^*(M)$ to discuss shift equivalence classes of $\mathcal{A}^*(M)$ and will show that the above theorem also holds for shift equivalence. On general infranilmanifolds, some weaker result of Theorem 2.28 will be given.

For the studying of other classes of noninvertible systems, one may refer to Coven-Reddy [12], Franke [24], Qian et al. [36] and Yang [50].

3 Metric Properties for Liftings of Coverings

Since we will study the problems using the universal covering spaces and many metric properties are exploited, we take certain pages in describing some properties of these. For the general knowledge on covering spaces, see Spanier [41].

As before, let M be a compact smooth connected Riemannian manifold. From algebraic topology, there is a universal covering space (\bar{M}, π) , where $\pi : \bar{M} \rightarrow M$ is the universal covering map. As M is a Riemannian manifold, one can endow \bar{M} with a Riemannian metric. In fact, let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on M with the corresponding metric d . Then on \bar{M} there is an induced Riemannian metric $\pi^*\langle \cdot, \cdot \rangle$ given by

$$\pi^*\langle u, v \rangle \stackrel{\text{def}}{=} \langle T\pi \cdot u, T\pi \cdot v \rangle, \quad u, v \in T\bar{M}.$$

Let \bar{d} be the induced metric on \bar{M} by $\pi^*\langle \cdot, \cdot \rangle$. We call $(\bar{M}, \pi^*\langle \cdot, \cdot \rangle)$ the universal covering manifold of M , see Boothby [7]. Moreover, there exists $\rho_0 > 0$ such that

$$\pi : B(\bar{x}, \delta) \rightarrow B(\pi\bar{x}, \delta) \quad \text{is an isometry} \quad (3.1)$$

for any $\bar{x} \in \bar{M}$, $\delta < \rho_0$. Here $B(\bar{x}, \delta)$ denotes the closed ball centred at \bar{x} with radius δ .

From algebraic topology, all covering transformations $\gamma : \bar{M} \rightarrow \bar{M}$, i.e., $\pi \circ \gamma = \pi$, constitute a group, for which we denote by $\Gamma = \Gamma(\bar{M}, \pi)$. Then Γ is isomorphic to the fundamental group $\pi_1(M)$ and any $\gamma \in \Gamma$ is an isometry on \bar{M} .

Let $f \in C^0(M)$. A lifting of f to \bar{M} is a map $F : \bar{M} \rightarrow \bar{M}$ such that $\pi \circ F = f \circ \pi$. In terminology of dynamical systems, F and f are semi-topologically conjugate:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{F} & \bar{M} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

Fix a lifting F of f . Then F yields a homomorphism ϕ on Γ by

$$F \circ \gamma = \phi(\gamma) \circ F, \quad \gamma \in \Gamma.$$

In fact such a homomorphism ϕ is just the induced homomorphism f_* of f on $\pi_1(M)$ if one identifies Γ with $\pi_1(M)$.

Proposition 3.1 *Let $F \in C^0(\bar{M})$. Then F is a lifting of some $f \in C^0(M)$ if and only if there exists a homomorphism $\phi : \Gamma \rightarrow \Gamma$ such that $F \circ \gamma = \phi(\gamma) \circ F$ for all $\gamma \in \Gamma$. \square*

Let $f \in C^0(M)$. As M is compact, f is uniformly continuous. Although \bar{M} may not be compact, we have

Proposition 3.2 *Let $F \in C^0(\bar{M})$ be a lifting of some $f \in C^0(M)$. Then F is uniformly continuous with respect to \bar{d} .*

Proof Take a fundamental domain $K \subset \bar{M}$, i.e., K is compact and $\pi(K) = M$. Then there exists $\beta > 0$ such that $K_\beta = \{y \in \bar{M} : \bar{d}(y, K) \leq \beta\}$ is compact. Thus there exists for any $\varepsilon > 0$ some $\delta = \delta(\varepsilon) > 0$ ($\delta < \beta$) such that

$$\bar{d}(F(x), F(y)) < \varepsilon, \quad x, y \in K_\beta, \quad \bar{d}(x, y) < \delta. \quad (3.2)$$

Let now $x, y \in \bar{M}$ be such that $\bar{d}(x, y) < \delta$. As K is a fundamental domain, there exists some $\gamma \in \Gamma$ such that $\gamma(x) \in K$. As γ is an isometry, $\bar{d}(\gamma(y), \gamma(x)) = \bar{d}(y, x) < \delta$. Thus $\gamma(x), \gamma(y) \in K_\beta$. It follows now from (3.2) that

$$\varepsilon > \bar{d}(F \circ \gamma(y), F \circ \gamma(x)) = \bar{d}(\phi(\gamma) \circ F(y), \phi(\gamma) \circ F(x)) = \bar{d}(F(y), F(x)).$$

This proves that F is uniformly continuous on \bar{M} . \square

If $f \in \mathcal{E}^0(M)$ is a local diffeomorphism, then f is a covering map of finite degree because M is compact. As a result any lifting F of f to \bar{M} is a diffeomorphism, see Shub [38] and Coven-Reddy [12].

Proposition 3.3 *Let $f \in \mathcal{E}^0(M)$. Then any lifting F of f to \bar{M} has the property that $F^{-1} : \bar{M} \rightarrow \bar{M}$ is uniformly continuous.*

Proof Let K, K_β be as above. As f is a covering map, $k = \text{card } f^{-1}(y) < \infty$ (independent of y). In this case, $f_*(\Gamma)$ is a subgroup of Γ of index k . Let $\{\gamma_1, \dots, \gamma_k\}$ be representatives of right cosets of $f_*(\Gamma)$ in Γ , i.e.,

$$\Gamma = \bigcup_{i=1}^k f_*(\Gamma) \cdot \gamma_i.$$

Let

$$T_\beta = \bigcup_{i=1}^k \gamma_i(K_\beta).$$

Then T_β is compact. Hence for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) (< \beta)$ such that

$$\bar{d}(F^{-1}(x), F^{-1}(y)) < \varepsilon, \quad x, y \in T_\beta, \quad \bar{d}(x, y) < \delta.$$

If $x, y \in \bar{M}$ with $\bar{d}(x, y) < \delta$, then there exists $\gamma \in \Gamma$ such that $\gamma(x) \in K, \gamma(y) \in K_\beta$. Since $\gamma^{-1} \in \Gamma$, there exists some $1 \leq i \leq k$ such that $\gamma^{-1} \in f_*(\Gamma) \cdot \gamma_i$, namely there exists $\gamma' \in \Gamma$ such that $\gamma^{-1} = f_*(\gamma')\gamma_i$. It follows from $F \circ \gamma' = f_*(\gamma') \circ F$ that $\gamma' \circ F^{-1} = F^{-1} \circ f_*(\gamma')$. Hence

$$\begin{aligned} \bar{d}(F^{-1}x, F^{-1}y) &= \bar{d}(F^{-1} \circ \gamma^{-1} \circ \gamma x, F^{-1} \circ \gamma^{-1} \circ \gamma y) \\ &= \bar{d}(F^{-1} \circ f_*(\gamma') \circ \gamma_i \circ \gamma x, F^{-1} \circ f_*(\gamma') \circ \gamma_i \circ \gamma y) \\ &= \bar{d}(\gamma' \circ F^{-1} \circ \gamma_i \circ \gamma x, \gamma' \circ F^{-1} \circ \gamma_i \circ \gamma y) \\ &= \bar{d}(F^{-1}(\gamma_i \circ \gamma x), F^{-1}(\gamma_i \circ \gamma y)). \end{aligned}$$

As $\gamma_i \circ \gamma x, \gamma_i \circ \gamma y \in T_\beta$ with $\bar{d}(\gamma_i \circ \gamma x, \gamma_i \circ \gamma y) < \delta$, one has $\bar{d}(F^{-1}(\gamma_i \circ \gamma x), F^{-1}(\gamma_i \circ \gamma y)) < \varepsilon$. \square

As we want to consider perturbations of systems, we give the following

Proposition 3.4 *Let $f, g \in C^0(M)$ be such that g is C^0 close to f , i.e., $d_0(f, g)$ is small. Let F and G be liftings of f and g such that $\bar{d}(F x_0, G x_0)$ is small for some $x_0 \in \bar{M}$. Then G is C^0 close to F . Furthermore, if f and g are local homeomorphisms, then G^{-1} is also C^0 close to F^{-1} .*

Proof Let K be a fundamental covering domain such that $x_0 \in K$. Since g is C^0 close to f , G and F induce the same homomorphism ϕ on Γ . Thus

$$\begin{aligned} \bar{d}_0(F, G) &= \sup\{\bar{d}(Fx, Gx) : x \in \bar{M}\} \\ &= \sup\{\bar{d}(F \circ \gamma x, G \circ \gamma x) : x \in K, \gamma \in \Gamma\} \\ &= \sup\{\bar{d}(\phi(\gamma) \circ Fx, \phi(\gamma) \circ Gx) : x \in K, \gamma \in \Gamma\} \\ &= \sup\{\bar{d}(Fx, Gx) : x \in K\} \\ &= \sup\{d(\pi \circ Fx, \pi \circ Gx) : x \in K\} \\ &= \sup\{d(f \circ \pi x, g \circ \pi x) : x \in K\} \\ &= \sup\{d(fy, gy) : y \in M\}, \end{aligned}$$

where (3.1) is used. Thus $\bar{d}_0(F, G)$ is small.

