Bounds on the (Laplacian) spectral radius of graphs

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Abstract

The spectral radius of a graph is the largest eigenvalue of adjacency matrix of the graph and its Laplacian spectral radius is the largest eigenvalue of the Laplacian matrix which is the difference of the diagonal matrix of vertex degrees and the adjacency matrix. Some sharp bounds are obtained for the (Laplacian) spectral radii of connected graphs. As consequences, some (sharp) upper bounds of the Nordhaus–Gaddum type are also obtained for the sum of (Laplacian) spectral radii of a connected graph and its connected complement.

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1. Introduction

The graphs in this paper are undirected and simple without loops and multiple edges. Let \( G = (V, E) \) be a simple undirected graph. The complement \( G^c \) of the graph \( G \) is the graph with the same set of vertices as \( G \), where two distinct vertices are adjacent if and only if they are independent in \( G \). For \( v \in V \), the degree of \( v \), denoted by \( d(v) = d_G(v) \), is the number of edges incident to \( v \). We denote by \( \Delta(G) \) and \( \delta(G) \) the maximum degree and minimum degree of vertices of \( G \), respectively. A graph is \( d \)-regular if \( \Delta(G) = \delta(G) = d \). Let \( u, v \in V \). A walk of \( G \) from \( u \) to \( v \) is a finite alternating sequence \( v_0(=u)e_1v_1e_2\ldots v_{k-1}e_kv_k(=v) \) of vertices and edges such that \( e_i = v_{i-1}v_i \) for \( i = 1, 2, \ldots, k \). The number \( k \) is the length of the walk. In particular, if the vertex \( v_i, i = 0, 1, \ldots, k \) in the walk are all distinct then the walk is called a path. The distance of \( u \) and \( v \) is the length of the shortest path between \( u \) and \( v \). The diameter \( d(G) \) of \( G \) is the maximum distance over all pairs of vertices.

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Let $A$ be a square matrix. The spectral radius $\rho(A)$ of $A$ is the maximum eigenvalue of $A$. Let $A(G)$ be the adjacency matrix of a graph $G$ and $D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. The spectral radius $\rho(G)$ of $G$ is the maximum eigenvalue of the adjacency matrix $A(G)$ and the Laplacian spectral radius $\mu(G)$ of $G$ is the maximum eigenvalue of the Laplacian matrix $L(G)$.

Since $A(G)$ is nonnegative, the Perron–Frobenius Theorem implies that $\rho(G)$ is associated to a nonnegative eigenvector. Moreover, if $G$ is connected, then $A(G)$ is irreducible and thus the Perron–Frobenius Theorem implies that $\rho(G)$ is simple and is associated to a positive eigenvector. The Geršgorin disc theorem implies that $L(G)$ is positive semidefinite and hence its eigenvalues are all nonnegative.

We study the (Laplacian) spectral radius of graphs in this paper. Many upper bounds on the (Laplacian) spectral radius of graphs have been obtained up to now. We will give some corresponding lower bounds and also state simpler proofs for some known upper bounds. This paper is organized as follows. In Section 2, we will give some sharp bounds on the spectral radius of connected graphs and some sharp upper bounds of the Nordhaus–Gaddum type on the sum of spectral radii of a connected graph and its connected complement. In Section 3, we will give some sharp bounds on the Laplacian spectral radius of connected graphs and an upper bound of the Nordhaus–Gaddum type. Now we introduce some lemmas which will be used later on.

**Lemma 1.1** [9]. Let $A$ be a nonnegative symmetric matrix and $x$ be a unit vector. If $\rho(A) = x^T A x$, then $A x = \rho(A) x$.

Let $B$ be a matrix. Denote by $s_i(B)$ the $i$th row sum of $B$. The proof of Lemma 2.1 in [5] implies the following slightly stronger version.

**Lemma 1.2.** Let $B$ be a real symmetric $n \times n$ matrix, and let $\lambda$ be an eigenvalue of $B$ with an eigenvector $x$ all of whose entries are nonnegative. Then
\[
\min_{1 \leq i \leq n} s_i(B) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(B).
\]
Moreover, if all entries of $x$ are positive then either of the equalities holds if and only if the row sums of $B$ are all equal.

**Lemma 1.3.** Let $B$ be a real symmetric $n \times n$ matrix, and let $\lambda$ be an eigenvalue of $B$ with an eigenvector $x$ whose entries are all nonnegative. Let $p$ be any polynomial. Then
\[
\min_{1 \leq i \leq n} s_i(p(B)) \leq p(\lambda) \leq \max_{1 \leq i \leq n} s_i(p(B)).
\]
Moreover, if all entries of $x$ are positive then either of the equalities holds if and only if the row sums of $p(B)$ are all equal.

**Proof.** This follows from Lemma 1.2 by noting that $p(B)$ has $x$ as an eigenvector for the eigenvalue $p(\lambda)$. \qed

2. The spectral radius

In this section, we study the spectral radius of graphs. We will state some known upper bounds on the spectral radius of graphs and give the corresponding lower bounds. The simpler proofs for upper bounds are also stated.
2.1. Bounds on the spectral radius

**Theorem 2.1.** Let $G = (V, E)$ be a graph. Then

$$\min_{v \in V} \left( \sum_{uv \in E} d(u) \right)^{1/2} \leq \rho(G) \leq \max_{v \in V} \left( \sum_{uv \in E} d(u) \right)^{1/2}.$$ 

Moreover, if $G$ is connected then either of the equalities holds if and only if $\sum_{uv \in E} d(u)$ is the same for all $v \in V$.

The upper bound in Theorem 2.1 is due to Favaron et al. [6], see also Cao [2].

