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TITLE: Some Problems on Anosov Endomorphisms

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0 Introduction

In the history of differentiable dynamical systems, several classes of systems of diffeomorphisms such as Anosov systems, quasi-Anosov systems and Axiom A systems, which have certain 'hyperbolicity', have been well understood in the 1960's and 1970's, see Smale [14], Mãñé [9, 10], and Bowen [2]. In recent years, some noninvertible differentiable maps have been also studied. For example, Shub [13] and Zhang [17] have proved the structural stability of expanding maps and expanding invariant sets, respectively. Mãñé & Pugh [11] have proved the instability for 'general' Anosov endomorphisms. Some research interests on differentiable maps lie in the following two aspects. The first one is on the similar problems as diffeomorphisms, and the second one is on the chaotical phenomena of 1- or 2-dimensional noninvertible maps. The later would still be a hot topics in dynamical systems.

This thesis is some study on the first aspect above.

In §1, we state some results on the relationship between manifolds and covering manifolds. Then we consider some relations between dynamical systems and their lifting systems. We will prove that a system has the same topological entropy as that of the lifting system.

In §2, we will give two hyperbolic toral automorphisms A on $\mathbb{T}^2$ such that A and A' are not topologically conjugate. These examples explain the difficulty in finding the complete set of invariants for Anosov diffeomorphisms under topological conjugacy.

The π_1 property was introduced by Shub [15] and Franks [5]. It was found in Shub [13] and Franks [5], Manning [8] that it is a useful concept in studying the stability of expanding maps and classification of Anosov diffeomorphisms. In §3, we will use the stability results in [11] to study the π_1 property of Anosov endomorphisms and will prove the following result.

Theorem 3.9 *Let A be a hyperbolic toral endomorphism on $\mathbb{T}^m$. Then A has π_1 property if and only if A is either a hyperbolic toral automorphism or an expanding toral endomorphism.*

This is a partial answer to a problem posed by F. Franks [5, p. 74].

In §3, we will also use the stability result in [12] to prove

Theorem 3.13 *Let $m \geq 2$. Then there is an infinite sequence of Anosov endomorphisms such that they are mutually not topologically conjugate and they are not topologically conjugate to any hyperbolic toral endomorphism.*

In §4, we introduce some works of Mãñé [9, 10] on quasi-Anosov diffeomorphisms. Then we will introduce the concept of quasi-Anosov endomorphisms. Some basic properties on quasi-Anosov endomorphisms will be stated without proofs and some problems on quasi-Anosov endomorphisms are also proposed.

This thesis was completed under the guidance of Professor LIAO Shantao. I would like to express my many thanks to him for his constant help and guidance!

1 Covering Spaces and Dynamical Systems

In this section we review some relations between manifolds and their covering manifolds. In basis of this, we give some dynamics connections between dynamical systems and their lifting systems. For example, a system and its lifting system have the same topological entropy. For the application of covering spaces to dynamical systems, see Shub [13], Franks [5], Mãné & Pugh [11] and Coven & Reddy [3].

Throughout this section, we always assume that topological spaces are path connected and locally path connected. For the basic theory on covering spaces, see books on general topology such as E. Spanier [15].

Definition 1.1 Let $\bar{B}$, B be topological spaces and $p : \bar{B} \to B$ be a continuous map. Suppose that for any $x \in B$, there exists an open neighborhood U of x in B such that U is path connected, $p^{-1}(U) = \cup_{i\in\Lambda} U_i$, where U_i are path connected components of $p^{-1}(U)$. Moreover, $p|U_i : U_i \to U$ are homeomorphisms. In this case, we say that $(\bar{B}, p)$ is a covering space of B. B is called the base space and p the covering map. Such a U is called an admissible neighborhood of x. If, in addition, $\bar{B}$ has the trivial fundamental group: $\pi_1(\bar{B}) = \{0\}$, then $(\bar{B}, p)$ is called a universal covering space.

We need some properties for covering spaces of connected Riemannian manifolds.

Proposition 1.2 *Let $(\bar{M}, p)$ be a covering space of a connected C^r manifold M. Then $\bar{M}$ is also a C^r manifold, for which p is a local diffeomorphism. Moreover, if M has a Riemannian metric g, then $\bar{M}$ may have a Riemannian metric $\bar{g}$, for which p is a local isometry.*

Theorem 1.3 *Let M be a connected compact differentiable manifold. Suppose that $f : M \to M$ is a local diffeomorphism, i.e., $T_x f : T_x M \to T_{f(x)} M$ is an isomorphism for each $x \in M$. Then (M, f) is a covering space of M.*

We are interested in the universal covering spaces for manifolds. Let M be a manifold. We denote by $(\tilde{M}, \pi)$ the universal covering manifolds of M. Since any manifold possesses the local semi-simple connectedness, it follows from [15, p.83] that the universal covering manifolds do exist.

