Risk-averse decision making in overbooking problem

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Abstract

Traditional literature studying overbooking problems focuses on risk-neutral decision makers. In this paper, we propose a multi-period overbooking model incorporating risk-aversion and extend well-known structural results (the 3-region policy) under the risk-neutral case to the risk-averse one on the basis of an exponential utility function. We also show that the optimal policy for the risk-neutral decision maker can be obtained by letting the risk-aversion parameter approach to zero under the risk-averse case. Therefore, the extant results under the risk-neutral case can be interpreted as a special case of ours. We also investigate how the optimal policy changes with some cost parameters and the decision maker's degree of risk-aversion. Numerical results suggest that the optimal bounds in the 3-region policy may increase or decrease with the decision maker's degree of risk-aversion.

Keywords: revenue management; risk-averse; overbooking; marketing; inventory

Introduction

The idea of revenue management (RM) in general is to increase revenues by more effective pricing and allocation of service capacity (see Chiang *et al*, 2007 for a review of recent researches on RM). Traditional literature on RM focuses on risk-neutral decision makers, who make decisions to maximize the expected profit or minimize the expected cost. Evidently, not all the decision makers are risk-neutral since many are willing to avoid possible huge losses at the expense of some profits. According to Schweitzer and Cachon (2000), for the so-called high-profit products, decision makers exhibit risk-averse behavior. Recently, there is an

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increasing interest on incorporating risk-aversion into models on RM. Feng and Xiao (1999, 2008), Koenig and Meissner (2008a), Levin *et al* (2008) and Lim and Shanthikumar (2007) focused on pricing decisions for the risk-averse decision makers. Chen *et al* (2007) introduced risk-aversion into inventory models, in which other issues, such as pricing, hedging risks with financial tools, had also been addressed. By a mean-variance analysis, Mitra and Wang (2005) investigated network revenue management in a specific traffic engineering model for bandwidth provisioning and route selection. Barz and Waldmann (2007) extended well-known results in classic static and dynamic capacity control problems from the perspective of a risk-averse decision maker with an exponential utility function, but they didn’t consider customer cancellations or no-shows. Barz (2007) also analyzed capacity control models with additive time-separable and atemporal utility functions. Barz (2006), Huang and Chang (2008), Koenig and Meissner (2008b) and Weatherford (2004) studied capacity control or seat inventory control problems with risk-averse decision makers by simulation.

In the field of RM, overbooking is a classic and important topic. As estimated in Smith *et al* (1992), approximately 50% of all reservations in the airline industry resulted in cancellations or no-shows, and 15% of seats on sold-out flights would be lost if overbooking were not practised. In literature, overbooking problems have been studied extensively (see Liberman and Yechiali, 1978; Chatwin, 1999 and Karaesmen and Ryzin, 2004 for some examples). However, although there is literature on overbooking problem that moves away from the risk-neutrality assumption (e.g. Lan *et al*, 2008 investigated overbooking problem with robust optimisation concept), to our knowledge, sparse literature considers the overbooking problem with a risk-averse decision maker. In this paper, we propose a multi-period overbooking model in which the decision maker is risk-averse with an exponential utility function. A critical structural policy, i.e., the 3-region policy (please refer to Section ‘Model’ for details), which is first derived in Liberman and Yechiali (1978) for the risk-neutral decision maker, is proved to be also optimal for the risk-averse decision maker. It is also shown that the optimal policy for the risk-neutral decision maker can be obtained by letting the risk-aversion parameter approach to zero under the risk-averse case. Therefore, the results of Liberman and Yechiali (1978) under the risk-neutral case can be interpreted as a special case of ours. In addition, we investigate how the optimal bounds in the 3-region policy change with some cost parameters and the decision maker’s degree of risk-aversion. Numerical results demonstrate that the bounds in the 3-region policy may increase or decrease as the decision maker's degree of risk-aversion increases.

The rest of the paper is organized as follows. In the next section, the model and the assumptions are presented. In Section 'Single-Period case', we derive the optimal policy for the risk-averse decision maker, characterize the relationship between the risk-averse and risk-neutral models and discuss how the optimal policy changes with some cost parameters and the decision maker's degree of risk-aversion based on singe-period assumption. In Section 'Multi-Period case', we generalize the results in the previous section to the multi-period case. In the final section, some discussions are conducted and the conclusions are summarized. All the proofs are included in 'Appendix'.

Model

Consider the similar overbooking problem as in Liberman and Yechiali (1978), faced by a hotel manager with all rooms being identical (the overbooking problem faced by the airline company is similar for a specified class of seats). Let M be the number of rooms available on *the target day*, $t_0=0$. The hotel begins its booking T days from t_0. The booking horizon, i.e., the time interval between T and t_0, is divided into n periods by intermediate time points:

$$T = t_n > t_{n-1} > \ldots > t_1 > t_0=0.$$

The time interval $[t_i, t_{i-1})$ is called period i ($i=n$, $n-1, \ldots$, 1). The events which happen sequentially in period i ($i=n$, $n-1, \ldots$, 1) are as follows (See Figure 1 for a graphical description):

1) At time t_i, the hotel management reviews both the inventory level X_i (which is the number of uncancelled reservations previously confirmed; denote $X_n = 0$) and the total number of not-yet-confirmed new requests in period $i+1$, Y_{i+1} (denote $Y_{n+1}=0$). Then decisions are made to increase or decrease the inventory level to Z_i by taking the following actions:
   (a) Confirming some new requests from Y_{i+1} without any cost;
   (b) Cancelling some confirmed reservations from X_i with a cost of b_i per unit;
   (c) Selling some capacities to some agencies (e.g. Hotwire) at a lower price or attracting some customers to reserve rooms by advertising. We refer to these actions as *acquiring*

additional reservations and refer to both the revenue loss caused by the lower selling price and the cost generated by advertising as *acquiring cost* (c_i per unit).

Actions (a) and (c) are taken to increase the inventory level, while action (b) is taken to decrease the inventory level. Obviously, these two types of actions cannot be taken at the same time, and action (c) is only possible to be taken after the new requests Y_{i+1} are completely confirmed by action (a).

2) At time $t_i - 0$, i.e., the time immediately after the hotel management takes actions at t_i, customer cancellations happen. The inventory level falls to $X_{i-1} = P_i Z_i$, where P_i is the fraction of uncancelled reservations.
3) During the time interval (t_i, t_{i-1}), new requests for rooms, Y_i, arrive.

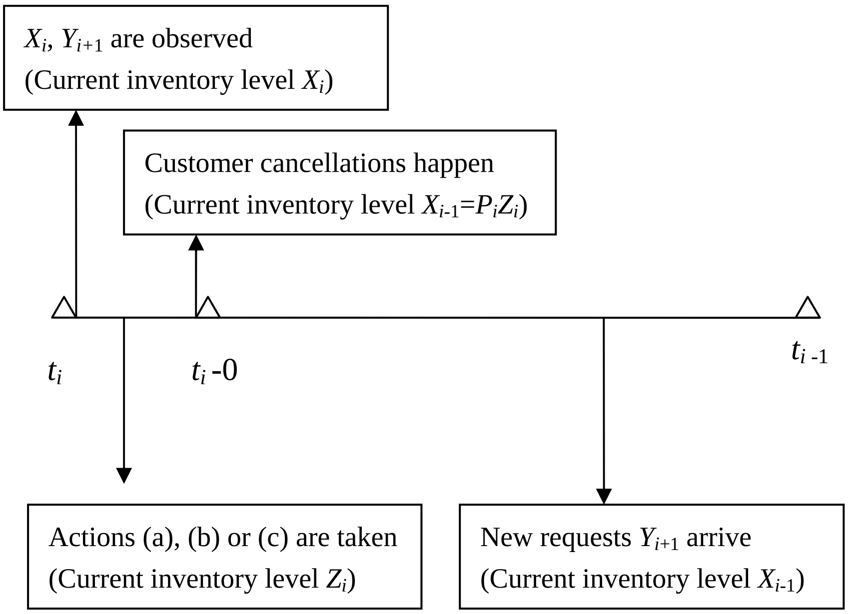

Figure 1. Sequence of events and inventory level in period i.