If $f, g \in \mathcal{E}^0(M)$, by the uniform continuity of F^{-1} , there exists for any $\varepsilon > 0$ some $\delta = \delta(\varepsilon) > 0$ such that

$$\bar{d}(Fx, Fy) < \varepsilon, \quad x, y \in \bar{M}, \quad \bar{d}(x, y) < \delta.$$

If g is sufficiently C^0 close to f , one has

$$\bar{d}(Fx, Gx) < \delta, \quad x \in \bar{M}.$$

Namely,

$$\bar{d}(F(G^{-1}x), x) < \delta, \quad x \in \bar{M}.$$

Thus

$$\bar{d}(G^{-1}x, F^{-1}x) = \bar{d}(F^{-1}(F(G^{-1}x)), F^{-1}x) < \varepsilon. \quad \square$$

We remark that F_n uniformly converges to F_0 does not imply that F_n^{-1} uniformly converges to F_0^{-1} , even when F_n and F_0 are uniformly continuous. For example, $F_n(x) = x^{1/3} + \frac{1}{n}$, $F_0(x) = x^{1/3}$, $x \in \mathbb{R}$.

By Proposition 3.4 and the definition of C^r topology, we have

Proposition 3.5 *Let $f, g \in C^r(M)$ ($r \geq 1$) and F, G be as above. Suppose that $\bar{d}(Fx_0, Gx_0)$ is small for some $x_0 \in \bar{M}$. Then G is C^r close to F . Furthermore, if $f, g \in \mathcal{E}^r(M)$, then G^{-1} is also C^r close to F^{-1} . \square*

This proposition means that once the base points are suitably chosen, the procedure of lifting $f \in C^r(M)$ to $F \in C^r(\bar{M})$ is continuously dependent on f .

4 A Topological Equivalence Theorem

In this section we will prove a topological conjugacy theorem for lifting systems of Anosov endomorphisms, which is similar to structural stability theorem for Anosov diffeomorphisms. Such a theorem will be used in Section 6. A corollary of this theorem will be given in this section.

Let M and (\bar{M}, π) be as in Section 3. Let $f \in \mathcal{A}(M)$ be an Anosov endomorphism and $F : \bar{M} \rightarrow \bar{M}$ be a lifting of f . By Proposition 2.19, F is an Anosov diffeomorphism on \bar{M} . It is well known that Anosov diffeomorphisms on compact manifolds are ε -structurally stable. Although $f \in \mathcal{A}(M)$ is not structurally stable, the lifting Anosov diffeomorphism F to universal covering manifold \bar{M} has some stability. More precisely, let $f, g \in \mathcal{A}(M)$ and F, G be their liftings. If G, G^{-1} are C^1 close to F, F^{-1} respectively, there will exist a homeomorphism $H : \bar{M} \rightarrow \bar{M}$ such that $H \circ F = G \circ H$, where H is C^0 close to $\text{id}_{\bar{M}}$ (Mäné-Pugh [26]). Since F and G are lifting maps, we have the following stronger result.

Theorem 4.1 *Let f, g, F, G be as above. Then the conjugating equation*

$$H \circ F = G \circ H \tag{4.1}$$

has a unique solution H in $C^0(\bar{M})$ with $\bar{d}_0(H, \text{id}_{\bar{M}})$ small. Moreover, H is a homeomorphism and H, H^{-1} are uniformly continuous. \square

The proof is similar to that for structural stability theorem of Anosov diffeomorphisms on compact manifolds (Smale [40] and Zhang [54]). We use the approach in Chapters 13 and 14 in [54].

Let $\exp : T\bar{M} \rightarrow \bar{M}$ be the exponential map. Since \bar{M} is a lifting manifold, there exists some constant $\rho > 0$ such that for any $x \in \bar{M}$

$$\exp_x : \{v \in T_x\bar{M} : \|v\| < \rho\} \rightarrow \bar{M}$$

is a diffeomorphism onto its image.

Suppose that (4.1) has a solution $H \in C^0(\bar{M})$ such that H is C^0 close to $\text{id}_{\bar{M}}$. Then

$$\gamma(x) = \exp_x^{-1} H(x)$$

is a continuous section of \bar{M} and satisfies

$$\gamma(x) = \exp_x^{-1} \circ G \circ \exp_{F^{-1}(x)} \gamma(F^{-1}(x)), \quad x \in \bar{M}. \tag{4.2}$$

We introduce a nonlinear operator on section space by

$$\mathcal{T}_G(\gamma) \stackrel{\text{def}}{=} \text{the right-hand side of (4.2)}.$$

Then (4.1) is the following fixed point equation

$$\gamma = \mathcal{T}_G(\gamma). \tag{4.3}$$

If $G = F$, equation (4.3) has a trivial fixed point $\gamma = \tilde{0}$ (zero section). As F is an Anosov diffeomorphism, one then can check that $\tilde{0}$ is a hyperbolic fixed point. When g is C^1 close to f , then G is also C^1 close to F (Proposition 3.5). Thus \mathcal{T}_G is C^1 close to \mathcal{T}_F . As a result,

\mathcal{T}_G has a (hyperbolic) fixed point $\gamma \sim \tilde{0}$ by the stability theorem for hyperbolic fixed points. In order to ensure that H is uniformly continuous, we are trying to find fixed point γ in the space of bounded, uniformly continuous sections.

As in [54], let $\tau_{\bar{M}} : T\bar{M} \rightarrow \bar{M}$ be the bundle projection. Let

$$\Gamma_{ub}^0(T\bar{M}) = \{\sigma : \bar{M} \rightarrow T\bar{M} : \sigma \text{ is uniformly continuous, } \sigma \text{ is a section} \\ (\tau_{\bar{M}} \circ \sigma = \text{id}_{\bar{M}}), \text{ and } \|\sigma\| := \sup_{x \in \bar{M}} \|\sigma(x)\| < \infty\}.$$

Then $\Gamma_{ub}^0(T\bar{M})$ with $\|\cdot\|$ is a Banach space.

In order to check hyperbolicity of the fixed point $\tilde{0}$ of \mathcal{T}_F , we need to compute the derivative which may be realized using the Palais Lemma in nonlinear analysis. In order to be applied to this case, we need a generalized version of Palais Lemma.

Let $u : \bar{M} \rightarrow \bar{M}$ be a diffeomorphism such that u, u^{-1} are uniformly continuous. Let $U : T\bar{M} \rightarrow T\bar{M}$ be a fibre-preserving map covering u , i.e.,

$$\begin{array}{ccc} T\bar{M} & \xrightarrow{U} & T\bar{M} \\ \tau_{\bar{M}} \downarrow & & \downarrow \tau_{\bar{M}} \\ \bar{M} & \xrightarrow{u} & \bar{M} \end{array}$$

Suppose further that U is uniformly continuous. Then U induces a map $\tilde{U} : \Gamma_{ub}^0(T\bar{M}) \rightarrow \Gamma_{ub}^0(T\bar{M})$ by

$$\tilde{U}(\sigma) = U \circ \sigma \circ u^{-1}, \quad \sigma \in \Gamma_{ub}^0(T\bar{M}). \quad (4.4)$$

Then one has the following

Lemma 4.2 (Palais Lemma) *Let \bar{M}, u, U, \tilde{U} be as above. Assume the derivative of U along fibres*

$$\tilde{D}U(\xi) \stackrel{\text{def}}{=} D\left(U|_{T_{\tau_{\bar{M}}(\xi)}\bar{M}}\right)(\xi)$$

is uniformly continuous. Then \tilde{U} is differentiable and its derivative is given by

$$(D\tilde{U}(\sigma) \cdot \tau)(x) = \tilde{D}U(\sigma(u^{-1}(x))) \cdot \tau(u^{-1}(x)), \quad x \in \bar{M}, \sigma, \tau \in \Gamma_{ub}^0(T\bar{M}). \quad (4.5)$$

Proof We refer to the proof of Theorem 2.2 in Chapter 13 of [54]. Note that although \bar{M} is not compact, the uniform continuity of \tilde{U} can help us overcome all difficulties caused by noncompactness. See also Remark 2.3 following that theorem in [54]. \square