Proposition 1.4 *Any C^r manifold M has universal covering manifolds $(\tilde{M}, \pi)$. Moreover, any two of universal covering manifolds with the same base M are C^r diffeomorphic.*

If M is a Riemannian manifold, then $\tilde{M}$ is also a Riemannian manifold by Proposition 1.2. Let us use d and $\tilde{d}$ to denote the metrics induced by the Riemannian metrics on M and $\tilde{M}$ respectively. Let $\Gamma(\tilde{M}, \pi) = \{\alpha : \tilde{M} \to \tilde{M} \,|\, \alpha \text{ is a diffeomorphism and } \pi\alpha = \pi\}$. Then it is a subgroup of $\mathrm{Diff}(M)$, the group of all C^1 self-diffeomorphisms of M. From algebraic topology, $\Gamma(\tilde{M}, \pi) \cong \pi_1(M, m_0)$, the fundamental group of M, where $m_0 \in M$. It is obvious that any element from $\Gamma(\tilde{M}, \pi)$ is an isometry of $\tilde{M}$.

In dynamical systems theory, the dynamics of $f : M \to M$ is related with that of the lifting system $\tilde{f}$ on $\tilde{M}$.

Proposition 1.5 [15, p.76] *Let $f : M \to N$ be a C^r mapping with $f(m) = n$. Suppose that $(\tilde{M}, \pi_M)$ and $(\tilde{N}, \pi_N)$ are universal covering manifolds of M and N respectively. Let $\tilde{m} \in \tilde{M}$ and $\tilde{n} \in \tilde{N}$ be such that $\pi_M(\tilde{m}) = m$ and $\pi_N(\tilde{n}) = n$. Then there exists a unique lifting of f to $\tilde{M}$: $\tilde{f} : \tilde{M} \to \tilde{N}$ such that $\tilde{f}(\tilde{m}) = \tilde{n}$ and $f \circ \pi_M = \pi_N \circ \tilde{f}$.*

For Riemannian manifolds, we have

Theorem 1.6 *Suppose that M is a compact connected Riemmanian manifold and $\pi : \tilde{M} \to M$ is a universal covering manifold. Let d and $\tilde{d}$ be the metrics on M and $\tilde{M}$ respectively. Then there exists $\delta_0 > 0$ such that*

$$d(\pi\tilde{p}, \pi\tilde{q}) = \tilde{d}(\tilde{p}, \tilde{q})$$

for all $\tilde{p},\ \tilde{q} \in \tilde{M}$ with $\tilde{d}(\tilde{p}, \tilde{q}) < \delta_0$.

This can be proved using the following facts.

Definition 1.7 Let $\pi : \tilde{M} \to M$ be a universal covering manifold with M compact. A compact subset $K \subset \tilde{M}$ is called a covering domain if K satisfies $\pi(K) = M$ and $\overline{\mathrm{int}(K)} = K$. Such covering domains do exist.

Lemma 1.8 [17] *Let M be a Riemannian manifold with the metric d. For any compact subset $K \subset M$, there exists $\rho > 0$ such that if $p \in K$ and $r \in (0, \rho]$, then the subset $\bar{B}_r(p) = \{q \in M \mid d(q, p) \le r\}$ satisfies the geodesic convexity, i.e., for any $x,\ y \in \bar{B}_r(p)$, there exists a unique shortest geodesics γ_{xy} which connects x and y, and $\gamma_{xy} \subset \bar{B}_r(p)$ and* $\mathrm{length}(\gamma_{xy}) = d(x, y)$.

As applications of Theorem 1.6, we have the following two results.

Theorem 1.9 *Let $f : M \to M$ be a local diffeomorphism on a compact connected manifold M. Let $\tilde{f} : \tilde{M} \to \tilde{M}$ be a lifting to the universal covering manifold $\tilde{M}$. Then $\tilde{f}$ is a diffeomorphism.*

From this and Theorem 1.6, we can prove that the system and its lifting system have the topological entropy. Topological entropy of systems is an important quantity which reflects the complexity of dynamical systems. It has several equivalent definitions. We will adopt that using generating sets.

Definition 1.10 [16, §7.2] Let (X, d) be a metric space and $f : X \to X$ a uniformly continuous mapping. For any $n \ge 1$, define

$$d_n(x, y) = \max_{0 \le k \le n-1} d(f^k(x), f^k(y)), \qquad x,\, y \in X.$$

Then d_n are also metrics of X. For any compact subset $K \subset X$ any $n \ge 1$, and any $\delta > 0$, a subset $E \subset K$ is called an (n, δ)-generating set of K if for any $x \in K$ there exists

$y \in E$ such that $d_n(x, y) \le \delta$. Let us use $r_n(K, \delta)$ to denote the minimal cardinality of all (n, δ)-generating sets of K. Define

$$h_d(f, K) = \lim_{\delta \to 0+} \limsup_{n \to +\infty} \frac{1}{n} \log r_n(K, \delta),$$

which is called the entropy of f with respect to K. Then the topological entropy of the system f (relative with the metric d) is defined as

$$h_d(f) = \sup_{\substack{K \subset X \\ K \text{ is compact}}} h_d(f, K).$$

Lemma 1.11 [16, p.199] *Let (X, d) and $(\tilde{X}, \tilde{d})$ be metric spaces and $\pi : \tilde{X} \to X$ be an onto continuous mapping. Suppose that there exists $\delta > 0$ such that $\pi|_{B_\delta(\tilde{x})} : B_\delta(\tilde{x}) \to B_\delta(\pi\tilde{x})$ is an isometry for any $\tilde{x} \in \tilde{X}$, where $B_r(z)$ denotes the ball of radius r centered at z. If $f : X \to X$ and $\tilde{f} : \tilde{X} \to \tilde{X}$ are uniformly continuous mappings such that $\pi \circ \tilde{f} = f \circ \pi$, then $h_d(f) = h_{\tilde{d}}(\tilde{f})$.*