Finally, at time t_0, if current inventory level is lower than M, the hotel management confirms some new requests from Y_1 provided that the inventory level after the confirmation doesn't exceed M; if current inventory level is higher than M, the hotel management decreases it to M by cancelling some of the confirmed reservations. At this time, a penalty cost b_0 is incurred for cancelling a confirmed reservation and a profit a_0 is made for each occupied room. We assume that at time t_0, no additional reservations can be acquired (i.e., action (c) is not available at this time), and no customer cancellations happen.

It has been proved in Liberman and Yechiali (1978) that the optimal booking policy for the risk-neutral decision maker is the 3-region policy, which is characterized by three sets of bounds Z_i^*, U_i^* and V_i^* ($U_i^* \le Z_i^* \le V_i^*$, $i = 1,2,...,n$):

(i) if $X_i + Y_{i+1} < U_i^*$, all Y_{i+1} new requests should be confirmed and $U_i^* - X_i - Y_{i+1}$ additional reservations should be acquired;

(ii) if $U_i^* \le X_i + Y_{i+1} \le Z_i^*$, all Y_i new requests should be confirmed;

(iii) if $Z_i^* < X_i + Y_{i+1}$ and $X_i \le V_i^*$, $\max\{0, Z_i^* - X_i\}$ new requests should be confirmed;

(iv) if $V_i^* < X_i$, $X_i - V_i^*$ confirmed reservations should be cancelled.

Rather than the risk-neutral decision maker in Liberman and Yechiali (1978), we consider the risk-averse decision maker of the hotel management with an exponential utility function as

$$u(x) = -\exp(-rx),$$

where $r > 0$ is the risk-aversion parameter reflecting the degree of risk-aversion (larger r stands for higher degree of risk-aversion). According to Howard (1988) and Kirkwood (2004), exponential utility functions "satisfactorily treat a wide range of individual and corporate risk preferences" and are in most cases very good approximations for general utility functions. Although the exponential utility doesn't produce linear utility directly, the expected exponential utility and the expected profit are related as follows (Feng and Xiao, 2008): Suppose that Π is the (random) profit, $W = E(-\exp(-r\Pi))$ is the expected utility and $R = E(\Pi)$ is the expected profit. Since

$$\lim_{r \to 0} -\frac{1}{r}\log(-W) = E\Pi = R, \tag{1}$$

the risk-neutral case can be regarded as a special case of the risk-averse one considered in our paper.

Notation.

a_0: The profit from an occupied room at time $t_0=0$, $a_0>0$.

c_i: The cost of acquiring an additional reservation at time t_i, $c_i>0$.

b_i: The penalty for cancelling a confirmed reservation at time t_i, $b_i>0$, ($i=0,1,\dots n$).

Z_i: The inventory level of confirmed reservations at time t_i, immediately after the hotel management takes actions according to its decisions.

X_i: The inventory level during the time interval (t_{i+1}, t_i), i.e., between the times immediately after customer cancellations of period $i+1$ and just before actions taken by the hotel management in period i.

Y_i: The number of new requests for rooms accumulated during period i. Y_i is random at time t_i but is known at time t_{i-1}.

$Q_i(\cdot)$: The CDF of Y_i.

P_i: A positive random variable (with mean $E_{P_i}(P_i)$) defined over [0,1] such that $X_{i-1} = P_i Z_i$.

$F_i(\cdot)$: The CDF of P_i.

$G_{i,r}(Z)$: The expected utility over the time interval $(t_i, t_0]$ with the risk-aversion parameter r, starting with inventory level Z and following an optimal strategy (for the risk-averse case).

$G_i(Z)$: The expected profit over the time interval $(t_i, t_0]$, starting with inventory level Z and following an optimal strategy (for the risk-neutral case).

Without loss of generality, we assume that the initial wealth for the hotel is 0. For ease of representation, we assume that Z_i, X_i and Y_i are continuous variables. The results for discrete variables can also be derived. The main purposes of this paper are deriving the optimal booking policy for the risk-averse decision maker and compare it with that for the risk-neutral decision maker.

Singe-Period case

In this section, we will derive the optimal policy for the risk-averse decision maker under the single-period case (i.e., the case of n=1 in the model described in the previous section) and characterize the relationship between the risk-averse and risk-neutral models. How the optimal policy changes with some cost parameters and the decision maker's degree of risk-aversion will also be discussed.

Suppose that Z is the inventory level at time t_1, immediately after the hotel management takes actions. For given realizations of P_1 and Y_1, the profit of the hotel is

$$\pi(Z,P_1,Y_1)=\begin{cases} a_0(P_1Z+Y_1), \text{ if } P_1Z<M-Y_1, \\ a_0M, \text{ if } M-Y_1\le P_1Z<M, \\ a_0M-b_0(P_1Z-M), \text{ if } P_1Z\ge M. \end{cases} \tag{2}$$

Thus, for any realization of Y_1, taking expectation of $-\exp\left(-r\pi(Z,P_1,Y_1)\right)$ with respect to P_1, we get a function of Z and Y_1:

$$\begin{aligned} G_{1,r}(Z,Y_1) &= E_{P_1}\left[-\exp\left(-r\pi(Z,P_1,Y_1)\right)\right] \\ &= -\int_0^{\min\{(M-Y_1)/Z,1\}}\exp\left(-ra_0(pZ+Y_1)\right)dF_1(p)-\int_{\min\{(M-Y_1)/Z,1\}}^{\min\{M/Z,1\}}\exp(-ra_0M)dF_1(p) \\ &\quad -\int_{\min\{M/Z,1\}}^{1}\exp\left(-r(a_0+b_0)M+rb_0pZ\right)dF_1(p). \end{aligned}$$

Therefore, the expected utility of the decision maker over the time interval $(t_1,t_0]$ is $G_{1,r}(Z)=E_{Y_1}G_{1,r}(Z,Y_1)=E_{Y_1}E_{P_1}\left[-\exp\left(-r\pi(Z,P_1,Y_1)\right)\right]$.

Lemma 1. *(a)* $G_{1,r}(Z)$ *is a concave function of Z.*

(b) For any real number e, $e^2G_{1,r}(Z)+2eG_{1,r}^{'}(Z)+G_{1,r}^{''}(Z)<0$.

Proof. The proof of this Lemma, as well as those for other lemmas, propositions and theorems, is provided in Appendix.

Remark 1. From Eqn. (1), we have that the expected profit over the time interval $(t_1,t_0]$ is

$$G_1(Z)=\lim_{r\to 0}-\left(1/r\right)\log\left(-G_{1,r}(Z)\right). \tag{3}$$

This fact, together with Part (a) of Lemma 1, indicates $G_1(Z)$ is also concave, which is consistent with the result in Lemma 1 of Liberman and Yechiali (1978).