**Proof.** Note that $s_v(A^2)$ is exactly the number of walks of length 2 in $G$ with an end at $v$. This equals the sum of degree of $u$ over all $u$ with $uv \in E$. If $G$ is connected, then the nonnegative adjacency matrix $A$ of $G$ is irreducible. The Perron–Frobenius Theorem guarantees that the spectral radius $\rho(G)$ is associated to a positive eigenvector. Then the conclusion follows from Lemma 1.3. □

**Corollary 2.1.** Let $G$ be a graph with $n$ vertices, $m$ edges and no isolated vertex. Let $A = A(G)$ and $\delta = \delta(G)$. Then

$$(2m - \Delta n + \Delta \delta + \Delta - \delta)^{1/2} \leq \rho(G) \leq (2m - \delta n + \Delta \delta - \Delta + \delta)^{1/2}.$$ 

Moreover, if $G$ is connected then the first equality holds if and only if $G$ is regular and the second holds if and only if $G$ is a regular graph or a star.

The upper bound in Corollary 2.1 is due to Cao [2], see also Das et al. [4].

**Proof.** Theorem 2.1 gives that

$$\rho(G) \geq \min_{v \in V(G)} \left( \sum_{uv \in E(G)} d(u) \right)^{1/2}$$

$$= \min_{v \in V(G)} \left( 2m - d(v) - \sum_{uv \not\in E(G)} d(u) \right)^{1/2}$$

$$\geq \min_{v \in V(G)} \left( 2m - d(v) - (n - d(v) - 1)\Delta \right)^{1/2}$$

$$= \min_{v \in V(G)} \left( 2m + (\Delta - 1)d(v) - \Delta(n - 1) \right)^{1/2}$$

$$\geq [2m + (\Delta - 1)\delta - \Delta(n - 1)]^{1/2}$$

$$= (2m - \Delta n + \Delta \delta + \Delta - \delta)^{1/2}.$$ 

If $\rho(G)$ attains the lower bound then all equalities in the argument must hold. Now if $G$ is connected then $A(G)$ is irreducible and the Perron–Frobenius Theorem implies that $\rho(G)$ is associated to a positive eigenvector. Lemma 1.3 implies that for all $v \in V(G)$,

$$\sum_{uv \not\in E(G)} d(u) = (n - d(v) - 1)\Delta.$$
Hence either \( d(v) = n - 1 \) or \( d(u) = \delta \), for all \( u \) with \( uv \notin E(G) \). This shows that the graph \( G \) is regular. Conversely if \( G \) is \( d \)-regular then clearly \( \sum_{uv \in E(G)} d(u) = d \) for all \( v \in V(G) \) and thus \( \rho(G) \) attains the lower bound.

Similarly for the upper bound, we have

\[
\rho(G) \leq \max_{v \in V(G)} \left( \sum_{uv \in E(G)} d(u) \right)^{1/2} \\
= \max_{v \in V(G)} \left( 2m - d(v) - \sum_{uv \in E(G)} d(u) \right)^{1/2} \\
\leq \max_{v \in V(G)} \left[ 2m - d(v) - (n - d(v) - 1)\delta \right]^{1/2} \\
= \max_{v \in V(G)} \left[ 2m + (\delta - 1)d(v) - \delta(n - 1) \right]^{1/2} \\
\leq \left[ 2m + (\delta - 1)A - \delta(n - 1) \right]^{1/2} \\
= (2m - \delta n + A\delta - A + \delta)^{1/2}. \\
\tag{1}
\]

If \( \rho(G) \) attains the upper bound then all equalities in the argument must hold. Now if \( G \) is connected then \( \rho(G) \) is associated to a positive eigenvector and thus Lemma 1.3 with (1) implies that for all \( v \in V(G) \),

\[
\sum_{uv \in E(G)} d(u) = (n - d(v) - 1)\delta. \\
\]

Hence either \( d(v) = n - 1 \) or \( d(u) = \delta \), for all \( u \) with \( uv \notin E(G) \). If \( d(v) = n - 1 \) for some \( v \) then by (2) we have \( d(u) = n - 1 \) or \( 1 \) for \( u \in V(G) - v \). Thus \( G \) is a star or a regular graph. Conversely if \( G \) is regular then clearly \( \rho(G) \) attains the upper bound. Now let \( G \) be a star. If \( d(v) = n - 1 \) then

\[
\sum_{uv \in E(G)} d(u) = \sum_{uv \in E(G)} 1 = n - 1. \\
\]

If \( d(v) = 1 \) then

\[
\sum_{uv \in E(G)} d(u) = n - 1. \\
\]

Thus \( \sum_{uv \in E(G)} d(u) = n - 1 \) for all \( v \in V(G) \) and hence \( \rho(G) \) attains the upper bound. \( \Box \)

**Theorem 2.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( \Lambda = \Lambda(G) \) and \( \delta = \delta(G) \). If \( A < 2n - 1 - [(2n - 1)^2 - 8m - 1]^{1/2} \), then

\[
\rho(G) \leq \frac{1}{2} \left( A - 1 - \sqrt{(A + 1)^2 + 4(2m - An)} \right) \\
\tag{3}
\]

or

\[
\frac{1}{2} \left( A - 1 + \sqrt{(A + 1)^2 + 4(2m - An)} \right) \\
\leq \rho(G) \leq \frac{1}{2} \left( \delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)} \right). \\
\tag{4}
\]
Moreover, if $G$ is connected, then the upper bound in (3) is strict, the first equality in (4) holds if and only if $G$ is regular, and the second in (4) holds if and only if $G$ is either a regular graph or a bidegreed graph with all vertices of degree $\delta$ or $n-1$.

The upper bound in (4) for connected graphs is due to Hong et al. [8].