From these we have

Theorem 1.12 *Le M be a compact connected Riemannian manifold with the universal covering manifold $\pi : \tilde{M} \to M$. Then for any continuous mapping $f : M \to M$ and its lifting mapping $\tilde{f} : \tilde{M} \to \tilde{M}$, we have $h_{\tilde{d}}(\tilde{f}) = h_d(f)$.*

Example 1.13 [16] Let A an endomorphism of $\mathbb{T}^m$. The lifting mapping of A to the universal covering manifold $\mathbb{R}^m$ of $\mathbb{T}^m$ is just a linear mapping which is also denoted by A. Then the topological entropy of A is

$$h_d(A) = \sum_{\substack{\lambda \in \sigma(A) \\ |\lambda| > 1}} \log |\lambda|,$$

where $\sigma(A)$ denotes the spectrum A, counting the multiplicity.

2 Conjugacy Invariants of Anosov Diffeomorphisms

In the study of the classification of Anosov diffeomorphisms on $\mathbb{T}^m$, A. Manning [8] proved that any Anosov diffeomorphism on $\mathbb{T}^m$ is topologically conjugate to some hyperbolic toral automorphism. There remains the following two problems in the complete classification of Anosov diffeomorphisms.

(I) Under what conditions, two Anosov diffeomorphisms are topologically conjugate?

(II) What are the complete set of invariants for classification of Anosov diffeomorphisms under topological conjugacy?

Let $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. Then any hyperbolic toral automorphism can be viewed as an element in $\mathrm{Sl}(m,\mathbb{Z})$. For the first problem, we have the following proposition.

Proposition 2.1 *Let A, $B \in \mathrm{Sl}(m,\mathbb{Z})$ be hyperbolic. Then A and B are topologically conjugate if and only if there exists $H \in \mathrm{Sl}(m,\mathbb{Z})$ such that $AH = HB$, i.e., A and B are similar in $\mathrm{Sl}(m,\mathbb{Z})$.*

We can give a simple proof to the above proposition as follows. Suppose that there exists such an $H \in \mathrm{Sl}(m,\mathbb{Z})$ with $AH = HB$. Then H can be viewed as a homeomorphism on $\mathbb{T}^m$ and it yields a conjugacy between A and B. Conversely, suppose that there exists a homeomorphism $h : \mathbb{T}^m \to \mathbb{T}^m$ such that $A \circ h = h \circ B$. Then taking the homotopy at 0, we have $Ah_{*0} = h_{*0}B$. Thus $H = h_{*0} : \pi_1(\mathbb{T}^m) \to \pi_1(\mathbb{T}^m) \cong \mathbb{Z}^m$ is the required element in $\mathrm{Sl}(m,\mathbb{Z})$. □

As for the problem (II), the situation is more complicated. Let us consider the case $m = 2$. For hyperbolic A, $B \in \mathrm{Sl}(2,\mathbb{Z})$, if their eigenvalues coincide, then they are similar in $\mathrm{Gl}(2,\mathbb{R})$. However, if one considers A and B as systems on $\mathbb{T}^2$, the situation is quite different.

Arnold & Avez [1, p. 11] has proposed the following example. Consider

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

as an Anosov system on $\mathbb{T}^2$. Are A and A' topologically conjugate? Let

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{Z}).$$

Then $AH = HA'$. Thus A and A' are topologically conjugate for this example.

More generally, we have the following problem:

(III) For any hyperbolic toral automorphism $A \in \mathrm{Sl}(2,\mathbb{Z})$, are A and A' topologically conjugate?

As described before, if one considers A and A' as systems on $\mathbb{R}^2$, they are always topologically conjugate because A and A' have the same eigenvalues. In this section, we use examples to describe that they may not be topologically conjugate when they are regarded as systems on $\mathbb{T}^2$.

Generally, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{Z})$$

be hyperbolic. Then A and A' are topologically conjugate is equivalent to the existence of

$$H = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{Z})$$

such that $AH = HA'$. A simple computation shows that

$$H = \begin{pmatrix} x & y \\ y & \frac{cx+(d-a)y}{b} \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{Z}).$$

(Note that $b \neq 0$ by the hyperbolicity of A.) Thus it is equivalent to the solvability of the following Diophantine equations:

$$\begin{cases} b \,|\, cx + (d-a)y, \\ cx^2 + (d-a)xy - by^2 = \pm b. \end{cases} \tag{2.1}$$

Example 2.2 Let

$$A = \begin{pmatrix} 2 & 5 \\ 7 & 18 \end{pmatrix}.$$

Then $\det A = +1$ and A is hyperbolic. In this case, (2.1) becomes

$$5 \,|\, 7x + 16y \qquad \text{and} \qquad 7x^2 + 16xy - 5y^2 = \pm 5.$$

Let $2x + y = 5u$. Then the above equations are equivalent to

$$9(x - 2u)^2 - 11u^2 = \pm 1.$$

Let $v = x - 2u$. Then A and A' are topologically conjugate is equivalent to the solvability of the following equation:

$$9v^2 - 11u^2 = \pm 1. \tag{2.2$^{\pm}$}$$

Let us first consider $(2.2)^-$. Taking mod 11, we have $9v^2 \equiv 10 \pmod{11}$. However, $9v^2 \pmod{11}$ can take only values $0,\ 9,\ 3,\ 4,\ 1,\ 5,\ 6$. Therefore $(2.2)^-$ has no solutions.