Remark 2. Part (b) of Lemma 1, which doesn't appear under the risk-neutral case in Liberman and Yechiali (1978), is a critical property for deriving the optimal policy for the risk-averse decision maker.

As mentioned in the previous section, it has been proved in Liberman and Yechiali (1978) that the optimal policy for the risk-neutral decision maker is the 3-region policy. An interesting question is whether this structural policy remains to be optimal for the risk-averse decision maker.

The next theorem provides a positive answer.

Theorem 1. *Assume the decision maker is risk-averse with an exponential utility* $u(x) = -\exp(-rx)$, $c_1 < a_0 E(P_1)$ *and* $b_1 < b_0 E(P_1)$. *Suppose that* X_1 *be the inventory level at time* t_1 *and* Y_2 *be the new requests during the time interval* (t_2, t_1).

(a) There exist numbers $Z^*_{1,r}$, $U^*_{1,r}$ *and* $V^*_{1,r}$ *independent of* X_1 *and* Y_2 , *such that* $U^*_{1,r} \le Z^*_{1,r} \le V^*_{1,r}$, *and that the optimal policy is as follows:*

(i) if $X_1 + Y_2 < U^*_{1,r}$, *all* Y_2 *new requests should be confirmed and* $U^*_{1,r} - X_1 - Y_2$ *additional reservations should be acquired;*

(ii) if $U^*_{1,r} \le X_1 + Y_2 \le Z^*_{1,r}$, *all* Y_2 *new requests should be confirmed;*

(iii) if $Z^*_{1,r} < X_1 + Y_2$ *and* $X_1 \le V^*_{1,r}$, $\max\{0, Z^*_{1,r} - X_1\}$ *new requests should be confirmed;*

(iv) if $V^*_{1,r} < X_1$, $X_1 - V^*_{1,r}$ *confirmed reservations should be cancelled.*

(b) The optimal numbers $Z^*_{1,r}$, $U^*_{1,r}$ *and* $V^*_{1,r}$ *satisfy* $G'_{1,r}(Z^*_{1,r}) = 0$, $rc_1 G_{1,r}(U^*_{1,r}) + G'_{1,r}(U^*_{1,r}) = 0$ *and* $-rb_1 G_{1,r}(V^*_{1,r}) + G'_{1,r}(V^*_{1,r}) = 0$ *respectively.*

Remark 3. Part (a) of Theorem 1 remains true even if the assumptions $c_1 < a_0 E(P_1)$ and $b_1 < b_0 E(P_1)$ fail to hold. If $c_1 \ge a_0 E(P_1)$, one only needs to set $U^*_{1,r}$ to 0. If $b_1 \ge b_0 E(P_1)$, one only needs to set $V^*_{1,r}$ to $+\infty$.

Theorem 1, together with Theorem 2 in Liberman and Yechiali (1978), indicates that both the risk-averse and risk-neutral decision makers have the same type of optimal policy: the 3-region policy, which is characterized by three bounds: $Z^*_{1,r}$ (or Z^*_1), $U^*_{1,r}$ (or U^*_1) and $V^*_{1,r}$ (or V^*_1). However, the values of the bounds are different for different risk attitudes (risk-averse or risk-neutral). How are these bounds related under risk-averse and risk-neutral cases? The answer is provided in Theorem 2, which is based on the following lemma.

Lemma 2. *Suppose that for any* ξ_0, $H(x,\xi)$ *uniformly (for all* $x \in [l_1, l_2]$*) converges to* $H(x,\xi_0)$ *as* ξ *approaches to* ξ_0, *and* $h(x)$ *has a continuous inverse function* $h^{-1}(x)$. *Define*

$x^*(\xi)$ *such that* $H\left(x^*(\xi),\xi\right)=0$ *and* x_0^* *such that* $h(x_0^*)=0$. *If* $\lim_{\xi\to 0} H(x,\xi)=h(x)$ *for any* $x\in[l_1,l_2]$, *then* $\lim_{\xi\to 0} x^*(\xi)=x_0^*$.

The uniform convergence property of $H(x,\xi)$ in Lemma 2 means that for any small $\varepsilon>0$, there exists a common $\delta(\varepsilon)>0$ for all $x\in[l_1,l_2]$, such that $\left|H(x,\xi)-H(x,\xi_0)\right|<\varepsilon$ for any $\xi\in\left(\xi_0-\delta(\varepsilon),\xi_0+\delta(\varepsilon)\right)$. This is a condition that is stronger than pointwise convergence property.

Theorem 2. *Suppose* $Z_{1,r}^*$, $U_{1,r}^*$ *and* $V_{1,r}^*$ *are the optimal bounds for the risk-averse decision maker with r>0, and* Z_1^*, U_1^* *and* V_1^* *are the optimal bounds for the risk-neutral decision maker. Then* $\lim_{r\to 0} Z_{1,r}^*=Z_1^*$, $\lim_{r\to 0} U_{1,r}^*=U_1^*$, $\lim_{r\to 0} V_{1,r}^*=V_1^*$.

Theorem 2 states that the optimal bounds under the risk-neutral case can be obtained by letting the risk-aversion parameter approach to zero under the risk-averse case. This indicates that the extant result of the risk-neutral case can be interpreted as a special case of ours. The next propositions investigate how the optimal policy of the risk-averse decision maker changes with some model parameters.

Lemma 3. *Suppose* $H(x,\xi)$ *is decreasing in x. Define* $x^*(\xi)$ *such that* $H\left(x^*(\xi),\xi\right)=0$. *If* $H(x,\xi)$ *is increasing (decreasing) in* ξ, *then* $x^*(\xi)$ *is increasing (decreasing) in* ξ.

Proposition 1. *(a)* $Z_{1,r}^*$ *is decreasing with respect to* b_0*; (b)* $U_{1,r}^*$ *is decreasing with respect to* c_1*; (c)* $V_{1,r}^*$ *is increasing with respect to* b_1.

Next we are concerned about the influence of the decision maker’s degree of risk-aversion on the optimal policy. As stated in Pratt (1964), increase of the degree of risk-aversion is represented by an increasing concave transformation (in expected utility theory). Therefore, an exponential utility reflects a higher degree of risk-aversion than a linear utility (risk-neutral) and an exponential utility with larger *r* reflects higher degree of risk-aversion than that with a smaller *r*. In Eeckhoudt *et al* (1995), the authors investigated how the optimal ordering quantity changed with the decision maker’s degree of risk-aversion in a newsvendor model. They found that the order quantity of a newsvendor became less when the decision maker’s degree of risk-aversion increased. Surprisingly, in our model, the similar conclusion doesn't hold any more.

Example 1. Consider the case that $\Pr\{Y_1 = 0\} = 1$, $a_0 = 10$, $b_0 = 20$, $b_1 = 5$, $c_1 = 3$ $M = 100$. Numerical calculations are conducted under three different risk attitudes (risk-neutral, $r_1 = 0.01$, $r_2 = 0.02$), and two different CDFs of P_1 (beta distribution with parameters 10, 1 and truncated norm distribution with mean 0.7 and standard deviation 0.1). The results are listed in Table 1.