Now we give the **proof of Theorem 4.1**. Let $B_\rho = \{\xi \in T\bar{M} : \|\xi\| < \rho\}$. Then B_ρ is an open subset of $T\bar{M}$. Define a map $U_G : B_\rho \rightarrow T\bar{M}$ by

$$U_G(\xi)(x) = \exp_{F(x)}^{-1} \circ G(\exp_x(\xi)) \quad x = \tau_{\bar{M}}(\xi), \xi \in B_\rho,$$

Then U_G is a fibre-preserving map which covers F . Since \exp is a local isometry, U_G satisfies the requirements of Lemma 4.2. It is easy to see that the map \mathcal{T}_G in (4.2) is just \tilde{U}_G . Thus

$$T_{\tilde{0}}\mathcal{T}_F \cdot \tau = TF \cdot \tau \cdot F^{-1}, \quad \tau \in \Gamma_{ub}^0(T\bar{M}).$$

Since F is an Anosov diffeomorphism, \mathcal{T}_F has $\tilde{0}$ as its hyperbolic fixed point. When g is C^1 close to f , \mathcal{T}_G has a unique fixed point $\gamma \in B_0 = \{\sigma \in \Gamma_{ub}^0(T\bar{M}) : \|\sigma\| < \rho\}$. Let now

$H(x) = \exp_x \gamma(x)$. Then H is uniformly continuous, and $\bar{d}_0(H, \text{id}_{\bar{M}}) = \|\gamma\| \ll 1$. Moreover H satisfies (4.1). In fact H is a homeomorphism which can be proved as in [54]. As for the uniform continuity of H^{-1} , our construction, together with the proof in [54], can guarantee this. \square

We remark that $H : \bar{M} \rightarrow \bar{M}$ to M cannot be in general projected onto M because f may be not structurally stable.

Now we give an application of Theorem 4.1. It is well known that for a uniform continuous map f from X to X one can define a numerical invariant — topological entropy $\text{ent}(f)$, which describes the complexity of iterations of f . This can be defined using spanning sets or separate sets, see Walters [44]. For a review of the studying of topological entropy, see Liu [23].

Let $f \in \mathcal{A}(M)$ be an Anosov endomorphism. Then topological entropy is preserved under small perturbations of f although f is in general not structurally stable.

Theorem 4.3 *Let $f \in \mathcal{A}(M)$. Then there exists a neighborhood \mathcal{U} of f in $C^1(M)$ such that $\text{ent}(g) = \text{ent}(f)$ for all $g \in \mathcal{U}$.*

Proof We sketch the proof as follows.

(1) Since $\pi : \bar{M} \rightarrow M$ is a local isometry, cf. (3.1), it follows from Theorem 8.12 in [44] that for $f \in C^0(M)$ and its lifting $F \in C^0(\bar{M})$ one has

$$\text{ent}_{\bar{d}}(F) = \text{ent}_d(f) (= \text{ent}(f)).$$

(2) Topological entropy is an invariant under uniform continuous topological conjugacy. More precisely, let f, g be uniformly continuous maps. If there exists a homeomorphism h such that $h \circ f = g \circ h$ and both h and h^{-1} is uniformly continuous, then $\text{ent}(f) = \text{ent}(g)$.

Now it follows from (1), (2) and Theorem 4.1 that

$$\text{ent}(g) = \text{ent}_d(g) \stackrel{(1)}{=} \text{ent}_{\bar{d}}(G) \stackrel{\text{Theorem 4.1 and (2)}}{=} \text{ent}_{\bar{d}}(F) \stackrel{(1)}{=} \text{ent}_d(f) = \text{ent}(f).$$

So the neighborhood \mathcal{U} can be chosen so that on which Theorem 4.1 holds. \square

We remark that the author of [50] proved a special case of Theorem 4.3 when f is a strong Anosov endomorphism using a different method.

5 Shift Equivalence

In this section we describe the role of shift equivalence in classification of hyperbolic attractors. Then we will study shift equivalence for certain Anosov endomorphisms. It is proved that topological conjugacy for general hyperbolic attractors cannot be in general reduced to shift equivalence.

5.1 Hyperbolic attractors and shift equivalence

Let us begin with an example. Consider the solid torus \mathbb{T} in \mathbb{R}^3 . We expand one direction along the torus and contracts other two directions of the torus. Then we put it into the torus in the way that the new one winds the original one two times. See Figure 1. In this way we obtain a diffeomorphism $f : \mathbb{T} \rightarrow \mathbb{T}$ (onto its image). Let

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(\mathbb{T}).$$

Then $f(\Lambda) = \Lambda$ and Λ is an attractor of f . Moreover f is hyperbolic on Λ . Such a construction is usually called a Solenoid.

Figure 1. Solenoid

Let $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the expanding map given by $g(z) = z^2$. Then there exists a homeomorphism $h : \Lambda \rightarrow \Sigma_g$ such that

$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ h \downarrow & & \downarrow h \\ \Sigma_g & \xrightarrow{\sigma_g} & \Sigma_g \end{array}$$

Namely the dynamics of the attractor $(\Lambda, f|_{\Lambda})$ can be described using the inverse limit system of g . This can be explained roughly as follows. View each vertical disc in \mathbb{T} (a stable manifold of f) as a point. Then \mathbb{T} is reduced into \mathbb{S}^1 , while $f|_{\Lambda}$ is the inverse limit system (Σ_g, σ_g) .

Since $\Sigma_g \cong \mathbb{S}^1 \times C$, where C is a Cantor set, the topological dimension of Λ is 1, which is coincident with the dimension of expanding directions of f . In this sense Λ is called an expanding attractor.

Williams [47, 48] systematically developed such an idea for more general hyperbolic attractors. Let M be a compact manifold and $f \in \mathcal{D}(M)$. A hyperbolic attractor Λ of f means a compact set such that Λ has an open neighborhood U in M such that

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(U),$$

and Λ is a hyperbolic invariant set of f .

Since Λ is hyperbolic, one has a stable foliation: For each $x \in \Lambda$, there corresponds to a stable manifold $W^s(x)$. W^s can be extended to a neighborhood of Λ . For some suitable closed neighborhood \mathcal{N} of Λ , we identify (\sim) each connected component of $\mathcal{N} \cap W^s(x)$ as a single point. Then \mathcal{N}/\sim is a manifold with branched points. See Figures 2 and 3.

Figure 2. Branched manifold

Figure 3. Branched manifold

In Figure 2, $\mathcal{N} \subset \mathbb{R}^2$. In Figure 3, $\mathcal{N} \subset \mathbb{R}^3$ with vertical lines as foliations.

In this way one obtains a branched manifold $K = \mathcal{N}/\sim$. Such a manifold has a unique tangent space at each point although there may be branched points. Now the map f is reduced to a map $g : K \rightarrow K$ by noticing the invariance of stable manifolds. Moreover, g has also some hyperbolicity.

Such a reduction can convert hyperbolic attractors Λ of higher dimensional systems $f \in \mathcal{D}(M)$ to the inverse limit systems (Σ_g, σ_g) of g 's on lower dimensional branched manifolds, where g 's have some hyperbolicity. Moreover, the topological conjugacy for attractors can be converted into easily-studied shift equivalence.

Theorem 5.1 (Williams [47]) *Let K be a one-dimensional branched manifold and g, g' be expanding maps on K . Then $g \overset{o}{\sim} g' \iff g \overset{s}{\sim} g'$. \square*

That is the reason for Williams to introduce the concept of shift equivalence. He studied in [48] general expanding attractors and gave in [49] a geometric explanation to the Lorenz attractor. For some related results on expanding attractors, see also Bothe [8], Farrell-Jones [13], Jones [18] and Plykin [34].

Recently, L. Wen [45] proved the following result on hyperbolic maps on two-dimensional manifolds. Let $g : K \rightarrow K$ be an Anosov endomorphism on branched surface K . Then there exists some $f \in \mathcal{A}(\mathbb{T}^2)$ such that $g \overset{s}{\sim} f$ (under some additional assumption on K).

These results lead us to consider shift equivalence for Anosov endomorphisms.

5.2 Shift equivalence for Anosov endomorphisms

Let $f \in C^0(X)$ and $g \in C^0(Y)$. Then f and g are shift equivalent if there exist maps $r \in C^0(X, Y)$, $s \in C^0(Y, X)$ and an integer $m \geq 0$ such that the following shift equivalence equations are satisfied:

$$r \circ f = g \circ r, \quad f \circ s = s \circ g, \quad s \circ r = f^m, \quad r \circ s = g^m. \quad (\text{SE})$$

In most case we will encounter maps $f, g \in C^0(M)$. We will consider lifting equations of (SE) to the universal covering manifold \bar{M} . Now we give a standard lifting procedure. Suppose that f has a fixed point x_0 . Let $y_0 = r(x_0)$. Then

$$\begin{aligned} g(y_0) &= g \circ r(x_0) = r \circ f(x_0) = r(x_0) = y_0. \\ s(y_0) &= s \circ r(x_0) = f^m(x_0) = x_0. \end{aligned}$$

Therefore, for any $\bar{x}_0 \in \pi^{-1}(x_0)$ and any $\bar{y}_0 \in \pi^{-1}(y_0)$, f, g, r, s have lifting maps F, G, R, S such that

$$F(\bar{x}_0) = \bar{x}_0, \quad G(\bar{y}_0) = \bar{y}_0, \quad R(\bar{x}_0) = \bar{y}_0, \quad S(\bar{y}_0) = \bar{x}_0.$$

It follows from (SE) that F, G, R, S and m satisfy the following lifting shift equivalence equations:

$$R \circ F = G \circ R, \quad F \circ S = S \circ G, \quad S \circ R = F^m, \quad R \circ S = G^m. \quad (\text{LSE})$$

When we encounter lifting shift equivalence equations, we always mean the equations (LSE) obtained from the above procedure.