For $(2.2)^+$, we will prove that it also has no solutions. Otherwise, assume that (u_0, v_0) is a solution of $(2.2)^+$. One may assume that $u_0 > 0$ and $v_0 > 0$. In this case, one has

$$(3v_0)^2 - 11u_0^2 = +1.$$

Consider the Pell equation

$$y^2 - 11x^2 = +1. \tag{2.3}$$

Then $(x, y) = (3, 10)$ is a solution and $x,\ y$ are minimal. From [7, p. 277] one knows that all solutions of (2.3) are given by

$$y + x\sqrt{11} = \pm(10 + 3\sqrt{11}\,)^n, \qquad n \in \mathbb{Z}.$$

In particular, there exists $n \in \mathbb{Z}$ such that

$$3v_0 + u_0\sqrt{11} = \pm(10 + 3\sqrt{11}\,)^n.$$

As $u_0 > 0$ and $v_0 > 0$, one takes $+$ and $n > 0$ in above equation. Expanding this equation, we have

$$3v_0 = 10^n + \sum_{k=1}^{[\frac{n}{2}]} C_n^{2k} \cdot 10^{n-2k} \cdot 11^k \cdot 3^{2k}.$$

Note that the right-hand side is not a multiple of 3. We get a contradiction. As a result, $(2.2)^+$ has no solutions. Consequently, A and A' are not topologically conjugate. $\square$

Example 2.3 Let

$$A = \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix}.$$

Then $\det A = -1$ and A is hyperbolic. In this case, (2.1) becomes

$$3 \,|\, 7x + 8y \qquad \text{and} \qquad 7x^2 + 8xy - 3y^2 = \pm 3.$$

Let $x - y = 3u$. Then the above equations are equivalent to

$$4y^2 + 22uy + 21u^2 = \pm 1. \tag{2.4}$$

We will prove that (2.4) has no solutions.

The technique adopted here is typical in algebraic number theory, see Chapters 26 and 28 in [6]. If (2.4) has a solution (y_0, u_0). Then u_0 is odd. Since

$$y_0 = \frac{-11u_0 \pm \sqrt{37u_0^2 \pm 4}}{4},$$

one has $37u_0^2 \pm 4 = v_0^2$ for some $v_0 \in \mathbb{Z}$. Consider the equation

$$37u^2 \pm 4 = v^2, \tag{2.5}$$

which has a solution (u_0, v_0). It is obvious that v_0 is also odd.

Consider the quadratic field* $\mathbb{Q}(\sqrt{37}\,) = \{r + s\sqrt{37} \,|\, r,\ s \in \mathbb{Q}\}$. Since the prime number $37 \equiv 1 \pmod 4$, then the integral ring is generated by 1 and $\frac{1+\sqrt{37}}{2}$, see [6, p. 499]. Thus the integral ring of $\mathbb{Q}(\sqrt{37}\,)$ is $\mathbb{Z}\left(\frac{1+\sqrt{37}}{2}\right)$. The basic unit ε_1 of $\mathbb{Z}\left(\frac{1+\sqrt{37}}{2}\right)$ can be found as follows. In $37u^2 \pm 4$, letting $u = 1, 2, \cdots$, we find that the first complete square number is $37 \cdot 2^2 - 4 = 12^2$. Thus

$$\varepsilon_1 = \frac{12 + 2\sqrt{37}}{2} = 6 + \sqrt{37},$$

see [6, p. 544].

Since u_0 and v_0 are odd, one has

$$\frac{v_0 + u_0\sqrt{37}}{2} = \left(\frac{v_0 - 1}{2} - \frac{u_0 - 1}{2}\right) + u_0 \frac{1 + \sqrt{37}}{2} \in \mathbb{Z}\left(\frac{1 + \sqrt{37}}{2}\right).$$

* A further reference for algebraic fields is: E. Hecke, *Lecture on the Theory of Algebraic Numbers*, Springer, NY, 1981. — Footnote added in 1987.

Similarly,

$$\frac{v_0 - u_0\sqrt{37}}{2} \in \mathbb{Z}\left(\frac{1+\sqrt{37}}{2}\right).$$

By (2.5),

$$\frac{v_0 + u_0\sqrt{37}}{2} \cdot \frac{v_0 - u_0\sqrt{37}}{2} = \pm 1.$$

Thus $\frac{v_0+u_0\sqrt{37}}{2}$ is also a unit of $\mathbb{Z}\left(\frac{1+\sqrt{37}}{2}\right)$, which has necessarily the form $\pm\varepsilon_1^n$ for some $n \in \mathbb{Z}$, see [6, p. 553]. As a result, there exists $n \in \mathbb{Z}$ such that

$$\frac{v_0 + u_0\sqrt{37}}{2} = \pm(6+\sqrt{37}\,)^n. \tag{2.6}$$

Since $(6+\sqrt{37}\,)^{-1} = -6+\sqrt{37}$, one sees that the coefficients of 1 and $\sqrt{37}$ in the right-hand side of (2.6) are integers, which contradicts the fact that u_0 and v_0 are odd. This also shows that (2.4) has no solutions and A, A' are not topologically conjugate. □

Remark As for the problem (II), the eigenvalues of hyperbolic toral automorphisms are invariants of topological conjugacy. However, these invariants are not a complete set of invariants from the above examples.