**Proof.** Note that $s_v(A) = d(v)$ for all $v \in V(G)$. By the proof of Theorem 2.1, we have for all $v \in V(G)$,

$$s_v(A^2) = \sum_{uv \in E(G)} d(u) = 2m - d(v) - \sum_{uv \in E(G)} d(u) \geq 2m - d(v) - (n - d(v) - 1)\Delta$$

$$= 2m + (\Delta - 1)d(v) - \Delta(n - 1) = 2m + (\Delta - 1)s_v(A) - \Delta(n - 1).$$

Hence

$$s_v[A^2 - (\Delta - 1)A] \geq 2m - \Delta(n - 1).$$

This with Lemma 1.3 implies that

$$\rho(G)^2 - (\Delta - 1)\rho(G) \geq 2m - \Delta(n - 1).$$

Solving this quadratic inequality, we obtain that if $\Delta < 2n - 1 - [(2n - 1)^2 - 8m - 1]^{1/2}$ then

$$\rho(G) \leq \frac{1}{2} (\Delta - 1 - \sqrt{(\Delta + 1)^2 + 4(2m - \Delta n)})$$

or

$$\rho(G) \geq \frac{1}{2} (\Delta - 1 + \sqrt{(\Delta + 1)^2 + 4(2m - \Delta n)}).$$

Either of the two equalities holds implies that all equalities in the argument must hold. As in the proof of Corollary 2.1, if $G$ is connected then $G$ is regular. Conversely, if $G$ is $d$-regular, then $\rho(G) = d$ attains the lower bound. This also implies that for connected graph $G$ the upper bound in (3) can never be attained. $\Box$

**Theorem 2.3.** Let $G = (V, E)$ be a connected graph with $m$ edges. Then

$$\rho(G) \geq \left[ \sum_{v \in V} \left( \sum_{uv \in E} \sqrt{d(u)} \right)^2 / (2m) \right]^{1/2}.$$ 

The equality holds if and only if $\sum_{uv \in E} \sqrt{d(u)/d(v)}$ is the same for all $v \in V$ or $G$ is a bipartite graph and $\sum_{uv \in E} \sqrt{d(u)/d(v)}$ is the same for all $v$ in the same part of $G$.

**Proof.** Let $A$ be the adjacency matrix of $G$ and $e = (\sqrt{d(v)}) / \sqrt{2m}$ a positive unit column vector. Then

$$Ac = \left( \sum_{uv \in E} \sqrt{d(u)} \right) / \sqrt{2m}.$$

Since $G$ is connected, $A$ is irreducible. The Perron–Frobenius Theorem implies that $\rho(G)$ is simple. By Rayleigh’s principle, we have that

$$\rho(G) \geq \left[ \sum_{v \in V} \left( \sum_{uv \in E} \sqrt{d(u)} \right)^2 / (2m) \right]^{1/2}.$$
If the equality holds, then \( \rho(A^2) = c^T A^2 c \). By Lemma 1.1, we have \( A^2 c = \rho(A^2) c \). If the multiplicity of \( \rho(A^2) \) is one, then \( c \) is also an eigenvector of \( \rho(G) \). This implies that
\[
\sum_{uv \in E} \sqrt{d(u)} = \rho(G) \sqrt{d(v)}, \text{ i.e. } \sum_{uv \in E} \sqrt{d(u)/d(v)} = \rho(G).
\]
Otherwise the multiplicity of \( \rho(A^2) = \rho(A)^2 \) is two. This implies that \( -\rho(G) \) is also an eigenvalue of \( G \). Then \( G \) is a connected bipartite graph (cf. Theorem 3.4 in [3]). Thus we may assume that
\[
A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},
\]
where \( B \) is an \( n_1 \times n_2 \) matrix with \( n_1 + n_2 = n \). We write correspondingly \( c = (c_1, c_2) \) where \( c_1 \) is an \( n_1 \)-vector and \( c_2 \) is an \( n_2 \)-vector. Since
\[
A^2 = \begin{pmatrix} B B^T & 0 \\ 0 & B^T B \end{pmatrix},
\]
we have
\[
B B^T c_1 = \rho(A^2) c_1 \text{ and } B^T B c_2 = \rho(A^2) c_2.
\]
Noting that \( B B^T \) and \( B^T B \) have the same nonzero eigenvalues, \( \rho(A^2) \) is the spectral radius of \( B B^T \) and its multiplicity is one. Since
\[
B B^T B c_2 = B \rho(A^2) c_2 = \rho(A^2) B c_2,
\]
we have that \( B c_2 \) is also an eigenvector of \( B B^T \) for \( \rho(A^2) \) and hence \( B c_2 = p_1 c_1 \) where \( p_1 \) is a scalar. This implies that for all \( v \) in the part of order \( n_1 \), we have
\[
\sum_{uv \in E} \sqrt{d(u)/d(v)} = p_1.
\]
Similarly we may obtain that \( B c_1 = p_2 c_2 \) where \( p_2 \) is a scalar. This implies that for all \( v \) in the part of order \( n_2 \), we have
\[
\sum_{uv \in E} \sqrt{d(u)/d(v)} = p_2.
\]
Conversely, if for all \( v \),
\[
\sum_{uv \in E} \sqrt{d(u)/d(v)} = p,
\]
then \( A c = p c \). It is well known that any positive eigenvector of a nonnegative matrix is of the spectral radius of the matrix. Hence
\[
\rho(G) = p = \left[ \sum_{v \in V} \left( \sum_{uv \in E} \sqrt{d(u)} \right)^2 / (2m) \right]^{1/2}.
\]
Now assume that \( G = (U, W; E) \) is a bipartite graph with \( |U| = n_1, |W| = n_2 \) and its adjacency matrix
\[
A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, \text{ where } B = (b_{uv}) \text{ is an } n_1 \times n_2 \text{ matrix}
\]
and for all $v \in U$,
\[
\sum_{uv \in E} \sqrt{d(u)/d(v)} = p_1;
\]
for all $v \in W$,
\[
\sum_{uv \in E} \sqrt{d(u)/d(v)} = p_2.
\]
Let $c_1 = (\sqrt{d(v)})_{v \in U}$ and $c_2 = (\sqrt{d(v)})_{v \in W}$. Then for each $v \in U$, the $v$th element of $BB^T c_1$ is
\[
rv(BB^T c_1) = \sum_{u \in U} \sum_{w \in W} b_{uw}b_{uv} \sqrt{d(u)} = \sum_{u \in U} b_{uw} \sum_{v \in U} b_{uv} \sqrt{d(u)}
\]
\[
= \sum_{u \in U} b_{uw} p_1 \sqrt{d(u)} = p_1 p_2 \sqrt{d(v)}.
\]
Similarly, $ru(B^T B c_2) = p_1 p_2 \sqrt{d(w)}$ for all $u \in W$. Hence $A^2 c = p_1 p_2 c$ where $c = \left( \frac{c_1}{c_2} \right) / \sqrt{2m}$.