CDFs of P_1	Beta(10,1)			Truncated Norm(0.7,0.1)		
risk attitudes	risk-neutral	r=0.01	r=0.02	risk-neutral	r=0.01	r=0.02
U_1^*	102	104	108	125	124	123
Z_1^*	104	106	111	132	128	126
V_1^*	107	108	113	144	132	128

Table 1. Bounds in the optimal policy v.s. risk attitudes under the single-period case

From the example above, we have that the optimal bounds in the 3-region policy may increase or decrease as the degree of risk-aversion increases. How the bounds change with the degree of risk-aversion here is a rather complicated problem. It may depend on interactions of several factors, such as cost parameters (a_0, b_0, b_1, c_1), and distributions of P_1 and Y_1. Next we provide a set of sufficient conditions (in Proposition 2) under which $Z_{1,r}^*$ is decreasing with the decision maker's degree of risk-aversion.

Proposition 2. *Denote the PDF of the continuous random variable* P_1 *by* $f_1(p)$. *Then* $Z_{1,r}^*$ *is decreasing with respect to r, if the following conditions hold:* $r \in (0, 1/a_0 M]$, $b_0 \geq a_0 / [(1 - r a_0 M) e^{-r a_0 M}]$ *and* $p f_1(p)$ *is increasing with respect to p;*

Remark 4. Under the set of conditions in Proposition 2, one can easily have $Z_{1,r}^* \leq Z_1^*$ for any $r \in (0, 1/a_0 M]$ by Theorem 2. Clearly, there exist parameters and the CDFs of P_1 satisfying this set of conditions.

Multi-Period case

In this section, we are trying to generalize the results in the previous section to the multi-period case. Theorems 1 and 2 can be proved to remain true for *n*-period case (see Theorems 3 and 4). However, under the multi-period case, it is a challenging task to find a set of sufficient conditions as in Proposition 2 for the optimal bounds being monotone with the decision maker's degree of risk-aversion. We will leave it for the future research.

Theorem 3. *Assume the decision maker is risk-averse with an exponential utility* $u(x) = -\exp(-rx)$,

$$c_1 < a_0 E_{P_1}(P_1),\ \ c_{i+1} < c_i E_{P_{i+1}}(P_{i+1}),\ \ i = 1,2,...,n-1, \tag{4}$$

$$b_{i+1} < b_i E_{P_{i+1}}(P_{i+1}),\ \ i = 0,1,...n-1. \tag{5}$$

(a) For $i = 1,2,...,n$, *there exist numbers* $Z^*_{i,r}$, $U^*_{i,r}$ *and* $V^*_{i,r}$ *such that* $U^*_{i,r} \le Z^*_{i,r} \le V^*_{i,r}$, *and that the optimal policy is as follows:*

(i) if $X_i + Y_{i+1} < U^*_{i,r}$, *all* Y_{i+1} *new requests should be confirmed and* $U^*_{i,r} - X_i - Y_{i+1}$ *additional reservations should be acquired;*

(ii) if $U^*_{i,r} \le X_i + Y_{i+1} \le Z^*_{i,r}$, *all* Y_i *new requests should be confirmed;*

(iii) if $Z^*_{i,r} < X_i + Y_{i+1}$ *and* $X_i \le V^*_{i,r}$, $\max\{0, Z^*_{i,r} - X_i\}$ *new requests should be confirmed;*

(iv) if $V^*_{i,r} < X_i$, $X_i - V^*_{i,r}$ *confirmed reservations should be cancelled.*

(b) Define $G_{i,r}(Z)$, *i*=2, 3, … , n, *recursively as the expected utility over the time interval* $(t_i, t_0]$ *if inventory level at time* t_i *is Z and the optimal policy stated as claim (a) is adopted over the time interval* $(t_{i-1}, t_0]$. *Then the numbers* $Z^*_{i,r}$, $U^*_{i,r}$ *and* $V^*_{i,r}$ *satisfy* $G'_{i,r}(Z^*_{i,r}) = 0$, $rc_i G_{i,r}(U^*_{i,r}) + G'_{i,r}(U^*_{i,r}) = 0$ *and* $-rb_i G_{i,r}(V^*_{i,r}) + G'_{i,r}(V^*_{i,r}) = 0$ *respectively.*

Remark 5. It can be easily proved that Part (a) of Theorem 3 remains true even if the assumptions (4) and (5) fail to hold. If $c_{i+1} \ge c_i E_{P_{i+1}}(P_{i+1})$ for some $i \in \{n-1, n-2, \cdots, 1\}$ (or $c_1 \ge a_0 E_{P_1}(P_1)$ for $i = 0$), one only needs to set the corresponding $U^*_{i+1,r}$ to 0. Similarly, if $b_{i+1} \ge b_i E_{P_{i+1}}(P_{i+1})$ for some $i \in \{n-1, n-2, \cdots, 0\}$, one only needs to set the corresponding $V^*_{i+1,r}$ to $+\infty$.

Theorem 4. *For any* i=1, 2, …, *n*,

(a) Suppose $Z_{i,r}^*$, $U_{i,r}^*$ *and* $V_{i,r}^*$ *are the optimal bounds of the risk-averse decision maker with r>0 at time* t_i, *and* Z_i^*, U_i^* *and* V_i^* *are the optimal bounds of the risk-neutral decision maker at time* t_i. *Then* $\lim_{r\to 0} Z_{i,r}^* = Z_i^*$, $\lim_{r\to 0} U_{i,r}^* = U_i^*$, $\lim_{r\to 0} V_{i,r}^* = V_i^*$.

(b) $G_i\,(Z) = \lim_{r\to 0} -\frac{1}{r}\log\left[-G_{i,r}\,(Z)\right]$.

Theorem 3 states that the well-known 3-region policy remains optimal for the risk-averse decision maker, even under the multi-period case. Theorem 4 characterizes the relationship between our model and the risk-neutral model and further convinces us that the extant results in the risk-neutral model can be regarded as a special case of ours.

It can be observed in Liberman and Yechiali (1978) that for the risk-neutral decision maker, the bound U_i^* in the optimal policy is decreasing in the acquiring cost c_i, while V_i^* is increasing in the penalty cost b_i (i=1, 2, …, n). Next we provide a proposition which demonstrates that the similar properties also hold for the risk-averse decision maker.

Proposition 3. *For any* $i = 1,2,...,n$, $U_{i,r}^*$ *is decreasing in* c_i, *and* $V_{i,r}^*$ *is increasing in* b_i.

Proposition 3 characterizes the monotonic relationship between the optimal bounds in the 3-region policy and some of the cost parameters. However, how the bounds are affected by the other cost parameters is a challenging problem and we will consider it in the future research.

Discussions and conclusions

In this paper, we propose a multi-period overbooking model in which the decision maker is risk-averse with an exponential utility function. We prove that the 3-region policy, which is shown to be optimal for the risk-neutral decision maker, is also optimal for the risk-averse decision maker. We also show that the optimal bounds in the 3-region policy for the risk-neutral decision maker can be obtained by letting the risk-aversion parameter approach to zero under the risk-averse case. Therefore, the extant results under the risk-neutral case can be interpreted as a special case of ours. Besides, we discuss how the optimal policy changes with the decision maker's degree of risk-aversion and some cost parameters. Numerical results demonstrate that the bounds in the 3-region policy may increase or decrease when the decision maker's degree of risk-aversion increases. We identify a set of sufficient conditions under which the optimal bound $Z_{1,r}^*$ is decreasing with the decision maker's degree of risk-aversion. It is also proved that for each period (period i, i=1,2,…,n), the optimal lower bound $U_{i,r}^*$ in the 3-region policy is decreasing in the cost of acquiring an additional reservation c_i, while the optimal upper bound $V_{i,r}^*$ is increasing in the penalty cost of cancelling a confirmed reservation b_i.