3 π_1 Property for Hyperbolic Toral Endomorphisms

In this section, we consider several problems for Anosov endomorphisms. Throughout this section, M is a connected smooth Riemannian manifold and, in most cases, is compact. Let $\mathrm{End}(M)$ be the space of all C^1 self-mappings of M equipped with the C^1 topology.

Definition 3.1 [11] Let $f \in \mathrm{End}(M)$. We say that f is a strong Anosov endomorphism, if f is a local diffeomorphism satisfying, for some Tf-invariant continuous decomposition $TM = E^s \oplus E^u$ and some positive constants c, C and $0 < \mu < 1 < \lambda$, that

$$\|Tf^n \cdot v\| \geq c\lambda^n \|v\|, \qquad \forall\, v \in E^u,\ \forall\, n \geq 0, \qquad \text{(expanding)},$$
$$\|Tf^n \cdot v\| \leq C\mu^n \|v\|, \qquad \forall\, v \in E^s,\ \forall\, n \geq 0, \qquad \text{(contracting)}.$$

Remark 1. If f is a diffeomorphism, then a strong Anosov endomorphism is just an Anosov diffeomorphism. If $E^s = \{0\}$, then a strong Anosov endomorphism is just an expanding endomorphism. 2. When M is compact, one may choose a compatible Riemannian metric on M so that the constants c and C can be 1. See [18, p. 17].

A more natural concept is the so-called (weak) Anosov endomorphisms, which are the objectives we will study in this section.

Definition 3.2 [11] Let $f \in \mathrm{End}(M)$. We say that f is a (weak) Anosov endomorphism, if f is a local diffeomorphism satisfying, for some Tf-invariant, continuous subbundle $E^s \subset TM$ such that Tf is contracting on E^s and the quotient map $\bar{T}f$ is expanding on the quotient bundle TM/E^s.

Remark Another equivalent definition for Anosov endomorphisms is given in [12].

Proposition 3.3 *Assume that M is compact and $f \in \mathrm{End}(M)$. Let $\tilde{f}$ be a lifting of f to the universal manifold $\tilde{M}$. Then f is a weak Anosov endomorphism if and only if $\tilde{f}$ is an Anosov diffeomorphism.*

Proof The necessity. Fix a Riemannian metric on M. Let E^s be the contracting subbundle which is continuous and is invariant under Tf. Let $E^{s\perp}$ be the orthogonal compliment of E^s. Then $TM = E^s \oplus E^{s\perp}$. Lifting E^s, $E^{s\perp}$ to $\tilde{M}$, we have

$$\tilde{E}^s_{\tilde{x}} = \left\{ v_{\tilde{x}} \in T_{\tilde{x}}\tilde{M} : T_{\tilde{x}}\pi \cdot v_{\tilde{x}} \in E^s_{\pi(\tilde{x})} \right\}.$$

$\tilde{E}^{s\perp}$ is analogously defined. Then $T\tilde{M} = \tilde{E}^{s\perp} \oplus \tilde{E}^s$.

Consider the tangent map $T\tilde{f} : T\tilde{M} \to T\tilde{M}$. Then $T\tilde{f}$ can be represented, in the above decomposition, by

$$T\tilde{f} = \begin{pmatrix} A & 0 \\ C & K \end{pmatrix}.$$

As $T\tilde{f}$ is the lifting of Tf, we know that A is expanding and K is contracting, while C is bounded. By Theorem 1.9, $\tilde{f}$ is a diffeomorphism. From the hyperbolicity theory of Hirsch-Pugh (see [18, p. 83]), there exists a $T\tilde{f}$-invariant subbundle $\tilde{E}^u \subset T\tilde{M}$ such that $\tilde{E}^s \oplus \tilde{E}^u = T\tilde{M}$. As $\bar{T}\tilde{f}$ is expanding on $T\tilde{M}/\tilde{E}^s$, then so does $T\tilde{f}$ on $\tilde{E}^u$. As a result, $\tilde{f}$ is an Anosov diffeomorphism.

The sufficiency can be proved similarly. □

Corollary 3.4 [11] *Let M be a compact manifold. Then the set $A(M)$ of all Anosov endomorphisms on M is an open subset of* $\mathrm{End}(M)$.

This corollary comes from the fact that the set of all Anosov diffeomorphisms on $\tilde{M}$ is open in $\mathrm{Diff}(\tilde{M})$ and that the lifting is continuous in the systems.

For $f \in A(M)$, we have the following stability result on the perturbation of f in $\mathrm{End}(M)$.

Theorem 3.5 [11, 12] *Let $f \in A(M)$. If f is ε-structurally stable, then f is either an Anosov diffeomorphism or an expanding endomorphisms. The converse is also true.*

This is a sharp result, from which we can derive some interesting results.

