SOLUTIONS FOR BARGAINING GAMES WITH INCOMPLETE INFORMATION: GENERAL TYPE SPACE AND ACTION SPACE

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(Communicated by the associate editor name)

Abstract. A Nash bargaining solution for Bayesian collective choice problem with general type and action spaces is built in this paper. Such solution generalizes the bargaining solution proposed by Myerson who uses finite sets to characterize the type and action spaces. However, in the real economics and industries, types and actions can hardly be characterized by a finite set in some circumstances. Hence our generalization expands the applications of bargaining theory in economic and industrial models.

1. Introduction. The bargaining problem with complete information is first discussed in Nash’s paper [11], where the Nash solution is proposed based on four axioms which represent efficiency and fairness. After this seminal work, different axioms which represent different views of fairness are studied (see, e.g., [6] and [13]). Harsanyi and Selten are the pioneers to extend the Nash solution to bargaining problems with incomplete information [9]. Following their work, Myerson introduces an arbitrator who uses incentive compatible mechanisms to coordinate all players [8]. Under the assumption that the players in the bargaining game are allowed to vote and change the mechanism after they have learnt their private information [4], Holmstrom and Myerson show that some incentive compatible mechanisms are no longer stable (i.e., players will vote to change the mechanism after they have learnt their private information). Hence in their paper, stable (durable) mechanisms are studied, and the existence of such mechanisms is given. Later, Myerson proposes a bargaining solution concept that generalizes the Nash solution and the Shapley non-transferable-utility value (NTU value) for cooperative game with incomplete information [9]. Myerson also gives another generalization of the two-person Nash solution with incomplete information by introducing different axioms [10].

However, in all the above solutions for bargaining games with incomplete information, the type space and the action space are assumed to be finite sets. Such an assumption simplifies the discussion of the solutions, but narrows the applications of bargaining theory in economic and industrial models. In the real world, private information of a decision maker may be various, complex and can hardly be characterized by a finite set. For example, in supply chain models, downstream member may have private forecast demand information [12], and upstream member’s

2010 Mathematics Subject Classification. Primary: 91A12, 91B26; Secondary: 91B02.

Key words and phrases. Bargaining, Nash solution, incentive compatibility, incomplete information, type space, action space.

This work has been supported by the National Natural Science Foundation of China under Projects Nos. 71210002 and 71671099. The authors are grateful to the anonymous referees for their constructive comments and suggestions.

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additional contracting profit may be privately known by himself [1]; in reverse auction, bidders may hold private cost information (see, e.g., [2] and [5]); and in principal-agent problems, the agent may privately observe a real valued Markov process [7]. Also, action spaces in these examples need not to be finite sets. Therefore, considering type space and action space to be more general sets in bargaining problems can significantly expand the applications of bargaining theory in economic and industrial models, such as in supply chain models, inventory models and so on.

In this paper, we aim to extend the bargaining solution to the incomplete information bargaining games with more general type and action spaces. In Section 2, we introduce the choice problem of the arbitrator together with the incentive compatible mechanisms used by the arbitrator. The bargaining solution and the axioms are described in Section 3. The existence and the uniqueness of the solution are also shown in Section 3. In Section 4, an example in supply chain management is introduced. Finally, this paper is concluded in Section 5.

2. The bargaining problem and incentive compatible mechanisms. In this section, we will consider the bargaining problem as the Bayesian collective choice problem:

\[
(A, a^*, T, \pi = (\pi_1, \ldots, \pi_n), \mu = \mu_1 \times \cdots \times \mu_n).
\]  