In practice, service level constraints are usually used to avoid large losses. It is actually another concept that also reflects the risk attitude of the decision maker. Next we will compare this concept with that of exponential utility functions used in this paper. Consider the service level constraint that the probability of more than m customers with confirmed reservations being cancelled at time t_0=0 should not exceed $\alpha \in [0,1]$ (similar service level constraints are also considered in literature, e.g., Shlifer and Vardi, 1975). This constraint requires that the probability of event $\{P_1 Z \geq M + m\}$ should not be greater than α, where Z stands for the inventory level at time t_1 (the time before customer cancellation and just after the hotel management takes actions). By simple calculations, we have $Z \leq (M+m)/F_1^{-1}(1-\alpha)$, where $F_1^{-1}(\cdot)$ is the inverse function of the CDF of P_1. Let $\overline{Z}(m,\alpha) = (M+m)/F_1^{-1}(1-\alpha)$. Then the following result holds (a proof is provided in Appendix: Proof D1): for any m>0, $0 \leq \alpha \leq 1$, there exists a $\overline{b}_0$>0 such that the optimal bound $Z_{1,r}^*$ corresponding to $\overline{b}_0$ is equal to $\overline{Z}(m,\alpha)$. This result indicates that the service level constraint can be incorporated by properly setting the value of b_0.

It is also proved (a proof is provided in Appendix: Proof D2) that with the service level constraint, the 3-region policy is still optimal for the single-period model when the decision maker is risk-neutral (the same result for the risk-averse decision maker is also true) and the new bounds are

$$\tilde{U}_1^* = \min\{U_1^*, (M+m)/F_1^{-1}(1-\alpha)\},$$

$$\tilde{Z}_1^* = \min\{Z_1^*, (M+m)/F_1^{-1}(1-\alpha)\},$$

$$\tilde{V}_1^* = \min\{V_1^*, (M+m)/F_1^{-1}(1-\alpha)\}.$$

Clearly, $\tilde{U}_1^* \le U_1^*$, $\tilde{Z}_1^* \le Z_1^*$, $\tilde{V}_1^* \le V_1^*$, which indicate that the decision maker would always act more "conservatively" under the service level constraint than without this constraint. This tendency of conservation (i.e., smaller bounds in the 3-region policy) also appears in some scenarios (e.g. Case "Truncated Norm(0.7,0.1)" in Example 1) when the decision maker's risk-aversion is represented by an exponential utility function. In these scenarios, both the concepts of service level constraints and exponential utility functions are suitable to characterize the risk-aversion of the decision maker. However, there are also some scenarios (e.g. Case "Beta(10,1)" in Example 1 with $U_{1,r}^* \ge U_1^*$, $Z_{1,r}^* \ge Z_1^*$, $V_{1,r}^* \ge V_1^*$) in which the concept of service level constraints is not suitable for characterizing the decision maker's risk-aversion. This is because the concept of service level constraints only considers the loss caused by cancelling confirmed reservations but doesn't consider the loss of unsold rooms due to high proportion of customer cancellations, while the concept of exponential utility functions considers both of them.

There are two interesting questions that deserve future research: 1) Does the similar structural policy remain to be optimal for the risk-averse decision maker with general utility functions (increasing concave functions)? 2) Do similar results remain true when the model incorporates risk-aversion, overbooking and multi demand classes of customers?

Appendix.

Proof of Lemma 1. (a) Notice that $\pi(Z, P_1, Y_1)$ in Eqn. (2) is piecewise linear in *Z*, with slope changing from $a_0 P_1$ to 0 to $b_0 P_1$ as *Z* increases. Hence, $\pi(Z, P_1, Y_1)$ is a concave function of *Z* for any realization of Y_1 and P_1, i.e. the function is sample-path concave. This implies $-\exp(-r\pi(Z, P_1, Y_1))$ is also sample-path concave because it is the composition of two concave functions. Sample path concavity implies that $G_{1,r}(Z) = E_{Y_1} E_{P_1}[-\exp(-r\pi(Z, P_1, y))]$ is a concave function of *Z*. This completes the proof.

(b) For any realization of $Y_1 = y$, we can get the first and second order partial derivatives of $G_{1,r}(Z, y)$ with respect to *Z*:

$$\frac{\partial G_{1,r}(Z,y)}{\partial Z} = \int_0^{\min\{(M-y)/Z,1\}} ra_0 p \exp\bigl(-ra_0(pZ+y)\bigr)dF_1(p)$$
$$-\int_{\min\{M/Z,1\}}^{1} rb_0 p \exp\bigl(-r(a_0+b_0)M + rb_0 pZ\bigr)dF_1(p).$$
$$\frac{\partial^2 G_{1,r}(Z,y)}{\partial Z^2} = -\int_0^{\min\{(M-y)/Z,1\}} (ra_0 p)^2 \exp\bigl(-ra_0(pZ+y)\bigr)dF_1(p)$$
$$-\int_{\min\{M/Z,1\}}^{1} (rb_0 p)^2 \exp\bigl(-r(a_0+b_0)M + rb_0 pZ\bigr)dF_1(p)$$
$$-\operatorname{sgn}(Z-M+y)\frac{(M-y)^2}{Z^3} ra_0 \exp(-ra_0 M) f_1\left(\frac{M-y}{Z}\right)$$
$$-\operatorname{sgn}(Z-M)\frac{M^2}{Z^3} rb_0 \exp(-ra_0 M) f_1\left(\frac{M}{Z}\right),$$

where $f_1(\cdot)$ is he PDF of P_1 and

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Thus, for any real number e, one has

$$e^2 G_{1,r}(Z,y) + 2e\frac{\partial G_{1,r}(Z,y)}{\partial Z} + \frac{\partial^2 G_{1,r}(Z,y)}{\partial Z^2}$$
$$\le -\int_0^{\min\{(M-y)/Z,1\}} (e - ra_0 p)^2 \exp\bigl(-ra_0(pZ+y)\bigr)dF_1(p)$$
$$-\int_{\min\{M/Z,1\}}^{1} (e + rb_0 p)^2 \exp\bigl(-r(a_0+b_0)M + rb_0 pZ\bigr)dF_1(p)$$
$$< 0.$$

Therefore, it follows that

$$e^2 G_{1,r}(Z) + 2eG_{1,r}^{'}(Z) + G_{1,r}^{''}(Z)$$
$$= E_{Y_1}\left[e^2 G_{1,r}(Z,Y_1) + 2e\frac{\partial G_{1,r}(Z,Y_1)}{\partial Z} + \frac{\partial^2 G_{1,r}(Z,Y_1)}{\partial Z^2}\right]$$
$$< 0.$$

This ends the proof. □

Proof of Theorem 1. Notice that the proof of Theorem 2 (Case *n*=1) in Liberman and Yechiali (1978) relies significantly on: 1) the concavity of the function $G_1(Z)$; 2) the property that $H(U) = G_1(U) - c_1 U$ and $\tilde{H}(V) = G_1(V) + b_1$ possesses maximum, but not on specific form of these functions. One can prove this theorem by simply imitating the proof of Theorem 2 (Case *n*=1) in Liberman and Yechiali (1978). Here $G_1(Z)$ should be replaced by $G_{1,r}(Z)$. $H(U)$ and