The π_1 *property* was introduced by Shub [13] in studying stability of expanding endomorphisms and by Franks [4, 5] in studying classification of Anosov diffeomorphisms. Manning [8] exploited the techniques in [13, 4] to have proved that all Anosov diffeomorphisms on $\mathbb{T}^m$ are π_1 diffeomorphisms, from which one has

Theorem 3.6 [8] *Let $f : \mathbb{T}^m \to \mathbb{T}^m$ be an Anosov diffeomorphism. Then there exists a hyperbolic toral automorphism A such that f is topologically conjugate to A. Moreover, two Anosov diffeomorphisms f and g on $\mathbb{T}^m$ are topologically conjugate if and only if f and g are π_1-conjugate. (The definition of π_1-conjugacy will be introduced later.)*

Shub and Manning succeeded in applying π_1 property of systems to study the stability and classification problems. Conversely, we will apply stability results to study π_1 property for certain of Anosov endomorphisms.

Definition 3.7 Let $f : M \to M$ be a homeomorphism (or, a self-mapping). We say that f is a π_1 homeomorphism (or, a π_1 self-mapping, respectively) if f possesses the following property: For any CW-complex K and any self-homeomorphism (or, a self-mapping, respectively) $g : K \to K$, if there exists a mapping $h : K \to M$ such that the diagram

$$\begin{array}{ccc} \pi_1(M) & \xleftarrow{h_*} & \pi_1(K) \\ f_* \downarrow & & \downarrow g_* \\ \pi_1(M) & \xleftarrow{h_*} & \pi_1(K) \end{array}$$

is commutative, then there exists a unique base point preserving map $h' : K \to M$ such that h' is homotopic to h and $h' \circ g = f \circ h'$.

Remark For basic properties of π_1 mappings, see [5]. For example, if $f : M \to M$ and $g : N \to N$ are π_1 mappings, then f and g are topologically conjugate if and only if f and g are π_1 conjugate, i.e., if and only if there exists a group isomorphism $\phi : \pi_1(N) \to \pi_1(M)$ such that $f_*\phi = \phi g_*$, see [5, p.75].

Theorem 3.8 *Let $A : \mathbb{T}^m \to \mathbb{T}^m$ be a hyperbolic toral endomorphism, i.e., $A \in \mathrm{Gl}(n, \mathbb{Z})$ and A is hyperbolic. If A is a π_1 endomorphism, then A is ε-structurally stable under C^1 perturbations.*

From this, we have

Theorem 3.9 *Let $A : \mathbb{T}^m \to \mathbb{T}^m$ be a hyperbolic toral endomorphism. Then A is a π_1 endomorphism if and only if A is either an automorphism or an expanding endomorphism.*

Proof "⇒" By Theorem 3.8, A is ε-structurally stable. Hence it follows from Theorem 3.5 that A is either an automorphism or an expanding endomorphism.

"⇐" If A is a hyperbolic toral automorphism, then A is a π_1 homeomorphism. See [5, p.70]. When A is an expanding endomorphism, it follows from [13] or [5, p.90] that A is a π_1 endomorphism. □

Remark Theorem 3.9 gives a partial answer to a question posed by Franks [4, p.74].

Problem 3.10 Among all Anosov endomorphisms of $\mathbb{T}^m$, whether only Anosov diffeomorphisms and expanding endomorphisms have π_1 property?

In order to prove Theorem 3.8, we introduce some notations and some lemmas.

A hyperbolic toral endomorphism A on $\mathbb{T}^m$ can be viewed as a matrix $A \in \mathrm{Gl}(n, \mathbb{Z})$. The lifting of A to $\mathbb{R}^m$ is itself. Introduce the following spaces of functions:

$$\mathcal{H} = \{\eta \in C^0(\mathbb{R}^m, \mathbb{R}^m) \,|\, \eta \text{ has period 1 with respect to each component}\},$$
$$C_b^0 = C_b^0(\mathbb{R}^m) = \{\eta \in C^0(\mathbb{R}^m, \mathbb{R}^m) \,|\, \eta \text{ is bounded}\}.$$

We endow the spaces $\mathcal{H}$ and C_b^0 with the super-norm. Then they are both Banach spaces, while $\mathcal{H}$ s a closed subspace of C_b^0.

Let $A \in \mathrm{End}(\mathbb{T}^m)$ and $g : \mathbb{T}^m \to \mathbb{T}^m$ be a C^1 mapping which is C^1 near A. Then g has a lifting $\tilde{g} = A + \tilde{\varphi} : \mathbb{R}^m \to \mathbb{R}^m$, where $\tilde{\varphi} \in \mathcal{H}$ and $\|\tilde{\varphi}\|_{C^1} \ll 1$.

Lemma 3.11 [18, p.168] *Suppose that A is a hyperbolic automorphism of $\mathbb{R}^m$. Then for any $\varphi \in C_b^0$ with $\|\varphi\|_{C^0} \ll 1$ and $\mathrm{Lip}(\varphi) \ll 1$, there exists a unique $\eta \in C_b^0$ such that*

$$(\mathrm{id} + \eta) \circ (A + \varphi) = A \circ (\mathrm{id} + \eta). \tag{3.1}$$

Moreover, $h = \mathrm{id} + \eta : \mathbb{R}^m \to \mathbb{R}^m$ is a homeomorphism, and $\|\eta\|_{C^0}$ is small when $\|\varphi\|_{C^0}$ and $\mathrm{Lip}(\varphi)$ are small.