Since any positive eigenvector of a nonnegative matrix is of the spectral radius of the matrix, we have $\rho(A^2) = p_1 p_2$.

**Remark.** Theorem 2.3 is analogous to Theorem 4 in [20] and Theorem 3.1 in [9]. A more general theorem of this type may be found in the recent paper [15].

### 2.2. Bounds of the Nordhaus–Gaddum type

Nordhaus and Gaddum [16] first studied the sum of the chromatic number of a graph $G$ and its complement $G^c$. Let $G$ be a graph of order $n$. Nosal [17] gave a sharp lower bound and an upper bound on the spectral radius $\rho(G)$ of adjacency matrix $A(G)$ of the Nordhaus–Gaddum type:

\[
n - 1 \leq \rho(G) + \rho(G^c) \leq \sqrt{2(n - 1)}.
\]

Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Li [11] proved that

\[
\rho(G) + \rho(G^c) \leq \sqrt{1 + 2n(n - 1) - 4\delta(n - 1 - \Delta) - 1}.
\]

Here, we give some sharp upper bounds on the spectral radius of the Nordhaus–Gaddum type for a connected graph $G$ and its connected complement $G^c$.

**Theorem 2.4.** Let $G$ be a graph of order $n$ with $0 < \delta(G) \leq \Delta(G) < n - 1$. Then

\[
\rho(G) + \rho(G^c) \leq \sqrt{2(n - 1)^2 - 2\delta n + 2\Delta \delta - \Delta + 3\delta}.
\]

Moreover, if both $G$ and $G^c$ are connected, then the equality holds if and only if $G$ is $(n - 1)/2$-regular.

**Proof.** Let $f(m, \Delta, \delta) = (2m - \delta n + \Delta \delta - \Delta + \delta)^{1/2}$. Note that $\Delta(G^c) = n - 1 - \delta, \delta(G^c) = n - 1 - \Delta$ and $m(G^c) = \left( \frac{n}{2} \right) - m$. Corollary 2.1 gives that
Now let \( g(m) = f(m, \Delta) + f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - \Delta \right) \). Then

\[
\rho(G) + \rho(G^c) \leq g(m).
\]

Since

\[
\frac{dg}{dm} = 1/f(m, \Delta, \delta) - 1/f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - \Delta \right),
\]

it is easy to check that \( \frac{dg}{dm} \geq 0 \) if and only if \( f(m, \Delta, \delta) \leq f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - \Delta \right) \)
i.e. \( m \leq \left\lfloor (n - 1)^2 + A + \delta \right\rfloor /4 \). Thus

\[
\rho(G) + \rho(G^c) \leq g \left( \left\lfloor (n - 1)^2 + A + \delta \right\rfloor /4 \right).
\]

If the sum of spectral radii attains the upper bound, then the spectral radii of \( G \) and \( G^c \) both attain their upper bounds and \( m = \left\lfloor (n - 1)^2 + A + \delta \right\rfloor /4 \). Now if both \( G \) and \( G^c \) are connected, then Corollary 2.1 implies that \( \Delta = \delta \). Thus

\[
2\delta n = (n - 1)^2 + 2\delta.
\]

This implies that \( \delta = (n - 1)/2 \) and hence \( G \) is \((n - 1)/2\)-regular. Conversely, if \( G \) is \((n - 1)/2\)-regular, then \( \rho(G) + \rho(G^c) = n - 1 \). □

**Theorem 2.5.** Let \( G \) be a graph of order \( n \). Then

\[
\rho(G) + \rho(G^c) \leq \left\lfloor n - A + \delta - 3 + \sqrt{2[(n - A)^2 + 4n(A - \delta) + (\delta + 1)^2]} \right\rfloor /2.
\]

Moreover, if both \( G \) and \( G^c \) are connected, then the equality holds if and only if \( G \) is \((n - 1)/2\)-regular.

**Proof.** Let \( f(m, \Delta, \delta) = (\delta + 1)^2 + 4(2m - \delta n) \). Note that \( A(G^c) = n - 1 - \delta, \delta(G^c) = n - 1 - A \) and \( m(G^c) = \left( \frac{n}{2} \right) - m \). Theorem 2.2 gives that

\[
\rho(G) \leq \left\lfloor \delta + 1 + f(m, \Delta, \delta) \right\rfloor /2
\]

and

\[
\rho(G^c) \leq \left\lfloor n - A - 2 + f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - A \right) \right\rfloor /2.
\]

Now let \( g(m) = f(m, \Delta, \delta) + f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - \Delta \right) \). Then

\[
\rho(G) + \rho(G^c) \leq \left\lfloor n - A + \delta - 3 + g(m) \right\rfloor /2.
\]

Since

\[
\frac{dg}{dm} = 4/f(m, \Delta, \delta) - 4/f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - \Delta \right),
\]

it is easy to check that \( \frac{dg}{dm} \geq 0 \) if and only if \( f(m, \Delta, \delta) \leq f \left( \left( \frac{n}{2} \right) - m, n - 1 - \delta, n - 1 - \Delta \right) \)
i.e. \( m \leq \left\lfloor (n - A)^2 + 4n(A + \delta) - (\delta + 1)^2 \right\rfloor/16 \). Thus
\[
\rho(G) + \rho(G^c) \leq \{n - \delta + \sqrt{2(n - \delta)^2 + 4n(A - \delta) + (\delta + 1)^2}/2 \}
\]