$\tilde{H}(V)$ should be replaced by the following functions respectively:

$$\overline{G}_{1,r}(U) = \exp(rc_1S)G_{1,r}(X_1+Y_2+S) = \exp\left(-rc_1(X_1+Y_2)\right)\exp(rc_1U)G_{1,r}(U),$$

$$\overline{\overline{G}}_{1,r}(V) = \exp(rb_1S)G_{1,r}(X_1-S) = \exp(rb_1X_1)\exp(-rb_1V)G_{1,r}(V).$$

By lemma 1, one can verify $G_{1,r}(Z)$, $\overline{G}_{1,r}(Z)$ and $\overline{\overline{G}}_{1,r}(Z)$ are all concave functions with respect to *Z*. Therefore, Part (a) of the theorem holds. Part (b) can be easily proved by letting $G_{1,r}^{'}(Z_{1,r}^{*})=0$, $\overline{G}_{1,r}^{'}(U_{1,r}^{*})=0$, $\overline{\overline{G}}_{1,r}^{'}(V_{1,r}^{*})=0$. □

Proof of Lemma 2. By the uniform continuity of $H(x,\xi)$ with respect to ξ, we have that for any $x\in[l_1,l_2]$, and any $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ (independent of *x*) such that $\left|H(x,\xi^{'})-H(x,\xi)\right|<\varepsilon$ for any $\xi^{'}\in\left(\xi-\delta(\varepsilon),\xi+\delta(\varepsilon)\right)$. Let $\delta^{'}(\varepsilon)=\delta(\varepsilon)/2$. Then for any $\xi\in\left(-\delta^{'}(\varepsilon),\delta^{'}(\varepsilon)\right)$ and $\xi^{'}\in\left(-\delta^{'}(\varepsilon),\delta^{'}(\varepsilon)\right)$, we have $\left|\xi-\xi^{'}\right|<2\delta^{'}(\varepsilon)=\delta(\varepsilon)$. Thus, it follows that $\left|H\left(x^{*}(\xi^{'}),\xi\right)\right|=\left|H\left(x^{*}(\xi^{'}),\xi\right)-H\left(x^{*}(\xi^{'}),\xi^{'}\right)\right|<\varepsilon$, which indicates $\lim\limits_{\xi\to0,\xi'\to0}H\left(x^{*}(\xi^{'}),\xi\right)=0=h(x_0^{*})$.

By the condition $\lim\limits_{\xi\to0}H(x,\xi)=h(x)$ for any $x\in[l_1,l_2]$, we have $\lim\limits_{\xi\to0,\xi'\to0}H\left(x^{*}(\xi^{'}),\xi\right)=\lim\limits_{\xi'\to0}\lim\limits_{\xi\to0}H\left(x^{*}(\xi^{'}),\xi\right)=\lim\limits_{\xi'\to0}h\left(x^{*}(\xi^{'})\right)$. Hence, $\lim\limits_{\xi'\to0}h\left(x^{*}(\xi^{'})\right)=h(x_0^{*})$. By the continuity of $h^{-1}(x)$, we have $\lim\limits_{\xi'\to0}x^{*}(\xi^{'})=h^{-1}\left[\lim\limits_{\xi'\to0}h\left(x^{*}(\xi^{'})\right)\right]=h^{-1}\left[h(x_0^{*})\right]=x_0^{*}$. This completes the proof. □

Proof of Theorem 2. First, we prove $\lim\limits_{r\to0}Z_{1,r}^{*}=Z_1^{*}$. Consider function $H_1(Z,r)=-G_{1,r}^{'}(Z)/\left[rG_{1,r}(Z)\right]$. We restrict our consideration on $Z\in[0,L]$ where L is a sufficiently large number. For any $Z\in[0,L]$, we have $\left|-G_{1,r}^{'}(Z)/\left[rG_{1,r}(Z)\right]\right|\le\max\left\{G_{1,r}^{'}(0)/r,G_{1,r}^{'}(L)/r\right\}$. In addition, for any $r_0>0$, both $G_{1,r}^{'}(0)/r$ and $G_{1,r}^{'}(L)/r$ converge as *r* approaches to r_0. Therefore, one has that for any $r_0>0$, $H_1(Z,r)$ uniformly (for all $Z\in[0,L]$) converges to $H_1(Z,r_0)$ as *r* approaches to r_0. Besides, we have that $\lim\limits_{r\to0}H_1(Z,r)=G_1^{'}(Z)$ by differentiating Eqn. (3) and $H_1(Z_{1,r}^{*},r)=0$ by Part (b) of Theorem 1. In addition, it follows by Liberman and Yechiali (1978) that: $G_1^{'}(Z)$ is a strictly decreasing function with $G_1^{'}(Z_1^*) = 0$ and has a continuous inverse function on $[0, L]$ Therefore, by Lemma 2, we have $\lim_{r\to 0} Z_{1,r}^* = Z_1^*$.

The proof for $\lim_{r\to 0} U_{1,r}^* = U_1^*$ and $\lim_{r\to 0} V_{1,r}^* = V_1^*$ is similar with considering, respectively:

$$H_2(U, r) = \exp(rc_1 U)\left[c_1 G_{1,r}(U) + \frac{G_{1,r}^{'}(U)}{r}\right],$$

$$H_3(V, r) = \exp(-rv_1 V)\left[-b_1 G_{1,r}(V) + \frac{G_{1,r}^{'}(V)}{r}\right].$$ □

Proof of Lemma 3. Suppose $\xi_1 < \xi_2$. Since $H(x, \xi)$ is increasing (decreasing) in ξ, then $H\left(x^*(\xi_1), \xi_2\right) \geq (\leq) H\left(x^*(\xi_1), \xi_1\right) = 0 = H\left(x^*(\xi_2), \xi_2\right)$. Therefore, we have $x^*(\xi_1) \leq (\geq) x^*(\xi_2)$ by the condition that $H(x, \xi)$ is decreasing in *x*. □

Proof of Proposition 1. It's easy to verify that

$$G_{1,r}^{'}(Z) = E_{Y_1}\left\{\int_0^{\frac{M-Y_1}{Z}} ra_0 p e^{-ra_0 pZ} dF_1(p) - \int_{\frac{M}{Z}}^{1} rb_0 p e^{-r[a_0 M - b_0(pZ - M)]} dF_1(p)\right\} \tag{6}$$

is decreasing in *Z* and b_0. Thus, by Lemma 3, it follows that $Z_{1,r}^*$ is decreasing with respect to b_0. This completes the proof for Part (a). Proofs of Parts (b) and (c) are included in that of Proposition 3. □

Proof of Proposition 2. Since $Z_{1,r}^*$ is the maximum point of concave function $G_{1,r}(Z)$, we only need to prove that $\partial^2 G_{1,r}(Z) \big/ \partial Z \partial r < 0$ by Lemma 3. By definition, we have $G_{1,r}(Z) = E_{Y_1} E_{P_1}\left[-\exp\left(-r\pi(Z, P_1, Y_1)\right)\right]$, where $\pi(Z, P_1, Y_1)$ is defined in Eqn. (2). For any realization of P_1 and Y_1, we have

$$\frac{\partial\left[-\exp\left(-r\pi(Z, P_1, Y_1)\right)\right]}{\partial Z \partial r} = \begin{cases} a_0 P_1 (1 - r\pi) e^{-r\pi}, & \text{if } P_1 Z < M - Y_1, \\ -b_0 P_1 (1 - r\pi) e^{-r\pi}, & \text{if } P_1 Z \geq M, \end{cases}$$

where π is used to denote $\pi(Z, P_1, Y_1)$ for short. By Eqn. (2), we have $Z_{1,r}^* \geq M$. Then we focus on the case that $Z \geq M$. For any realization of Y_1, we have

$$\begin{aligned}&\frac{\partial^2 E_{P_1}\left[-\exp\left(-r\pi(Z,P_1,Y_1)\right)\right]}{\partial Z\partial r}\\&=\int_0^{\frac{M-Y_1}{Z}} a_0 p\left(1-r\pi\right)e^{-r\pi} f_1(p)dp-\int_{\frac{M}{Z}}^{1} b_0 p\left(1-r\pi\right)e^{-r\pi} f_1(p)dp.\end{aligned} \tag{7}$$