Lemma 3.12 [18, p.175] *Suppose that $A \in \mathrm{End}(\mathbb{T}^m)$ is a hyperbolic endomorphism. For any $\varepsilon > 0$, there exists $A \in \mathcal{U}_\varepsilon \subset_{\text{open}} \mathrm{End}(\mathbb{T}^m)$ such that any $g \in \mathcal{U}_\varepsilon$ has a unique (hyperbolic) fixed point $[x_g] \in \mathbb{T}^m$ with $\|x_g\| < \varepsilon$.*

Proof of Theorem 3.8. Let $A : \mathbb{T}^m \to \mathbb{T}^m$ be a hyperbolic endomorphism and $g : \mathbb{T}^m \to \mathbb{T}^m$ be C^1 near A. Consider the liftings to $\mathbb{R}^m$ (denoted by $\tilde{}$). Then $\tilde{g} = A + \tilde{\varphi}$, where $\tilde{\varphi}$ is C^1 small and $\tilde{g}$ has a unique fixed point $x_g \in \mathbb{R}^m$ with $\|x_g\|$ small. Let $\tilde{H} : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $\tilde{H}(x) = x + x_g$. Then the projection H of $\tilde{H}$ onto $\mathbb{T}^m$ is a homeomorphism and H is near $\mathrm{id}_{\mathbb{T}^m}$. Let $\bar{g}(x) = \tilde{H}^{-1} \circ \tilde{g} \circ \tilde{H} : \mathbb{R}^m \to \mathbb{R}^m$. Then

$$\bar{g}(x) = Ax + (\tilde{\varphi}(x + x_g) + (A - I)x_g) =: Ax + \bar{\varphi}(x),$$

and $\bar{\varphi} \in \mathcal{H}$. Moreover,

$$\|\bar{\varphi}\|_{C^1} \le \|\tilde{\varphi}\|_{C^1} + \|A - I\| \, \|x_g\|$$

is small. Applying Lemma 3.11 to $\bar{g} = \mathrm{id} + \tilde{\varphi}$, we know that there exists a unique $\tilde{\eta} \in C_b^0$ such that

$$A \circ (\mathrm{id} + \tilde{\eta}) = (\mathrm{id} + \tilde{\eta}) \circ (A + \bar{\varphi}), \tag{3.2}$$

and $\tilde{h} = \mathrm{id} + \tilde{\eta}$ is a homeomorphism of $\mathbb{R}^m$ with $\|\tilde{\eta}\|_{C^0}$ small.

Suppose now that A is a π_1 endomorphism. Then for any g which is C^1 near A, we have $\bar{g} = A + \bar{\varphi} : \mathbb{R}^m \to \mathbb{R}^m$, where $\bar{\varphi} \in C^1(\mathbb{R}^m, \mathbb{R}^m) \cap \mathcal{H}$ and $\bar{\varphi}(0) = 0$, $\|\bar{\varphi}\|_{C^1}$ small. Let $\bar{g}_t(x) = Ax + t\bar{\varphi}(x)$, $t \in [0,1]$. Then $\bar{g}_t$ is homotopic to A relative to 0.

Project all mappings to $\mathbb{T}^m$. Denote the projection of $\bar{g}$ by g_0. Then g_0 is homotopic to A relative to $[0]$. Hence $g_{0*} = A_{*0} = A : \pi_1(\mathbb{T}^m, [0]) \to \pi_1(\mathbb{T}^m, [0])$. Thus $\mathrm{id}_{\pi_1(\mathbb{T}^m,[0])}$ is a π_1 conjugacy between g_0 and A. It follows from the fact that A is a π_1 endomorphism that there exists a unique $h_0 : (\mathbb{T}^m, [0]) \to (\mathbb{T}^m, [0])$ such that $h_{0*} = \mathrm{id}_{\pi_1(\mathbb{T}^m,[0])}$ and

$$A \circ h_0 = h_0 \circ g_0. \tag{3.3}$$

Let us consider the liftings of the mappings in above equality. The lifting of g_0 is $\bar{g} = A + \bar{\varphi}$. Let $\tilde{h}_0$ be the lifting of h_0. Then $\tilde{h}_0(x + l) = l + \tilde{h}_0(x)$ for all $x \in \mathbb{R}^m$ and $l \in \mathbb{Z}^m$. Let $\tilde{h}_0 = \mathrm{id} + \tilde{\eta}_0$. Then $\tilde{\eta}_0 \in \mathcal{H}$. By (3.3) we have

$$A \circ (\mathrm{id} + \tilde{\eta}_0) = (\mathrm{id} + \tilde{\eta}_0) \circ (A + \bar{\varphi}). \tag{3.4}$$

Comparing (3.2) with (3.4), we know from the uniqueness of the solution of (3.2) that $\tilde{\eta} = \tilde{\eta}_0$ because $\tilde{\eta}_0 \in \mathcal{H} \subset C_b^0$. Therefore, $\tilde{h}_0 = \mathrm{id} + \tilde{\eta}_0 = \mathrm{id} + \tilde{\eta} : \mathbb{R}^m \to \mathbb{R}^m$ is a homeomorphism with the projection $h_0 : \mathbb{T}^m \to \mathbb{T}^m$ also being a homeomorphism. Note that h_0 is near $\mathrm{id}_{\mathbb{T}^m}$ when $\|\tilde{\varphi}\|_{C^1}$ is small.