From the proof of Proposition 2.16 we know that the maps R_i, S_j in the conclusion $f \stackrel{s}{\sim} g \implies f \stackrel{o}{\sim} g$ have some special structure. This shows that the converse cannot hold in general. We do prove this using Anosov endomorphisms in the following result.

Theorem 5.2 *Let $a : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a hyperbolic toral endomorphism and f be C^1 close to a . Then $f \stackrel{s}{\sim} a \iff f \stackrel{t}{\sim} a$.*

Proof By Proposition 2.16 we need only to prove that $f \stackrel{s}{\sim} a \implies f \stackrel{t}{\sim} a$. Assume that there exist $r, s \in C^0(\mathbb{T}^n)$ and integer $m \geq 0$ such that

$$r \circ f = a \circ r, \quad f \circ s = s \circ a, \quad s \circ r = f^m, \quad r \circ s = a^m. \quad (5.1)$$

Generally speaking, the maps r, s are not homeomorphisms. However we will construct from r a topological conjugacy between f and a .

Since $a([0]) = [0]$, we can use the above procedure to lift (5.1) to \mathbb{R}^n and obtain maps F, A, R, S such that

$$R \circ F = A \circ R, \quad F \circ S = S \circ A, \quad S \circ R = F^m, \quad R \circ S = A^m. \quad (5.2)$$

Here $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear hyperbolic automorphism of \mathbb{R}^n .

The lifting F may not be C^1 close to A because of the choice of bases. However there exists some $k \in \mathbb{Z}^n$ such that $F - k$ is C^1 close to A because f is C^1 close to a . Set $F = A + \varphi + k$. Then $\varphi : \mathbb{R}^n \rightarrow \varphi$ satisfies that $\|\varphi\|_{C^1}$ is small.

As usual, let us introduce the following spaces of maps.

$$\begin{aligned} C_b^0(\mathbb{R}^n) &= \{\xi : \mathbb{R}^n \rightarrow \mathbb{R}^n : \xi \text{ is a bounded continuous map}\}, \\ \mathcal{P}^0(\mathbb{R}^n) &= \{\xi \in C_b^0(\mathbb{R}^n) : \xi \text{ satisfies } \xi(x + \ell) = \xi(x) \text{ for all } x \in \mathbb{R}^n, \ell \in \mathbb{Z}^n\}. \end{aligned}$$

On the space $C_b^0(\mathbb{R}^n)$ the norm $\|\cdot\|_0$ is defined as follows:

$$\|\xi\|_0 = \sup\{\|\xi(x)\| : x \in \mathbb{R}^n\},$$

where $\|\cdot\|$ is a norm on \mathbb{R}^n . Then $C_b^0(\mathbb{R}^n)$ is a Banach space, while $\mathcal{P}^0(\mathbb{R}^n)$ is a closed subspace.

Consider the following functional equation

$$(\text{id} + \sigma) \circ A = (A + \varphi + k) \circ (\text{id} + \sigma). \quad (5.3)$$

It follows from the following Lemma 5.3 that equation (5.3) has a unique solution $\sigma = \sigma_0$ in $C_b^0(\mathbb{R}^n)$ which also guarantees that $\text{id} + \sigma_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism.

By the first equation in (5.2) and equation (5.3), we have

$$[R \circ (\text{id} + \sigma_0)] \circ A = A \circ [R \circ (\text{id} + \sigma_0)]. \quad (5.4)$$

Let $R = R_0 + \xi_0$, where $R_0 = r_*$ is an integral matrix and $\xi_0 \in \mathcal{P}^0(\mathbb{R}^n)$. It follows from (5.2) that R_0 and $S_0 = s_*$ satisfy the linear shift equivalence equations:

$$R_0 \circ A = A \circ R_0, \quad A \circ S_0 = S_0 \circ A, \quad S_0 \circ R_0 = A^m, \quad R_0 \circ S_0 = A^m. \quad (5.5)$$

By (5.4), $\xi := R \circ (\text{id} + \sigma_0) - R_0 = R_0 \circ \sigma_0 + \xi_0 \in C_b^0(\mathbb{R}^n)$ satisfies

$$(R_0 + \xi) \circ A = A \circ (\text{id} + \xi). \quad (5.6)$$

As $R_0 \circ A = A \circ R_0$ and A is hyperbolic, it is easy to see that (5.6) has a unique solution $\xi = 0$ in $C_b^0(\mathbb{R}^n)$. Thus we have

$$R \circ (\text{id} + \sigma_0) = R_0.$$

Thus

$$(\text{id} + \sigma_0)^{-1} = R_0^{-1} \circ R = \text{id} + R_0^{-1} \circ \xi_0 \in \text{id} + \mathcal{P}^0(\mathbb{R}^n),$$

because $R_0 \circ S_0 = A^m$ implies that R_0 is invertible in \mathbb{R} although it may not be invertible in \mathbb{Z} . Hence $\text{id} + \sigma_0 \in \text{id} + \mathcal{P}^0(\mathbb{R}^n)$, i.e., $\sigma_0 \in \mathcal{P}^0(\mathbb{R}^n)$. Thus the projection of $\text{id} + \sigma_0$ onto \mathbb{T}^n yields a homeomorphism h_0 . Projecting (5.3) onto \mathbb{T}^n , we have $h_0 \circ a = f \circ h_0$. Thus $f \stackrel{t}{\sim} a$. \square

Now we give Lemma 5.3 which is used in above proof.

Lemma 5.3 *Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a hyperbolic automorphism. Let $\varphi, \psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C^1 maps such that $\|\varphi\|_{C^1}, \|\psi\|_{C^1}$ are small. Then for any $j, k \in \mathbb{Z}^n$, the following functional equation*

$$(\text{id} + \sigma) \circ (A + \varphi + j) = (A + \psi + k) \circ (\text{id} + \sigma) \quad (5.7)$$

has a unique solution $\sigma = \sigma_0$ in $C_b^0(\mathbb{R}^n)$. Moreover, $\text{id} + \sigma_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism.

Proof Note that when $\varphi = \psi = 0$, equation (5.7) has a unique (constant) solution $\sigma = \sigma_0 = (I - A)^{-1}(k - j)$.

The proof of this lemma is similar to that for the Hartman-Grobman linearization theorem. The only difference is that we have two integral parameters $j, k \in \mathbb{Z}^n$ in (5.7). It is not difficult to see that the technique for the Hartman-Grobman theorem also applies to this case. We will not give the detail and refer the reader to Chapter 8 of Zhang [54] or Chapter 2 of Palis-de Melo [31]. Some similar ingredient can also be found in the next section. \square

We remark here that some results on shift equivalence for other classes of systems can be found in Williams [47, 48], Arteaga [5] and Parry [33].

By Theorem 5.2, we have

Theorem 5.4 *For general Anosov endomorphisms, inverse limit equivalence does not imply shift equivalence.*

Proof We take an $a \in \mathcal{H}_i^*(\mathbb{T}^n) = \mathcal{H}_l(\mathbb{T}^n) \cap \mathcal{A}^*(\mathbb{T}^n)$ ($n \geq 2$). Then a is o -stable by Theorem 2.21. If we assume that inverse limit equivalence would imply shift equivalence for all Anosov endomorphisms, then a is s -stable. By Theorem 5.2, this implies that a is t -stable, which contradicts Theorem 2.24. \square

As a corollary of Theorem 5.4, we have

Corollary 5.5 *Let $a \in \mathcal{H}_i^*(\mathbb{T}^n)$ ($n \geq 2$) and \mathcal{U} be a neighborhood of a in $C^1(\mathbb{T}^n)$. Then there must exist some $f \in \mathcal{U}$ such that $f \stackrel{s}{\not\sim} a$.* \square

6 Shift Equivalence Classes

In this section we discuss shift equivalence classes of Anosov endomorphisms. Since systems in $\mathcal{A}^*(M)$ lose st -stability, there would lead to the existence of ‘nonlinear’ Anosov endomorphisms under shift equivalence. We will prove that there will exist ‘many’ such nonlinear Anosov endomorphisms. On infranilmanifolds, such systems constitute at least a dense subset of $\mathcal{A}^*(M)$; and on tori, these systems constitute a residual subset of $\mathcal{A}^*(\mathbb{T}^n)$.