With the set of conditions in Proposition 2, we have

$$\begin{aligned}&\frac{\partial^2 E_{P_1}\left[-\exp\left(-r\pi(Z,P_1,Y_1)\right)\right]}{\partial Z\partial r}\\&\le a_0\frac{M-Y_1}{Z} f_1\left(\frac{M-Y_1}{Z}\right)-b_0\frac{M}{Z}\left(1-ra_0M\right)e^{-ra_0M} f_1\left(\frac{M}{Z}\right)\\&\le\frac{M}{Z} f_1\left(\frac{M}{Z}\right)\left(a_0-b_0\left(1-ra_0M\right)e^{-ra_0M}\right)\le 0.\end{aligned}$$

Therefore,

$$\partial^2 G_{1,r}(Z)\big/\partial Z\partial r=E_{Y_1}\left\{\frac{\partial^2 E_{P_1}\left[-\exp\left(-r\pi(Z,P_1,Y_1)\right)\right]}{\partial Z\partial r}\right\}<0.\quad\square$$

Proof of Theorem 3. We prove the theorem by induction. For *k*=1, claim (a) and (b) have been proved in Theorem 1. Assume that for all $k\le i$, the optimal policy at time t_k ($i\ge 1$) is as stated in (a) with $U_{k,r}^*$ and $V_{k,r}^*$ satisfying (b). We only need to prove claims (a) and (b) hold for time t_{i+1}. By definition of $G_{i+1,r}(Z)$, *i*=1, 2, . . . , $n-1$ and induction assumptions, we have

$$\begin{aligned}G_{i+1,r}(Z)=E_{Y_1}\Bigg\{&\int_0^{\min\left\{(U_{i,r}^*-Y_{i+1})/Z,1\right\}}\exp\left(rc_i(U_{i,r}^*-Y_{i+1}-pZ)\right)G_{i,r}(U_{i,r}^*)dF_{i+1}(p)\\&+\int_{\min\left\{(U_{i,r}^*-Y_{i+1})/Z,1\right\}}^{\min\left\{(Z_{i,r}^*-Y_{i+1})/Z,1\right\}}G_{i,r}(pZ+Y_{i+1})dF_{i+1}(p)+\int_{\min\left\{(Z_i^*-Y_{i+1})/Z,1\right\}}^{\min\left\{Z_i^*/Z,1\right\}}G_{i,r}(Z_{i,r}^*)dF_{i+1}(p)\\&+\int_{\min\left\{Z_{i,r}^*/Z,1\right\}}^{\min\left\{V_{i,r}^*/Z,1\right\}}G_{i,r}(pZ)dF_{i+1}(p)+\int_{\min\left\{V_{i,r}^*/Z,1\right\}}^{1}\exp\left(rb_i(pZ-V_{i,r}^*)\right)G_{i,r}(V_{i,r}^*)dF_{i+1}(p)\Bigg\}.\end{aligned}\tag{8}$$

Next we will prove:

(I) $G_{i+1,,r}(Z)$ is a strictly concave function of *Z* and

(II) $e^2G_{i+1,r}(Z)+2eG_{i+1,r}^{'}(Z)+G_{i+1,r}^{''}(Z)<0$ for any real number *e*.

To do this, we employ another inductive proof. By Lemma 1, we have $G_{1,r}(Z)$ satisfies Properties (I) and (II). Assume that Properties (I) and (II) hold for $G_{i,r}(Z)$. We need to prove Properties (I) and (II) also hold for $G_{i+1,r}(Z)$. Differentiating $G_{i+1,r}(Z)$, one has

$$
\begin{aligned}
G^{'}_{i+1,r}(Z) = E_{Y_{i+1}} \Bigg\{ & \int_{0}^{\min\{(U^{*}_{i,r}-Y_{i+1})/Z,1\}} - rc_i\, p \exp\big(rc_i(U^{*}_{i,r} - Y_{i+1} - pZ)\big) G_{i,r}(U^{*}_{i,r}) dF_{i+1}(p) \\
& + \int_{\min\{(U^{*}_{i,r}-Y_{i+1})/Z,1\}}^{\min\{(Z^{*}_{i,r}-Y_{i+1})/Z,1\}} pG^{'}_{i,r}(pZ + Y_{i+1}) dF_{i+1}(p) + \int_{\min\{Z^{*}_{i,r}/Z,1\}}^{\min\{V^{*}_{i,r}/Z,1\}} pG^{'}_{i,r}(pZ) dF_{i+1}(p) \\
& + \int_{\min\{V^{*}_{i,r}/Z,1\}}^{1} rb_i\, p \exp\big(rb_i(pZ - V^{*}_{i,r})\big) G_{i,r}(V^{*}_{i,r}) dF_{i+1}(p) \Bigg\}. \qquad (9)
\end{aligned}
$$

Differentiate again and one will get the following equation by claim (b) of induction assumption:

$$
\begin{aligned}
G^{''}_{i+1,r}(Z) = E_{Y_{i+1}} \Bigg\{ & \int_{0}^{\min\{(U^{*}_{i,r}-Y_{i+1})/Z,1\}} (rc_i\, p)^2 \exp\big(rc_i(U^{*}_{i,r} - Y_{i+1} - pZ)\big) G_{i,r}(U^{*}_{i,r}) dF_{i+1}(p) \\
& + \int_{\min\{(U^{*}_{i,r}-Y_{i+1})/Z,1\}}^{\min\{(Z^{*}_{i,r}-Y_{i+1})/Z,1\}} p^2 G^{''}_{i,r}(pZ + Y_{i+1}) dF_{i+1}(p) + \int_{\min\{Z^{*}_{i,r}/Z,1\}}^{\min\{V^{*}_{i,r}/Z,1\}} p^2 G^{''}_{i,r}(pZ) dF_{i+1}(p) \\
& + \int_{\min\{V^{*}_{i,r}/Z,1\}}^{1} (rb_i\, p)^2 \exp\big(rb_i(pZ - V^{*}_{i,r})\big) G_{i,r}(V^{*}_{i,r}) dF_{i+1}(p) \Bigg\}. \qquad (10)
\end{aligned}
$$