If the sum of spectral radii attains the upper bound, then the spectral radii of \( G \) and \( G^c \) both attain their upper bounds and 
\[
m = [(n - \delta)^2 + 4n(A + \delta) - (\delta + 1)^2]/16.
\]
Now if both \( G \) and \( G^c \) are connected, then Theorem 2.2 implies that \( A = \delta \). Thus
\[
8\delta n = (n - \delta)^2 + 8n - (\delta + 1)^2.
\]
This implies that \( \delta = (n - 1)/2 \) and hence \( G \) is \( (n - 1)/2 \)-regular. Conversely, if \( G \) is \( (n - 1)/2 \)-regular, then \( \rho(G) + \rho(G^c) = n - 1. \)

**Remark.** It is easy to see that our upper bounds are incomparable to the bounds of Nosal and Li. However, if \( A = o(n) \) or \( \delta = n - o(n) \), then Theorem 2.5 implies that \( \rho(G) + \rho(G^c) = O(\sqrt{2} + 1)n/2 \) which is better than the bounds \( O(\sqrt{2}n) \) of Nosal and Li and of Theorem 2.4.

### 3. The Laplacian spectral radius

In this section, we study the Laplacian spectral radius of graphs.

#### 3.1. Bounds on the Laplacian spectral radius

**3.1.1. Bounds by degrees**

In this section, we will state some known upper bounds on the Laplacian spectral radius of graphs and give the corresponding lower bounds for bipartite graphs. The simpler proofs for upper bounds are also stated. An upper bound for irregular graphs will be given in the end.

Let \( G \) be a graph with the degree diagonal matrix \( D(G) \) and the adjacency matrix \( A(G) \).

Let \( Q(G) = D(G) + A(G) \). Then \( Q \) is nonnegative and hence the Perron–Frobenius Theorem guarantees that the spectral radius \( \rho(Q) \) of \( Q \) is associated to a nonnegative eigenvector.

**Lemma 3.1** [23]. Let \( G \) be a graph. Then \( \mu(G) \leq \rho(Q) \). Moreover, if \( G \) is connected then the equality holds if and only if \( G \) is a bipartite graph.

**Theorem 3.1.** Let \( G = (V, E) \) be a graph. Then 
\[
\mu(G) \leq \sqrt{2} \max_{v \in V} \left( d(v)^2 + \sum_{uv \in E} d(u) \right)^{1/2}.
\]
Moreover, if \( G \) is connected then the equality holds if and only if \( G \) is bipartite and \( d(v)^2 + \sum_{uv \in E} d(u) \) is the same for all \( v \in V \). In particular, if \( G \) is bipartite then
\[
\mu(G) \geq \sqrt{2} \min_{v \in V} \left( d(v)^2 + \sum_{uv \in E} d(u) \right)^{1/2}.
\]
Moreover, if \( G \) is connected then the equality holds if and only if \( d(v)^2 + \sum_{uv \in E} d(u) \) is the same for all \( v \in V \).
Proof. Since $s_v(Q) = 2d(v)$ and $s_v(AD) = s_v(A^2) = \sum_{uv \in E} d(u)$, we have
\[ s_v(Q^2) = s_v(DQ + AD + A^2) = d(v)s_v(Q) + 2 \sum_{uv \in E} d(u) = 2d(v)^2 + 2 \sum_{uv \in E} d(u). \]

Lemma 3.1 and Lemma 1.3 imply that
\[ \mu(G) \leq \rho(Q) \leq \sqrt{2} \max_{v \in V} \left( d(v)^2 + \sum_{uv \in E} d(u) \right)^{1/2} \]
and for connected graph $G$, the equality holds if and only if $G$ is bipartite and $d(v)^2 + \sum_{uv \in E} d(u)$ is the same for all $v \in V$.

If $G$ is bipartite then $D - A$ and $D + A$ have the same eigenvalues and $D + A$ is a nonnegative irreducible symmetric matrix. The Perron–Frobenius Theorem and Lemma 1.3 imply that
\[ \mu(G) = \rho(Q) \geq \sqrt{2} \min_{v \in V} \left( d(v)^2 + \sum_{uv \in E} d(u) \right)^{1/2}, \]
and for connected graph $G$, the equality holds if and only if $d(v)^2 + \sum_{uv \in E} d(u)$ is the same for all $v \in V$. \qed

**Corollary 3.1.** Let $G$ be a graph with $n$ vertices, $m$ edges and no isolated vertex. Let $\Lambda = \Lambda(G)$ and $\delta = \delta(G)$. Then
\[ \mu(G) \leq [2\Lambda^2 + 4m - 2\delta(n - 1) + 2\Lambda(\delta - 1)]^{1/2}. \]
Moreover, if $G$ is connected then the equality holds if and only if $G$ is a regular bipartite graph. In particular, if $G$ is bipartite then
\[ \mu(G) \geq [2\delta^2 + 4m - 2\Lambda(n - 1) + 2\delta(\Lambda - 1)]^{1/2}. \]
Moreover, if $G$ is connected then the equality holds if and only if $G$ is regular.

The upper bound in Corollary 3.1 is due to Li et al. [10].