6.1 Shift equivalence classes of Anosov endomorphisms

We continue the discussion in last section. Let $a \in \mathcal{H}_l^*(\mathbb{T}^n)$, where $n \geq 2$. By Theorems 5.2 and 5.4, for any neighborhood \mathcal{U}_a of a in $C^1(\mathbb{R}^n)$, there must exist some $f_a \in \mathcal{U}_a \cap \mathcal{A}^*(\mathbb{T}^n)$ such that f_a is not shift equivalent to a . Furthermore we have

Lemma 6.1 *If \mathcal{U}_a is sufficiently small, the above f_a is not shift equivalent to any hyperbolic toral endomorphism.*

Proof Let \mathcal{U}_a be sufficiently small such that $\mathcal{U}_a \subset \mathcal{A}^*(\mathbb{T}^n)$ and

$$f_* = a_*, \quad f \in \mathcal{U}_a, \quad (6.1)$$

where f_* is the induced homomorphism of f in $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$. Assume that there exists some $a' \in \mathcal{H}_l(\mathbb{T}^n)$ such that $f_a \stackrel{s}{\sim} a'$, i.e., there exist $r, s \in C^0(\mathbb{T}^n)$ and $m \geq 0$ such that

$$f_a \circ r = r \circ a', \quad s \circ f_a = a' \circ s, \quad s \circ r = (a')^m, \quad r \circ s = f_a^m. \quad (6.2)$$

By (6.1) we can obtain from (6.2) that

$$a_* \circ r_* = r_* \circ a'_*, \quad s_* \circ a_* = a'_* \circ s_*, \quad s_* \circ r_* = (a'_*)^m, \quad r_* \circ s_* = a_*^m. \quad (6.3)$$

If we view r_*, s_* as maps on \mathbb{T}^n , we get from (6.3) that $a \stackrel{s}{\sim} a'$. As a result,

$$f_a \stackrel{s}{\sim} a' \stackrel{s}{\sim} a.$$

This contradicts the choice of f_a . □

A similar argument shows that

Lemma 6.2 *Let $a, a' \in \mathcal{H}_l^*(\mathbb{T}^n)$. If $a \not\stackrel{s}{\sim} a'$, then $f_a \not\stackrel{s}{\sim} f_{a'}$.* □

Thus we have

Theorem 6.3 *Let $n \geq 2$. Then there exists a sequence $\{f_i\}_{i=1}^\infty$ in $\mathcal{A}^*(\mathbb{T}^n)$ such that all f_i are mutually not shift equivalent and all f_i 's are not shift equivalent to any hyperbolic toral endomorphism. As a result, besides the linear ones in $\mathcal{A}^*(\mathbb{T}^n)$, there are infinitely many shift equivalence classes in $\mathcal{A}^*(\mathbb{T}^n)$.*

Proof By Theorem 4.8 of Williams [47], we know that if $a, a' \in \mathcal{H}_l^*(\mathbb{T}^n)$ satisfy $a \stackrel{s}{\sim} a'$, then a and a' have the same eigenvalues. Using this idea it is not difficult to construct a

sequence $\{a_i\} \subset \mathcal{H}_l^*(\mathbb{T}^n)$ such that all a_i 's have different eigenvalues so that $a_i \stackrel{s}{\not\sim} a_j$ if $i \neq j$. Now the corresponding f_{a_i} 's are the required Anosov endomorphisms. \square

From another view of point, there will be many nonlinear Anosov endomorphisms in $\mathcal{A}^*(\mathbb{T}^n)$ under shift equivalence. To this end, we notice that $\mathcal{A}^*(\mathbb{T}^n)$, $\mathcal{A}(\mathbb{T}^n)$ are open subsets of $C^1(\mathbb{T}^n)$.

Proposition 6.4 *Let $\mathcal{U} \subset \mathcal{A}(\mathbb{T}^n)$ be an open subset such that for any $f \in \mathcal{U}$ there exists $a = a_f \in \mathcal{H}_l(\mathbb{T}^n)$ with $f \stackrel{s}{\sim} a$. Then any $f \in \mathcal{U}$ is s -stable.*

Proof Fix an $f_1 \in \mathcal{U}$. Let $f_2 \in C^1(\mathbb{T}^n)$ be sufficiently C^1 close to f_1 . One then can assume that $f_2 \in \mathcal{U}$. By the assumption on \mathcal{U} , there exist $a_i \in \mathcal{H}_l(\mathbb{T}^n)$ such that

$$f_i \stackrel{s}{\sim} a_i, \quad i = 1, 2. \quad (6.4)_i$$

Namely, there exist $r_i, s_i \in C^0(\mathbb{T}^n)$ and $m_i \geq 0$ such that

$$r_i \circ f_i = a_i \circ r_i, \quad f_i \circ s_i = s_i \circ a_i, \quad s_i \circ r_i = f_i^{m_i}, \quad r_i \circ s_i = a_i^{m_i}.$$

As $a_i([0]) = [0]$, we use the standard procedure to obtain liftings A_i, F_i, R_i, S_i such that

$$R_i \circ F_i = A_i \circ R_i, \quad F_i \circ S_i = S_i \circ A_i, \quad S_i \circ R_i = F_i^{m_i}, \quad R_i \circ S_i = A_i^{m_i}. \quad (6.5)$$

As F_i, R_i, S_i are lifting maps, there exist $B_i, C_i, D_i \in M_n(\mathbb{Z})$ (integral matrixes) such that $F_i - B_i, R_i - C_i, S_i - D_i \in \mathcal{P}^0(\mathbb{R}^n)$, $i = 1, 2$. By (6.5) we have linear shift equivalence equations:

$$C_i \circ B_i = A_i \circ C_i, \quad B_i \circ D_i = D_i \circ A_i, \quad D_i \circ C_i = B_i^{m_i}, \quad C_i \circ D_i = A_i^{m_i}. (i = 1, 2) \quad (6.6)_i$$

Since B_i, C_i, D_i induce maps b_i, c_i, d_i on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, it follows from (6.6)_i that $b_i \stackrel{s}{\sim} a_i$, $i = 1, 2$.

As f_2 is C^1 close to f_1 and $\pi_1(\mathbb{T}^n)$ is commutative, we have $B_2 = B_1$ because f_2 is close to f_1 . Thus $b_2 = b_1$. Therefore

$$f_2 \stackrel{s}{\sim} a_2 \stackrel{s}{\sim} b_2 = b_1 \stackrel{s}{\sim} a_1 \stackrel{s}{\sim} f_1.$$

This yields that $f_2 \stackrel{s}{\sim} f_1$ and f_1 is s -stable. \square

Remark 6.5 One can generalize above proposition to infranilmanifolds. For example, let $M = G/\Gamma$ be a nilmanifold. We need only to modify the above proof a little. In fact $\pi_1(M) \cong \Gamma$ (left multiplication). Then F_i 's induce homomorphisms B_i 's:

$$F_i \circ \gamma = B_i(\gamma) \circ F_i, \quad i = 1, 2.$$

Consider B_i as maps from Γ to itself. Then B_i 's have the following extensions $\bar{B}_i : G \rightarrow G$:

$$\bar{B}_i(x) = F_i(x) \cdot F_i(e)^{-1}, \quad x \in G, \quad i = 1, 2,$$

where e is the unit element of G and $F_i(e)^{-1}$ means the inverse in G .

Then $\bar{B}_i|_\Gamma = B_i$ and \bar{B}_i 's satisfy

$$\begin{aligned}\bar{B}_i(\gamma x) &= F_i(\gamma x) \cdot F_i(e)^{-1} = B_i(\gamma) \circ F_i(x) \cdot F_i(e)^{-1} \\ &= B_i(\gamma) \circ \bar{B}_i(x), \quad \gamma \in \Gamma, x \in G.\end{aligned}$$

Hence \bar{B}_i 's induce maps \bar{b}_i 's on $G/\Gamma = M$. In this case $b_2 = b_1$ may not hold.

Since f_2 is C^1 close to f_1 , there exists some $\gamma_0 \in \Gamma$ such that $\gamma_0 \circ F_2$ is close to F_1 . Hence $\gamma_0 \circ F_2$ and F_1 induce the same homomorphism on Γ . Thus

$$\begin{aligned}(\gamma_0 \circ F_2)(\gamma x) &= \gamma_0 \circ F_2(\gamma x) = (\gamma_0 B_2(\gamma)) \circ F_2(x) \\ &= (\gamma_0 B_2(\gamma) \gamma_0^{-1})(\gamma_0 \circ F_2)(x), \quad \gamma \in \Gamma, x \in G.\end{aligned}$$

Hence $B_1(\gamma) = \text{Ad}_{\gamma_0} \circ \bar{B}_2(\gamma)$ for all $\gamma \in \Gamma$, where $\text{Ad}_{\gamma_0}(g) = \gamma_0 g \gamma_0^{-1}$ is the self-automorphism.