Since $G_{i,r}(Z) < 0$ for any Z and $G_{i,r}(Z)$ is strictly concave (by induction assumption), we have $G_{i+1,r}(Z)$ is also strictly concave according to Eqn. (10). In addition, from Eqns. (8), (9) and (10), we have

$$
\begin{aligned}
& e^2 G_{i+1,r}(Z) + 2eG^{'}_{i+1,r}(Z) + G^{''}_{i+1,r}(Z) \\
& = E_{Y_{i+1}} \Bigg\{ \int_{0}^{\min\{(U^{*}_{i,r}-Y_{i+1})/Z,1\}} (e - rc_i\, p)^2 \exp\big(rc_i(U^{*}_{i,r} - Y_{i+1} - pZ)\big) G_{i,r}(U^{*}_{i,r}) dF_{i+1}(p) \\
& + \int_{\min\{(U^{*}_{i,r}-Y_{i+1})/Z,1\}}^{\min\{(Z^{*}_{i,r}-Y_{i+1})/Z,1\}} p^2 \left[\left(\frac{e}{p}\right)^2 G_{i,r}(pZ + Y_{i+1}) + 2\left(\frac{e}{p}\right) G^{'}_{i,r}(pZ + Y_{i+1}) + G^{''}_{i,r}(pZ + Y_{i+1}) \right] dF_{i+1}(p) \\
& + \int_{\min\{Z^{*}_{i,r}/Z,1\}}^{\min\{V^{*}_{i,r}/Z,1\}} p^2 \left[\left(\frac{e}{p}\right)^2 G_{i,r}(pZ) + 2\left(\frac{e}{p}\right) G^{'}_{i,r}(pZ) + G^{''}_{i,r}(pZ) \right] dF_{i+1}(p) \\
& + \int_{\min\{V^{*}_{i,r}/Z,1\}}^{1} (e + rb_i\, p)^2 \exp\big(rb_i(pZ - V^{*}_{i,r})\big) dF_{i+1}(p) \Bigg\}.
\end{aligned}
$$

Thus, by induction assumptions, we have that $e^2 G_{i+1}(Z) + 2eG^{'}_{i+1}(Z) + G^{''}_{i+1}(Z) < 0$ for any real number *e*. This proves that Properties (I) and (II) hold for $G_{i+1}(Z)$. The proof of claims (a) and (b) being true for t_{i+1} can be carried out similarly as in Theorem 1. □

Proof of Theorem 4. We carry out the proof of Part (a) by induction, where Part (b) is proved as a by-product. For *k*=1, Part (a) is proved by Theorem 2. Assume Part (a) is true for all $k \le i$. We need to prove that Part (a) is also true for *k*=*i*+1. Let $x_i = (Z_1, U_1, V_1, Z_2, U_2, V_2, \ldots, Z_i, U_i, V_i)$,

$x_{i,r} = (Z^*_{1,r}, U^*_{1,r}, V^*_{1,r}, Z^*_{2,r}, U^*_{2,r}, V^*_{2,r}, \ldots, Z^*_{i,r}, U^*_{i,r}, V^*_{i,r})$, $x_{i,0} = (Z^*_1, U^*_1, V^*_1, Z^*_2, U^*_2, V^*_2, \ldots, Z^*_i, U^*_i, V^*_i)$. Denote $\Pi_{i+1}(Z, x_i, \omega)$ as the (random) profit of the hotel over $(t_{i+1}, 0]$ when the inventory level at time t_{i+1} is Z and the hotel management adopts the 3-region policy over $(t_i, 0]$ where bounds are characterized by x_i. The variable ω in $\Pi_{i+1}(Z, x_i, \omega)$ is the sample point. Once ω is determined, the random variables $Y_j(\omega)$ and $P_j(\omega)$, j=1,2,…, i+1, are determined. For any sample point ω, $\Pi_{i+1}(Z, x_i, \omega)$ is continuous in x_i, since it is an affine transformation of x_i. Thus, $E_{Y_1,P_1,\ldots,Y_{i+1},P_{i+1}}\left[\Pi_{i+1}(Z, x_i, \omega)\right]$ is also continuous in x_i . By definitions, $G_{i+1,r}(Z) = E_{Y_1,P_1,\ldots,Y_{i+1},P_{i+1}}\left[-\exp\left(-r\Pi_{i+1}(Z, x_{i,r}, \omega)\right)\right]$ and $G_{i+1}(Z) = E_{Y_1,P_1,\ldots,Y_{i+1},P_{i+1}}\left[\Pi_{i+1}(Z, x_{i,0}, \omega)\right]$. By Eqn. (1) and the continuity of $E_{Y_1,P_1,\ldots,Y_{i+1},P_{i+1}}\left[\Pi_{i+1}(Z, x_i, \omega)\right]$ with respect to x_i, we have

$$\begin{aligned}
&\lim_{r\to 0}\left|-\frac{1}{r}\log\left[-G_{i,r}(Z)\right] - G_{i+1}(Z)\right| \\
&\le \lim_{r\to 0}\left|-\frac{1}{r}\log\left[-G_{i,r}(Z)\right] - E_{Y_1,P_1,\ldots,Y_{i+1},P_{i+1}}\left[\Pi_{i+1}(Z, x_{i,r}, \omega)\right]\right| \\
&\quad + \lim_{r\to 0}\left|E_{Y_1,P_1,\ldots,Y_{i+1},P_{i+1}}\left[\Pi_{i+1}(Z, x_{i,r}, \omega)\right] - G_{i+1}(Z)\right| \\
&= 0,
\end{aligned}$$

Then it follows that $G_{i+1}(Z) = \lim_{r\to 0} -\frac{1}{r}\log\left[-G_{i+1,r}(Z)\right]$, which means Part (b) is true. As in Theorem 2, we can prove that $\lim_{r\to 0} Z^*_{i+1,r} = Z^*_{i+1}$, $\lim_{r\to 0} U^*_{i+1,r} = U^*_{i+1}$, and $\lim_{r\to 0} V^*_{i+1,r} = V^*_{i+1}$ (one just needs to replace index 1 by i+1). This completes the proof. □

Proof of Proposition 3. The proposition can be easily known by Lemma 3 and the fact that $\partial\left(rc_i G_i + G_i^{'}\right)/\partial c_i = rG_i \le 0$ and $\partial\left(-rb_i G_i + G_i^{'}\right)/\partial b_i = -rG_i \ge 0$. □

Proof D1. Recall $G^{'}_{1,r}(Z)$ in Eqn. (6). Take $G^{'}_{1,r}(Z)$ and $Z^*_{1,r}$ as functions of b_0 (i.e., $G^{'}_{1,r}(Z, b_0)$, $Z^*_{1,r}(b_0)$). When b_0=0, then $G^{'}_{1,r}(Z, 0)$ >0 for any $Z \ge M$, which implies $Z^*_{1,r}(0) = +\infty$. When b_0 approaches to infinity, then $G^{'}_{1,r}(Z, b_0) < 0$ for any $Z > M$. Thus, $\lim_{b_0\to+\infty} Z^*_{1,r}(b_0) = 0$. It's easy to verify that $Z^*_{1,r}(b_0)$ is a continuous function of b_0. Therefore, for any m>0 and $\alpha > 0$, there exists a $\bar{b}_0 > 0$ such that $Z^*_{1,r}(\bar{b}_0) = \bar{Z}(m, \alpha)$. □

Proof D2. With the service level constraint, all the bounds U_1^*, Z_1^*, V_1^* are subject to being less than $(M+m)/F_1^{-1}(1-\alpha)$. With similar arguments as in Lemmas 1 and 2 of Liberman and Yechiali (1978), one can easily prove that the 3-region policy is still optimal with the service level constraint and the new optimal bounds fall on $\tilde{U}_1^* = \min\{U_1^*, M+m\}$, $\tilde{Z}_1^* = \min\{Z_1^*, M+m\}$, $\tilde{V}_1^* = \min\{V_1^*, M+m\}$. Virtually, for the single-period model with the service level constraint, the 3-region policy is also optimal for the risk-averse decision maker. The argument for this is similar as that of the risk-neutral decision maker. □

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