**Proof.** Theorem 3.1 gives that
\[
\mu(G) \leq \sqrt{2} \max_{v \in V(G)} \left( d(v)^2 + \sum_{uv \in E(G)} d(u) \right)^{1/2} \\
\leq \max_{v \in V(G)} \left[ 2d^2 + 2 \left( 2m - d(v) - \sum_{uv \in E(G)} d(u) \right) \right]^{1/2} \\
\leq \max_{v \in V(G)} \left[ 2\Lambda^2 + 4m - 2d(v) - 2(n - d(v) - 1)\delta \right]^{1/2} \\
= \max_{v \in V(G)} \left[ 2\Lambda^2 + 4m + 2(\delta - 1)d(v) - 2\delta(n - 1) \right]^{1/2} \\
\leq [2\Lambda^2 + 4m - 2\delta(n - 1) + 2\Lambda(\delta - 1)]^{1/2}.
\]
The equality holds if and only if all equalities in the argument hold. This implies that for connected graph $G$, $\mu(G)$ attains the upper bound if and only if $G$ is a regular bipartite graph.
Similarly, if $G$ is bipartite then we have
\[
\mu(G) \geq \sqrt{2} \min_{v \in V(G)} \left( d(v)^2 + \sum_{uv \in E(G)} d(u) \right)^{1/2}
\]
\[
\geq \min_{v \in V(G)} \left[ 2\Delta^2 + 2 \left( 2m - d(v) - \sum_{uv \in E(G)} d(u) \right) \right]^{1/2}
\]
\[
= \min_{v \in V(G)} \left[ 2\Delta^2 + 4m - 2d(v) - 2(n - d(v) - 1)A \right]^{1/2}
\]
\[
\geq [2\Delta^2 + 4m - 2A(n - 1) + 2\delta(A - 1)]^{1/2}.
\]

For connected graph $G$, the equality holds if and only if $G$ is regular.

**Theorem 3.2.** Let $G$ be a bipartite graph with $n$ vertices and $m$ edges. Let $\Lambda = \Lambda(G)$ and $\delta = \delta(G)$. Then
\[
\mu(G) \leq \left[ \Lambda + \delta - 1 - [\Lambda + \delta - 1]^2 + 8(2m - \Lambda n + A) \right]^{1/2} / 2
\]
(5) \ \text{or} \ \mu(G) \leq \left[ \Lambda + \delta - 1 + [\Lambda + \delta - 1]^2 + 8(2m - \delta n + \delta) \right]^{1/2} / 2.
(6)

Moreover, if $G$ is connected then the upper bound in (5) is strict, and either of the equalities in (6) holds if and only if $G$ is regular.

The upper bound in (6) of Theorem 3.2 is due to Liu et al. [13].

**Proof.** As in the proof of Theorem 3.1, we have
\[
s_v(Q^2) = 2d(v)^2 + 2 \sum_{uv \in E(G)} d(u)
\]
\[
= d(v)s_v(Q) + 2 \left( 2m - d(v) - \sum_{uv \in E(G)} d(u) \right)
\]
\[
\geq \delta s_v(Q) + 4m - 2d(v) - 2(n - d(v) - 1)A
\]
\[
= 4m + s_v(Q)(\Lambda + \delta - 1) - 2A(n - 1).
\]

Hence for each $v \in V(G)$, we obtain
\[
s_v(Q^2 - (\Lambda + \delta - 1)Q) \geq 4m - 2A(n - 1).
\]

Then Lemma 1.3 with $\mu(G) = \rho(Q)$ implies that
\[
\mu(G)^2 - (\Lambda + \delta - 1)\mu(G) \geq 4m - 2A(n - 1).
\]
Solving the quadratic inequality, we obtain
\[ \mu(G) \leq \frac{\{A + \delta - 1 - [(A + \delta - 1)^2 + 8(2m - A\gamma + \lambda)]^{1/2}\}}{2} \]
or
\[ \mu(G) \geq \frac{\{A + \delta - 1 + [(A + \delta - 1)^2 + 8(2m - A\gamma + \lambda)]^{1/2}\}}{2}. \]

Either of the above equalities holds implies that all equalities in the argument must hold. For connected graph \( G \), Lemma 1.3 implies that for all \( v \in V(G) \),
\[ \sum_{uv \in E(G)} d(u) = (n - d(v) - 1)A. \]
Hence either \( d(v) = n - 1 \) or \( d(u) = A \), for all \( u \) with \( uv \in E(G) \). This shows that the graph \( G \) is regular. Conversely, if \( G \) is \( d \)-regular then \( \mu(G) = 2d \) attains the lower bound. This also implies that for connected graph \( G \) the upper bound in (5) can never be attained.

**Theorem 3.3.** Let \( G = (V, E) \) be a connected bipartite graph with \( m \) edges. Then
\[ \mu(G) \geq \frac{\sum_{v \in V} (d(v)^{3/2} \sqrt{\sum_{uv \in E} \sqrt{d(u)}})^2}{(2m)}^{1/2}. \]
The equality holds if and only if \( G \) is regular.

**Proof.** Let \( A \) be the adjacency matrix of \( G \) and \( D \) be the diagonal matrix of degrees. Let \( Q = D + A \) and \( e = (\sqrt{d(v)})_v/\sqrt{2m} \) a positive unit column vector. Then
\[ Qe = (d(v)^{3/2} + \sum_{uv \in E} \sqrt{d(u)})_v/\sqrt{2m}. \]
Note that \( Q \) is nonnegative and irreducible. The Perron–Frobenius Theorem implies that \( \rho(Q) \) is simple. By Rayleigh’s principle, we have that
\[ \mu(G) = \rho(Q) \geq \left[ \sum_{v \in V} (d(v)^{3/2} + \sum_{uv \in E} \sqrt{d(u)})^2 / (2m) \right]^{1/2}. \]
If the equality holds, then \( \rho(Q^2) = e^TQ^2e \). By Lemma 1.1, we have \( Q^2e = \rho(Q^2)e \). Since \( Q \) is a positive semidefinite matrix and \( \rho(Q^2) = \rho(Q)^2 \), the multiplicity of \( \rho(Q^2) \) is one. Thus \( e \) is also an eigenvector of \( \rho(Q) \). This implies that for all \( v \in V \),
\[ d(v)^{3/2} + \sum_{uv \in E} \sqrt{d(u)} = \mu(G) \sqrt{d(v)}, \quad \text{i.e.} \ d(v) + \sum_{uv \in E} \sqrt{d(u)/d(v)} = \mu(G). \]
Then
\[ A + \sqrt{4\delta} \leq \mu(G) \leq \delta + \sqrt{4\delta}. \]
Thus \( A = \delta \) and hence \( G \) is regular.
Conversely, if \( G \) is \( d \)-regular then
\[ d(v)^{3/2} + \sum_{uv \in E} \sqrt{d(u)} = 2d^{3/2}. \]
It follows that
\[ \mu(G) = 2d = \left[ \sum_{v \in V} \left( d(v)^{3/2} + \sum_{uv \in E} \sqrt{d(u)} \right)^2 / (2m) \right]^{1/2}. \]