Let $H_0 : G \rightarrow G$ be the following map:

$$H_0(x) = \gamma_0 x, \quad x \in G.$$

Then $H_0 \circ \gamma x = \text{Ad}_{\gamma_0}(\gamma) \circ H_0(x)$ and H_0 induces a homeomorphism h_0 on $M = G/\Gamma$.

As $\bar{B}_1 = H_0 \circ \bar{B}_2 \circ H_0^{-1}$, we know that $\bar{b}_1 = h_0 \circ \bar{b}_2 \circ h_0^{-1}$. As a result, $\bar{b}_1 \stackrel{t}{\sim} \bar{b}_2$. Now we have

$$f_2 \stackrel{s}{\sim} a_2 \stackrel{s}{\sim} \bar{b}_2 \stackrel{t}{\sim} \bar{b}_1 \stackrel{s}{\sim} a_1 \stackrel{s}{\sim} f_1.$$

On an infranilmanifold M , we need only to notice that M has a nilmanifold $G/G \cap \Gamma$ as its covering manifold. \square

From shift equivalence equations for Anosov endomorphisms f, g such as $r \circ s = f^m$ we know that r, s are local homeomorphisms onto M because f, g are local diffeomorphisms. Thus $f \stackrel{s}{\sim} g \implies f \stackrel{st}{\sim} g$. As any $f \in \mathcal{A}^*(M)$ is not semi-structurally stable (Theorem 2.25), we obtain from Proposition 6.4 the following result.

Theorem 6.6 *Let M be an infranilmanifold. Then there exists a dense subset \mathcal{U} in $\mathcal{A}^*(M)$ such that any $f \in \mathcal{U}$ is not shift equivalent to any hyperbolic endomorphism on M . \square*

6.2 A genericity theorem

In the remaining part of this section, we will prove that Theorem 6.6 has a stronger version when $M = \mathbb{T}^n$ is a torus.

Theorem 6.7 *Let $n \geq 2$. Then there exists a residual subset \mathcal{U} in $\mathcal{A}^*(\mathbb{T}^n)$ such that any $f \in \mathcal{U}$ is not shift equivalent to any hyperbolic toral endomorphism. \square*

By a residual set \mathcal{U} we mean that \mathcal{U} is the intersection of countable open dense subsets.

The proof of this theorem need several pages. We will insert several lemmas during the whole proof. The construction of open dense subsets is as in [52]. To this end, we need some notation.

Let

$$\begin{aligned} S &= \{f \in \mathcal{A}^*(\mathbb{T}^n) : f \text{ is not shift equivalent to any } a \in \mathcal{H}_l(\mathbb{T}^n)\}, \\ S' &= \{f \in \mathcal{A}^*(\mathbb{T}^n) : f \text{ is shift equivalent to some } a \in \mathcal{H}_l(\mathbb{T}^n)\}. \end{aligned}$$

Now we give the decomposition in several steps.

Step 1. As hyperbolic toral endomorphisms on \mathbb{T}^n are countable, let $\mathcal{H}_l(\mathbb{T}^n) = \{a_1, a_2, \dots\}$.

Let

$$Q_i = \{f \in \mathcal{A}^*(\mathbb{T}^n) : f \stackrel{s}{\sim} a_i\}.$$

Then

$$S' = \bigcup_{i=1}^{\infty} Q_i.$$

Generally speaking, Q_i are not closed.

Step 2. For any fixed $m \geq 0$, let

$$Q_{im} = \{f \in Q_i : \text{there exist } r, s \in C^0(\mathbb{T}^n) \text{ such that } r, s, m \text{ realize the shift equivalence between } f \text{ and } a_i\}.$$

Then

$$S' = \bigcup_{i=1}^{\infty} \bigcup_{m=0}^{\infty} Q_{im}.$$

Step 3. We need to specify maps r, s in the shift equivalence between f and a_i . Since all integral matrixes are countable, let $\{B_1, B_2, \dots\}$ be all integral matrixes in $M_n(\mathbb{Z})$ with $\det(B_j) \neq 0$. Let

$$Q_{imlj} = \{f \in Q_{im} : \text{there exist } r, s \in C^0(\mathbb{T}^n) \text{ such that } r_* = B_l, s_* = B_j \text{ and } r, s, m \text{ realize the shift equivalence between } f \text{ and } a_i\}.$$

Then

$$S' = \bigcup_{i=1}^{\infty} \bigcup_{m=0}^{\infty} \bigcup_{l=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{imlj},$$

or

$$S = \bigcap_{i=1}^{\infty} \bigcap_{m=0}^{\infty} \bigcap_{l=1}^{\infty} \bigcap_{j=1}^{\infty} (\mathcal{A}^*(\mathbb{T}^n) \setminus Q_{imlj}). \quad (6.7)$$

We have the following important conclusion.

Proposition 6.8 *Each Q_{imlj} is closed in $\mathcal{A}^*(\mathbb{T}^n)$ (with C^1 topology).*

Once this is proved, one can obtain Theorem 6.7 as follows. By Theorem 6.6, S is dense in $\mathcal{A}^*(\mathbb{T}^n)$ and then so does each $\mathcal{A}^*(\mathbb{T}^n) \setminus Q_{imlj}$, which is also open by Proposition 6.8. Now (6.7) fulfills Theorem 6.7.

The main task of the remaining part of this section is to prove Proposition 6.8. Let us first sketch the main idea for the proof. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence in Q_{imlj} such that f_k tends to some f_0 in $\mathcal{A}^*(\mathbb{T}^n)$, where i, m, l, j are fixed. We want to prove that $f_0 \in Q_{imlj}$. The maps r_0, s_0 for shift equivalence between f_0 and a_i will be constructed from

the limit of those r_k , s_k for shift equivalence between f_k and a_i . The lifting of the equations for r_k to \mathbb{R}^n yields a sequence of linear operators \mathcal{T}_k , $k \geq 0$. The difficulty is that the linear operators \mathcal{T}_k do not converge to \mathcal{T}_0 in the space $C_b^0(\mathbb{R}^n)$. However, the main idea is that \mathcal{T}_k , restricted on some subspace of $C_b^0(\mathbb{R}^n)$, do converge weakly to \mathcal{T}_0 . Such a weak convergence does fulfill the construction of r_0 and s_0 .

We introduce the following subspace of $C_b^0(\mathbb{R}^n)$

$$C_{ub}^0(\mathbb{R}^n) = \{\xi : \mathbb{R}^n \rightarrow \mathbb{R}^n : \xi \text{ is uniformly continuous and is bounded}\}.$$

Then we have

Lemma 6.9 $C_{ub}^0(\mathbb{R}^n)$ is a closed subspace of $C_b^0(\mathbb{R}^n)$ and therefore is a Banach space. Moreover, $\mathcal{P}^0(\mathbb{R}^n) \subset C_{ub}^0(\mathbb{R}^n) \subset C_b^0(\mathbb{R}^n)$. \square

We take for $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the fundamental domain $K = \{(x_1, \dots, x_n) : 0 \leq x_i \leq 1, 1 \leq i \leq n\}$. Then $\pi(K) = \mathbb{T}^n$. Now we begin to prove Proposition 6.8. For convenience, we drop indexes. Let $a \in \mathcal{H}_l(\mathbb{R}^n)$, $m \geq 0$ and non-singular (in \mathbb{R}) integral matrixes B_1 , B_2 be fixed. Set

$$Q = \{f \in \mathcal{A}^*(\mathbb{T}^n) : \text{there exist } r, s \in C^0(\mathbb{T}^n) \text{ such that } r_* = B_1, s_* = B_2 \\ \text{and } r, s, m \text{ realize the shift equivalence between } f \text{ and } a\}.$$

Let $\{f_k\}_{k=1}^\infty \subset Q$ such that $f_k \rightarrow f_0$ for some $f_0 \in \mathcal{A}^*(\mathbb{T}^n)$ (in the C^1 topology). By the definition of Q , for any $k \geq 1$, there exist $r_k, s_k \in C^0(\mathbb{T}^n)$ such that $r_{k*} = B_1, s_{k*} = B_2$ and

$$r_k \circ f_k = a \circ r_k, \quad f_k \circ s_k = s_k \circ a, \quad s_k \circ r_k = f_k^m, \quad r_k \circ s_k = a^m. \quad (k = 1, 2, \dots) \quad (6.8)_k$$

Let $x_k \in K$ such that $[x_k] = r_k([0])$, $k = 1, 2, \dots$ We use the standard lifting procedure in the last section to obtain lifting maps F_k, A, R_k, S_k such that

$$F_k(x_k) = x_k, \quad R_k(0) = x_k, \quad S_k(x_k) = 0, \quad (k = 1, 2, \dots) \quad (6.9)_k$$

and

$$R_k \circ F_k = A \circ R_k, \quad F_k \circ S_k = S_k \circ A, \quad S_k \circ R_k = F_k^m, \quad R_k \circ S_k = A^m. \quad (k = 1, 2, \dots) \quad (6.10)_k$$

As K is compact one may assume without loss of generality that $x_k \rightarrow x_0 \in K$. Therefore, it follows from Proposition 3.5 that if F_0 is the lifting of f_0 such that $F_0(x_0) = x_0$, by noticing that $f_k([x_k]) = [x_k]$ for all $k \geq 1$ imply that $f_0([x_0]) = [x_0]$, then F_k, F_k^{-1} are convergent to F_0, F_0^{-1} in the C^1 topology.