Since $g_0 : \mathbb{T}^m \to \mathbb{T}^m$ is the projection of $\bar{g} = \tilde{H}^{-1} \circ \tilde{g} \circ \tilde{H} : \mathbb{R}^m \to \mathbb{R}^m$ onto $\mathbb{T}^m$, we know that $g_0 = H^{-1} \circ g \circ H$. It now follows from (3.3) that

$$A \circ (h_0 \circ H^{-1}) = (h_0 \circ H^{-1}) \circ g.$$

Such a homeomorphism $h_0 \circ H^{-1}$ gives a required topological conjugacy between g and A. Since $h_0 \circ H^{-1}$ is near $\mathrm{id}_{\mathbb{T}^m}$ when $\|\tilde{\varphi}\|_{C^1}$ is small, A is thus ε-structurally stable under small C^1 perturbations. □

From Theorem 3.5 we can obtain a result for Anosov endomorphisms, which is different from that for Anosov diffeomorphisms.

Theorem 3.13 *Let $m \geq 2$. Then there is an infinite sequence of Anosov endomorphisms such that they are mutually not topologically conjugate and they are not topologically conjugate to any hyperbolic toral endomorphism.*

Proof Let A be a hyperbolic toral endomorphism on $\mathbb{T}^m$ which is neither an automorphism nor an expanding endomorphism. We assert, for any C^1 neighborhood $\mathcal{U}$ of A in $\mathrm{End}(\mathbb{T}^m)$, that there must be some $f \in \mathcal{U}$ such that f is an Anosov endomorphism and f is *not* topologically conjugate to *any* hyperbolic toral endomorphism.

By Theorem 3.4, we may assume that $f \in \mathcal{U} \subset A(\mathbb{T}^m)$. If the above assertion is not true, then for any $g \in \mathcal{U}$, there would be some hyperbolic endomorphism B and some homeomorphism $h_1 : \mathbb{T}^m \to \mathbb{T}^m$ such that

$$g \circ h_1 = h_1 \circ B. \tag{3.5}$$

Let $[x_1] = h_1([0])$. Then $[x_1]$ is a fixed point of g. By (3.5) we have

$$g_{*[x_1]} = h_{1*[0]} \cdot B \cdot h^{-1}_{1*[0]}. \tag{3.6}$$

As in the proof of Theorem 3.8, we know by shrinking $\mathcal{U}$ that any $g \in \mathcal{U}$ has a unique fixed point $[x_2]$ near $[x_1]$. Let $h_2([x]) = [x + x_2]$. Then

$$g_{*[x_2]} = h_{2*[0]} \cdot B \cdot h^{-1}_{2*[0]}. \tag{3.7}$$

From algebraic topology, there exists an automorphism $h_0 : \pi_1(\mathbb{T}^m, [x_1]) \to \pi_1(\mathbb{T}^m, [x_2])$ such that

$$h_0 \cdot g_{*[x_1]} = g_{*[x_2]} \cdot h_0. \tag{3.8}$$

Comparing (3.6)–(3.8) and letting $h = h^{-1}_{2*[0]} \cdot h_0 \cdot h_{1*[0]} : \pi_1(\mathbb{T}^m, [0]) \to \pi_1(\mathbb{T}^m, [0])$, we have

$$B = h^{-1} \cdot A \cdot h^{-1}.$$

Since h is an automorphism of $\pi_1(\mathbb{T}^m, [0]) \cong \mathbb{Z}^m$, $h \in \mathrm{Sl}(m, \mathbb{Z})$ induces a homeomorphism on $\mathbb{T}^m$. Substituting $B = h^{-1} \cdot A \cdot h$ into (3.5), we have

$$g \cdot (h_1 \cdot h^{-1}) = (h_1 \cdot h^{-1}) \cdot A,$$

i.e., g and A are topologically conjugate. Since $g \in \mathcal{U}$ is arbitrary, A is structurally stable. By Theorem 3.5 or [12, Theorem 4.7, p.279], A is either an automorphism or an expanding endomorphism, a contradiction to the assumption.

Now the theorem is proved by choosing an infinite sequence of hyperbolic toral endomorphisms A's which are neither Anosov diffeomorphisms nor expanding endomorphisms. Since different determinants of A's give different endomorphisms A's with respect to topological conjugacy. For each A, there is $A \in \mathcal{U}_A \subset_{\mathrm{open}} \mathrm{End}(\mathbb{T}^m)$ such that any $g \in \mathcal{U}_A$ satisfies (3.7). From the assertion, there is some $f_A \in \mathcal{U}_A$ such that f_A is an Anosov endomorphism and f_A is not topologically conjugate to any hyperbolic toral endomorphism. If A and A' are not topologically conjugate, it is easy to verify from (3.7) that f_A and $f_{A'}$ are also not topologically conjugate. □

Problem 3.14 Let $A'(\mathbb{T}^m) = \{f \in A(\mathbb{T}^m) \mid f$ is neither an Anosov diffeomorphism nor an expanding endomorphism$\}$. Let $\mathcal{X} = \{f \in A'(\mathbb{T}^m) \mid f$ is not topologically conjugate to any hyperbolic toral endomorphism$\}$. The problem is if $\mathcal{X}$ is a Baire subset of $A'(\mathbb{T}^m)$.

By the results of Przytycki [12] and the proof of Theorem 3.13, it can be proved that $\mathcal{X}$ is dense in $A'(\mathbb{T}^m)$.

4 Quasi-Anosov Endomorphisms

The contents of this section are omitted.

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