**Remark.** Theorem 3.3 is analogous to Theorem 9 in [20] which gives the following lower bound on the spectral radius of a bipartite connected graph \( G = (V, E) \):
\[ \mu(G) \geq \left[ \sum_{v \in V} \left( d(v)^2 + \sum_{uv \in E} d(u) \right)^2 / \sum_{v \in V} d(v)^2 \right]^{1/2}, \]
with equality if and only if \( \sum_{uv \in E} d(u)/d(v) \) is the same for all \( v \in V \) in the same part of \( G \).

However, it is easy to see that they are incomparable.

The spectral radius of a \( d \)-regular graph is \( d \) with \((1, 1, \ldots, 1)\) as its eigenvector. Recently, Stevanović [19], Zhang [22], as well as Liu and Shen [12] studied the spectral radius of irregular graphs. The current best result is due to Liu and Shen as follows: For an irregular graph \( G \) of order \( n \) with maximum degree \( \Delta \), \( \rho(G) \leq \Delta - (\Delta + 1)/[n(3n + 2\Delta - 4)]. \)

Similarly, the Laplacian spectral radius of a bipartite \( d \)-regular graph is precisely equal to \( 2d \).

We give an upper bound on the Laplacian spectral radius of irregular graphs analogous to that of the spectral radius.

**Theorem 3.4.** Let \( G \) be a connected irregular graph of order \( n \) with maximum degree \( \Delta \). Then \( \mu(G) < 2\Delta - 2/(2n^2 - n) \).

Since the diameter of a connected graph \( G \) satisfies \( d(G) < n \), Theorem 3.4 follows easily from the following result.

**Theorem 3.5.** Let \( G \) be a connected irregular graph of order \( n \) and diameter \( d \). Let \( \Lambda = \Lambda(G) \). Then
\[ \mu(G) < 2\Lambda - \frac{2}{(2d + 1)n}. \]

**Proof.** Let \( \mu = \mu(G) \), \( A = A(G) \), \( D = D(G) \) and \( Q = D + A \). By normalizing, we may assume that the positive eigenvector \( x \) of \( \rho(Q) \) is a unit vector. Let \( v_k \) and \( v_l \) be the vertices with \( x_k = \max_i x_i \) and \( x_l = \min_i x_i \) respectively. Then Lemma 3.1 implies that
\[ 2\Delta - \mu \geq 2\Delta - \rho(Q) = 2\Delta - x^T(D + A)x \]
\[ = 2 \sum_{i=1}^{n} (A - d_i)x_i^2 + \sum_{i=1}^{n} d_i x_i^2 - 2 \sum_{vi,vj \in E(G)} x_i x_j \]
\[ \geq 2x_l^2 + \sum_{vi,vj \in E(G)} (x_i - x_j)^2. \]
Let \( P \) be a shortest path of length \( a \leq d \) from \( v_k \) to \( v_l \). Then we have
\[
\sum_{v_iv_j \in E(G)} (x_i - x_j)^2 \geq \sum_{v_iv_j \in E(P)} (x_i - x_j)^2 \geq \left( \sum_{v_iv_j \in E(P)} (x_i - x_j) \right)^2 / a \geq (x_k - x_l)^2 / d.
\]
Thus by noting that \( x_k^2 \geq 1/n \) for \( G \) is irregular, we obtain
\[2A - \mu \geq 2x_k^2 + (x_k - x_l)^2 / d \geq 2x_k^2 / (2d + 1) > 2 / [(2d + 1)n].\]

The bound in Theorem 3.4 is asymptotically best possible when \( A \) is fixed. This is shown by the following example. The sum \( G_1 \sqcup G_2 \) of two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) is a graph with vertex set \( V_1 \times V_2 \) in which two vertices \((u_1, u_2)\) and \((v_1, v_2)\) are adjacent if and only if either \( u_1 = v_1 \) and \( u_2v_2 \in E_2 \) or \( u_1v_1 \in E_1 \) and \( u_2 = v_2 \). It is known \([1, 7]\) that the Laplacian spectrum of \((G_1 \sqcup G_2)\) consists of all possible sums \( \lambda(G_1) \) \( + \lambda(G_2) \) where \( \lambda(G_i) \) is in the Laplacian spectrum of \( G_i \) for \( i = 1, 2 \). In particular, \( \mu(G_1 \sqcup G_2) = \mu(G_1) + \mu(G_2) \). It is easy to see that a complete bipartite regular graph \( K_{\alpha, \beta} \) has the Laplacian spectral radius \( \mu(K_{\alpha, \beta}) = 2(\alpha - 2) \). It is also known \([18]\) that a path \( P_n \) of order \( n \) has the Laplacian spectral radius \( \mu(P_n) = 2 + 2 \cos(\pi/m) \). Then the Laplacian spectral radius of \( P_m \sqcup K_{\alpha, \beta} \) with \( n = 2(\alpha - 2)m \) vertices satisfies
\[
\mu(P_m \sqcup K_{\alpha, \beta}) = \mu(P_n) + \mu(K_{\alpha, \beta}) = 2 + 2 \cos(\pi/m) + 2(\alpha - 2) = 4 + 2 \sin^2(\pi/m).
\]

### 3.1.2. Bounds by covering numbers

Let \( G \) be a graph. A set of vertices \( C \) of \( G \) is called a cover of \( G \) if every edge of \( G \) is incident to some vertex in \( C \). The least cardinality of a cover of \( G \) is called the covering number of \( G \) and denoted by \( \tau(G) \). In order to give a lower bound for the Laplacian spectral radius of graphs in terms of the covering number, we first need a lemma due to Lu et al. \([14]\). The original form of this lemma is for connected graphs. However, the proof there applies also for disconnected graphs. Thus we have

**Lemma 3.2** \([14]\). Let \( G \) be a graph of order \( n \) and \( G_1 \) be an induced subgraph of \( G \) with \( n_1(n_1 < n) \) vertices and average degree \( r_1 \). Set \( d_1 = \sum_{v \in V(G_1)} d(v) / n_1 \). Then \( \mu(G) \geq n(d_1 - r_1) / (n - n_1) \).