Let $B = f_{0*} \in M_n(\mathbb{Z})$. Then one may assume that $f_{k*} = B$ for all $k \geq 1$. Hence we have

$$F_k = B + \varphi_k,$$

where $\varphi_k \in \mathcal{P}^1(\mathbb{R}^n) := \mathcal{P}^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, $k = 1, 2, \dots$

Consider the following functional equations

$$(B_1 + \xi) \circ (B + \varphi_k) = A \circ (B_1 + \xi), \quad k = 0, 1, 2, \dots \quad (6.11)_k$$

For any $k \geq 1$, R_k satisfies (6.10) $_k$. Since $r_{k*} = B_1$ for $k \geq 1$, we know that $\xi := R_k - B_1 \in \mathcal{P}^0(\mathbb{R}^n)$ satisfies (6.11) $_k$ for any $k \geq 1$. By (6.10) $_k$ we know that B , A , B_1 , B_2 satisfy the linear shift equivalence equations. In particular, we have $B_1 \circ B = A \circ B_1$. Hence (6.11) $_k$ is equivalent to

$$A \circ \xi - \xi \circ (B + \varphi_k) = B_1 \circ \varphi_k, \quad k = 0, 1, 2, \dots \quad (6.12)_k$$

Denote the left-hand side of (6.12) $_k$ by $\mathcal{L}_k(\xi)$. Then \mathcal{L}_k is a linear operator for each $k \geq 0$. Now (6.12) $_k$ becomes

$$\xi = \mathcal{L}_k^{-1}(B_1 \circ \varphi_k).$$

As it is not easy to analyze \mathcal{L}_k^{-1} , we reformulate (6.12) $_k$.

The following construction is similar to that for the proof of Hartman-Grobman theorem. As $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a hyperbolic automorphism, there exists a decomposition $\mathbb{R}^n = E^s \oplus E^u$ such that

- (i) $A(E^s) = E^s$ and $A(E^u) = E^u$; and
- (ii) there exist a norm $|\cdot|$ on \mathbb{R}^n and a constant $0 < \tau < 1$ such that

$$|Av| \leq \tau|v|, \quad v \in E^s; \quad |Av| \geq \tau^{-1}|v|, \quad v \in E^u.$$

Let $A^s = A|_{E^s}$ and $A^u = A|_{E^u}$. Let P^s and P^u be projections of $\mathbb{R}^n = E^s \oplus E^u$ onto E^s and E^u respectively. Correspondingly, we have decomposition:

$$C_{ub}^0(\mathbb{R}^n) = C_{ub}^s \oplus C_{ub}^u,$$

where

$$C_{ub}^{s(u)} = \{\xi \in C_{ub}^0(\mathbb{R}^n) : \xi(x) \in E^{s(u)} \text{ for all } x \in \mathbb{R}^n\}.$$

For any $\xi \in C_{ub}^0(\mathbb{R}^n)$, one has

$$\xi = \xi^s + \xi^u = P^s \xi + P^u \xi.$$

Let

$$|\xi| = \max\{|\xi^s|_0, |\xi^u|_0\}.$$

Then $|\cdot|$ is equivalent to $|\cdot|_0$ on $C_{ub}^0(\mathbb{R}^n)$.

Projecting (6.12) $_k$ onto E^s and E^u respectively, we have

$$\begin{cases} A^s \circ \xi^s - \xi^s \circ (B + \varphi_k) = P^s \circ B_1 \circ \varphi_k, \\ A^u \circ \xi^u - \xi^u \circ (B + \varphi_k) = P^u \circ B_1 \circ \varphi_k, \end{cases}$$

i.e.,

$$\begin{cases} \xi^s = A^s \circ \xi^s \circ F_k^{-1} - P^s \circ B_1 \circ \varphi_k \circ F_k^{-1} =: \mathcal{T}_k^s(\xi), \\ \xi^u = (A^u)^{-1} \circ \xi^u \circ F_k + (A^u)^{-1} \circ P^u \circ B_1 \circ \varphi_k =: \mathcal{T}_k^u(\xi), \end{cases} \quad k = 0, 1, 2, \dots \quad (6.13)_k$$

As F_k , F_k^{-1} , $\varphi_k \in \mathcal{P}^0(\mathbb{R}^n)$ are uniformly continuous, both $\mathcal{T}_k^s(\xi)$ and $\mathcal{T}_k^u(\xi)$ are uniformly continuous for any $\xi \in C_{ub}^0(\mathbb{R}^n)$. Therefore we can define a sequence of operators $\mathcal{T}_k : C_{ub}^0(\mathbb{R}^n) \rightarrow C_{ub}^0(\mathbb{R}^n)$ by

$$\mathcal{T}_k(\xi) \stackrel{\text{def}}{=} \mathcal{T}_k^s(\xi) + \mathcal{T}_k^u(\xi), \quad k = 0, 1, 2, \dots$$

Now (6.13) $_k$ become

$$\xi = \mathcal{T}_k(\xi), \quad k = 0, 1, 2, \dots \quad (6.14)_k$$

Lemma 6.10 *For any $k = 0, 1, 2, \dots$, \mathcal{T}_k is a contraction on $C_{ub}^0(\mathbb{R}^n)$. More precisely,*

$$|\mathcal{T}_k(\xi) - \mathcal{T}_k(\eta)| \leq \tau|\xi - \eta|, \quad \xi, \eta \in C_{ub}^0(\mathbb{R}^n), \quad k = 0, 1, 2, \dots \quad (6.15)_k$$

Proof We only consider the s -components. By (6.13)_k we have

$$\begin{aligned} |\mathcal{T}_k^s(\xi) - \mathcal{T}_k^s(\eta)|_0 &= \sup_{x \in \mathbb{R}^n} \left| A^s \left(\xi^s(F_k^{-1}(x)) - \eta^s(F_k^{-1}(x)) \right) \right| \\ &= \sup_{x \in \mathbb{R}^n} |A^s(\xi^s(x) - \eta^s(x))| \\ &\leq \tau \sup_{x \in \mathbb{R}^n} |\xi^s(x) - \eta^s(x)| \\ &= \tau |\xi^s - \eta^s|_0 \leq \tau |\xi - \eta|. \end{aligned}$$

For the u -components, there holds a similar inequality. So the lemma is proved. \square

By Lemma 6.10, for each $k = 0, 1, 2, \dots$, (6.14)_k, or equivalently (6.11)_k, has a unique solution $\xi = \xi_k$ in $C_{ub}^0(\mathbb{R}^n)$. It thus necessary that $\xi_k = R_k - B_1$ for $k = 1, 2, \dots$

Lemma 6.11 *When $k \rightarrow \infty$, \mathcal{T}_k is weakly convergent in $C_{ub}^0(\mathbb{R}^n)$ to \mathcal{T}_0 . Namely, for any $\xi \in C_{ub}^0(\mathbb{R}^n)$, one has*

$$\lim_{k \rightarrow \infty} |\mathcal{T}_k(\xi) - \mathcal{T}_0(\xi)| = 0.$$

Proof We only consider the s -components. Note that $\varphi_k = F_k - B_1$. Thus $\varphi_k \circ F_k^{-1} = \text{id} - B_1 \circ F_k^{-1}$. By (6.13)_k we have

$$\begin{aligned} |\mathcal{T}_k^s(\xi) - \mathcal{T}_0^s(\xi)|_0 &= \sup_{x \in \mathbb{R}^n} \left| A^s \left(\xi^s(F_k^{-1}(x)) - \xi^s(F_0^{-1}(x)) \right) \right. \\ &\quad \left. - P^s \circ B_1 \circ B(F_k^{-1}(x) - F_0^{-1}(x)) \right| \\ &\leq \tau \sup_{x \in \mathbb{R}^n} |\xi^s(F_k^{-1}(x)) - \xi^s(F_0^{-1}(x))| \\ &\quad + |P^s \circ B_1 \circ B| \sup_{x \in \mathbb{R}^n} |F_k^{-1}(x) - F_0^{-1}(x)|. \end{aligned}$$