The following theorem is an extension of results of Yuan \([21]\) and of Lu et al. (Corollary 11 in \([14]\)) which assert that \( \mu(G) \geq n / \tau(G) \).

**Theorem 3.6.** Let \( G \) be a graph of order \( n \) with minimum degree \( \delta \). Then
\[
\mu(G) \geq n \delta / \tau(G).
\]

**Proof.** Let \( C \) be a minimal covering set of \( G \) with \( |C| = \tau(G) \) and let \( m \) be the number of edges between \( C \) and \( V(G) \setminus C \). Then by the definition of covering set, we have \( m \geq \delta(n - \tau(G)) \).

Now Lemma 3.2 with the subgraph \( G_1 = G[C] \) induced by \( C \) implies that
\[\mu(G) \geq n(d_1 - r_1)/(n - \tau(G)) = n \left[ \sum_{v \in C} (d(v) - d_{G_1}(v)) \right] / [\tau(G)(n - \tau(G))] = mn / [\tau(G)(n - \tau(G))] \geq \delta n / \tau(G). \]

It is well known that \(\mu(G) \leq 2\Delta\). Thus by Theorem 3.6, we easily obtain a lower bound on the covering number of a graph.

**Corollary 3.2.** Let \(G\) be a graph of order \(n\) with maximum degree \(\Delta\) and minimum degree \(\delta\). Then
\[\tau(G) \geq \delta n / \mu(G) \geq \delta n / (2\Delta).\]

### 3.2. Bounds of the Nordhaus–Gaddum type

The well known bound \(\mu(G) \leq 2\Delta\) easily implies the simplest upper bound on the sum of Laplacian spectral radii of a graph \(G\) and its complement \(G^c\):
\[\mu(G) + \mu(G^c) \leq 2(n - 1) + 2(\Delta - \delta).\]
In particular, if both \(G\) and \(G^c\) are connected and irregular then Theorem 3.4 implies a slightly better upper bound as follows:
\[\mu(G) + \mu(G^c) \leq 2[n - 1 - 2/(2n^2 - n)] + 2(\Delta - \delta)\]

Recently, Liu et al. [13] proved that
\[\mu(G) + \mu(G^c) \leq n - 2 + \sqrt{(\Delta + \delta + 1 - n)^2 + n^2 + 4(\Delta - \delta)(n - 1)}.\]

Here we give another upper bound of the Nordhaus–Gaddum type.

**Theorem 3.7.** Let \(G\) be a graph of order \(n\) with \(0 < \delta(G) \leq \Delta(G) < n - 1\). Then
\[\mu(G) + \mu(G^c) \leq 2\sqrt{2(n - 1)^2 - 3\delta(n - 1) + (\Delta + \delta)^2 - \Delta + \delta}.\]
Moreover, if both \(G\) and \(G^c\) are connected then the upper bound is strict.

**Proof.** Let \(f(m, \Delta, \delta) = 2(\Delta^2 + 4m - 2\delta(n - 1) + 2\Delta(\delta - 1))^1/2.\) Note that \(\Delta(G^c) = n - 1 - \delta, \delta(G^c) = n - 1 - \Delta\) and \(m(G^c) = \binom{n}{2} - m\). Corollary 3.1 gives that
\[\mu(G) \leq f(m, \Delta, \delta) \quad \text{and} \quad \mu(G^c) \leq f \left( \binom{n}{2} - m, n - 1 - \delta, n - 1 - \Delta \right).\]

Now let \(g(m) = f(m, \Delta, \delta) + f \left( \binom{n}{2} - m, n - 1 - \delta, n - 1 - \Delta \right).\) Then
\[\mu(G) + \mu(G^c) \leq g(m).\]

Since
\[\frac{dg}{dm} = 2/f(m, \Delta, \delta) - 2/f \left( \binom{n}{2} - m, n - 1 - \delta, n - 1 - \Delta \right),\]
it is easy to check that \(\frac{dg}{dm} \geq 0\) if and only if \(f(m, \Delta, \delta) \leq f \left( \binom{n}{2} - m, n - 1 - \delta, n - 1 - \Delta \right)\)
i.e. \(m \leq [2(n - 1)^2 - \delta(n - 2) - \Delta^2 + \delta^2 + \Delta^1]/4). Thus
\[ \mu(G) + \mu(G^c) \leq g(2(n-1)^2 - \delta(n-2) - A^2 + \delta^2 + A/4) \\
= 2f(2(n-1)^2 - \delta(n-2) - A^2 + \delta^2 + A/4, A, \delta) \\
= 2\sqrt{2(n-1)^2 - 3\delta(n-1) + (A + \delta)^2 - A + \delta}. \]

If both \( G \) and \( G^c \) are connected, then either \( G \) or \( G^c \) fails to be a bipartite regular graph. Corollary 3.1 implies that the Laplacian spectral radius of either \( G \) or \( G^c \) fails to attain its upper bound and so does the sum. \( \square \)

It is easy to see that the bound in Theorem 3.7 is incomparable to the known bounds listed above.

Acknowledgments

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References