Since $\xi^s = P^s \circ \xi \in C_{ub}^0(\mathbb{R}^n)$ is uniformly continuous and

$$\lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^n} |F_k^{-1}(x) - F_0^{-1}(x)| = 0$$

(see Proposition 3.4), it is easy to see that

$$\lim_{k \rightarrow \infty} |\mathcal{T}_k^s(\xi) - \mathcal{T}_0^s(\xi)|_0 = 0$$

holds. Similarly,

$$\lim_{k \rightarrow \infty} |\mathcal{T}_k^u(\xi) - \mathcal{T}_0^u(\xi)|_0 = 0.$$

The lemma is thus proved. \square

Because of such a weak convergence, we have

Lemma 6.12 *R_k is uniformly convergent to $R_0 = B_1 + \xi_0$. In particular, $\xi_0 \in \mathcal{P}^0(\mathbb{R}^n)$.*

Proof As ξ_k 's are fixed points, we have

$$\begin{aligned} |\xi_k - \xi_0| &= |\mathcal{T}_k(\xi_k) - \mathcal{T}_0(\xi_0)| \\ &\leq |\mathcal{T}_k(\xi_k) - \mathcal{T}_k(\xi_0)| + |\mathcal{T}_k(\xi_0) - \mathcal{T}_0(\xi_0)| \\ &\leq \tau|\xi_k - \xi_0| + |\mathcal{T}_k(\xi_0) - \mathcal{T}_0(\xi_0)|. \end{aligned}$$

Thus

$$|\xi_k - \xi_0| \leq \frac{1}{1-\tau} |\mathcal{T}_k(\xi_0) - \mathcal{T}_0(\xi_0)|.$$

Now it follows from Lemma 6.11 that $|\xi_k - \xi_0| \rightarrow 0$. As $\xi_k = R_k - B_1 \in \mathcal{P}^0(\mathbb{R}^n)$ for all $k \geq 1$, we have $\xi_0 = \lim_{k \rightarrow \infty} \xi_k \in \mathcal{P}^0(\mathbb{R}^n)$. \square

From above lemma, we know that $R_0 = B_1 + \xi_0$ can be projected onto \mathbb{T}^n to obtain a map r_0 on \mathbb{T}^n . However, for convergence of $\{S_k\}$, we cannot obtain similarly from the equations $S_k \circ A = F_k \circ S_k$ because they are nonlinear equations. This can be achieved by considering convergence of $\{R_k^{-1}\}$.

Lemma 6.13 *The limiting map $R_0 = B_1 + \xi_0$ is a self-homeomorphism of \mathbb{R}^n .*

Proof We will apply Theorem 4.1. Since r_k, s_k satisfy shift equivalence equations, $k \geq 1$. By the equation $R_k \circ S_k = A^m$ in $(6.10)_k$, we know that $R_k \in \text{Homeo}(\mathbb{R}^n)$ for all $k \geq 1$.

As f_k is convergent to f_0 in the C^1 topology and F_k is C^1 close to F_0 , it follows from Theorem 4.1 that if $N \gg 1$ there exists $\sigma_N \in C_{ub}^0(\mathbb{R}^n)$ such that

$$(\text{id} + \sigma_N) \circ F_0 = F_N \circ (\text{id} + \sigma_N) \quad (6.16)$$

and $|\sigma_N|$ is small, $\text{id} + \sigma_N \in \text{Homeo}(\mathbb{R}^n)$.

By $(6.10)_N$ and (6.16) , we have

$$R_N \circ (\text{id} + \sigma_N) \circ F_0 = A \circ R_N \circ (\text{id} + \sigma_N).$$

Namely

$$\xi := R_N \circ (\text{id} + \sigma_N) - B_1 = B_1 \circ \sigma_N + \xi_N \circ (\text{id} + \sigma_N) \in C_{ub}^0(\mathbb{R}^n)$$

satisfies $(6.11)_0$. By the uniqueness we have

$$R_0 = B_1 + \xi_0 = R_N \circ (\text{id} + \sigma_N) \in \text{Homeo}(\mathbb{R}^n). \quad \square$$

Now we have the following result.

Lemma 6.14 *S_k converges uniformly to some $S_0 \in B_2 + \mathcal{P}^0(\mathbb{R}^n)$.*

Proof By $(6.10)_k$ we have $S_k = R_k^{-1} \circ A^m$ for all $k \geq 1$. By Proposition 3.4, S_k converges uniformly to $R_0^{-1} \circ A^m$. Since $S_k \in B_2 + \mathcal{P}^0(\mathbb{R}^n)$ for all $k \geq 1$, then $S_0 \in B_2 + \mathcal{P}^0(\mathbb{R}^n)$. \square

Finally, letting $k \rightarrow \infty$ in $(6.10)_k$, we have

$$R_0 \circ F_0 = A \circ R_0, \quad F_0 \circ S_0 = S_0 \circ A, \quad S_0 \circ R_0 = F_0^m, \quad R_0 \circ S_0 = A^m. \quad (6.17)$$

Since $R_0 - B_1, S_0 - B_2 \in \mathcal{P}^0(\mathbb{R}^n)$, we can project (6.16) onto \mathbb{T}^n . This yields a shift equivalence $f_0 \stackrel{s}{\sim} a$. By definition of Q , it is easy to see that $f_0 \in Q$. This proves Proposition 6.8. As explained before, the proof for Theorem 6.7 is complete.

7 Notes

For classification of dynamical systems, one may use different equivalence relations. In this dissertation, we deal with some of these. They have the following relations:

$$\begin{array}{ccccccc}
 \varepsilon\text{-topol. equiv.} & \Rightarrow & \text{topol. equiv.} & \Rightarrow & \text{shift equiv.} & \Rightarrow & \text{inverse limit equiv.} \\
 & & & & \Downarrow & & \text{(orbit equiv.)} \\
 & & & & \text{semi-topol. equiv.} & &
 \end{array}$$

These equivalence relations yield the corresponding concepts of stability, which have the reversed relations.

Generally speaking, the converses for the above relations do not hold. However, for Anosov endomorphisms we are interested, some of the converses do hold, see Theorem 5.2 and the work of Williams. For example, Williams proved that orbit equivalence does imply shift equivalence for one-dimensional expanding attractors, while we have proved shift equivalence does imply topological conjugacy for certain Anosov endomorphisms. When hyperbolic attractors are realized by Anosov endomorphisms, we have proved that for ‘true’ hyperbolic attractors, such a reduction does not hold. So the systems in $\mathcal{A}^*(M)$ and in $\mathcal{A}(M) \setminus \mathcal{A}^*(M)$ have quite different dynamical behaviour. For example, all systems in $\mathcal{A}(M) \setminus \mathcal{A}^*(M)$ are linear under topological conjugacy. As systems in $\mathcal{A}^*(M)$ lose structural stability (even, semi-structural stability), there will be many nonlinear ones under topological conjugacy or even under the weak shift equivalence.

As systems in $\mathcal{A}^*(M)$ and in $\mathcal{A}(M) \setminus \mathcal{A}^*(M)$ share the same orbit stability, one may expect that the following hold:

For any $f \in \mathcal{A}^(\mathbb{T}^n)$, there exists some $a \in \mathcal{H}_1^*(\mathbb{T}^n)$ such that $f \overset{\circ}{\sim} a$. Namely all f in $\mathcal{A}^*(\mathbb{T}^n)$ are linearizable under more weak equivalence relation — orbit equivalence.*

For this problem, there are two points worth to be mentioned.

The first is a negative argument. For $f \in \mathcal{A}^*(\mathbb{T}^n)$, Σ_f has no good ‘algebraic structure’ such as the fundamental group. Thus it is not easy to find the corresponding ‘algebraic condition’ for $f \overset{\circ}{\sim} g$.

The second is a positive argument. Since $f \in \mathcal{A}^*(\mathbb{T}^n)$ is o -stable, one may expect the following

- (a) for any $f \in \mathcal{A}^*(\mathbb{T}^n)$, $f_* : \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n \rightarrow \pi_1(\mathbb{T}^n)$ is hyperbolic; and
- (b) for any $f \in \mathcal{A}^*(\mathbb{T}^n)$, $f_* \in \mathcal{A}^*(\mathbb{T}^n)$ and $f \overset{\circ}{\sim} f_*$. (This can be achieved by only showing that f and f_* are in the same connected component of $\mathcal{A}^*(\mathbb{T}^n)$!)

All of these do hold for Anosov diffeomorphisms and expanding maps. However, for ‘true’ Anosov endomorphisms, these problems may be related with the structure of the centralizers of Anosov endomorphisms, see Kopell [9] and Palis-Yoccoz [32].

The affirmative answer to the above problem, combined with the work of Wen [45], will conclude that all 2-dimensional hyperbolic attractors are essentially the inverse limit systems of Anosov endomorphisms on 2-torus. This will be an important step to understand the famous Lorenz attractors.

8 References

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