In the above formula, \(n\) represents the number of players; \(A\) is the set that represents all players’ pure action space, which is assumed to be a Hausdorff space; \(a^* \in A\) is the disagreement point, which represents the outcome of all players if they fail to cooperate; \(T_i\) is the set that represents player \(i\)’s type space; \(\mathcal{F}_i\) is a \(\sigma\)-field on \(T_i\); and \(\mathcal{F}\) is the product \(\sigma\)-field on \(T\); bounded measurable function \(\pi_i : A \times T \to [0, +\infty)\) represents the payoff of player \(i\), and \(\pi_i(t, t)\) is continuous on \(A\) for any \(t \in T\); finally, \(\mu\) is a probability measure on \((T, \mathcal{F})\), representing the prior type information of player \(i\), and \(\mu\) is the product measure. This assumption indicates that the private information of players are independently distributed. It is assumed that nature will randomly select the types of all players (i.e., \(t \in T\)) with respect to \(\mu\), and player \(i\) will be privately informed by his type \(t_i\). Since we do not restrict the decisions to be pure actions, the decision can be chosen from the following set:

\[
PM(A, Borel(A)) = \{\nu | \nu\text{ is a probability measure on } (A, Borel(A))\},
\]  

where \(Borel(A)\) is the Borel \(\sigma\)-field of \(A\), and \(PM(A, Borel(A))\) is a metric space equipped with the total variation norm.

Following Myerson’s work [8], we also consider the case in which there is an arbitrator to make the decision. The decision is determined through the following manner: firstly, the arbitrator chooses a mechanism and makes it common knowledge to all players; then all players make responses to the arbitrator; finally, based on the mechanism, all players’ actions are automatically determined.

In this work, we assume that the mechanism used by the arbitrator is to ask each player a question, and player \(i\) will respond his question from a set \(R_i\) equipped with a \(\sigma\)-field \(\mathcal{R}_i\); and then, based on all players’ responses \(r = \{r_i\}_{i=1}^n\), the decision is made with respect to the probability measure \(m(r)\), where \(m : R = R_1 \times \cdots \times R_n \to PM(A, \mathcal{B}(A))\) is measurable with respect to \(\mathcal{R} = \bigotimes_{i=1}^n \mathcal{R}_i\). The function \(m\) is selected by the arbitrator, and \(m\) is common knowledge before players make responses. Hence, the resultant responses from the players will form a Bayesian Nash equilibrium. That is, if a measurable function \(s_i : T_i \to PM(R_i, \mathcal{R}_i)\) is player \(i\)’s mixed strategy, then \(\{s_i\}_{i=1}^n\) should form a Bayesian Nash equilibrium.

Remark 1. It should be noted that, when type spaces are infinite sets, the definition of the Bayesian Nash equilibrium is a little bit different from the case when type spaces are finite sets. This is because the expected payoff of a player is described by a conditional expectation, which is only defined under almost sure equivalence. The detailed definition in our work is given below.

Definition 2.1. A mixed strategy profile \(\{s_i\}_{i=1}^n\) forms a Bayesian Nash equilibrium if and only if for all \(1 \leq i \leq n\) and any mixed strategy \(\hat{s}_i \in PM(R_i, \mathcal{R}_i)\) the following inequality holds almost surely,

\[
E_{\mu} \left[ \int_R \int_A \pi_i(a, t)m(r)(da)s_1(t_1)(dr_1) \cdots s_n(t_n)(dr_n) | t_i \right] 
\geq E_{\mu} \left[ \int_R \int_A \pi_i(a, t)m(r)(da)s_1(t_1)(dr_1) \cdots \hat{s}_i(dr_i) \cdots s_n(t_n)(dr_n) | t_i \right].
\]  


The definition indicates that, player \( i \) will not deviate from the strategy \( s_i \) with probability one. And if the type spaces are finite or countably infinite sets, such definition reduces to the standard definition.

We say \( m \) is a Bayesian incentive compatible mechanism if \( (R_i, R_i) = (T_i, F_i) \) and the strategy profile in which all players report their true type (i.e., \( s_i(t_i) = 1(t_i = t_i) \)) forms a Bayesian Nash equilibrium (i.e. the truth telling Bayesian Nash equilibrium). Myerson shows that in his setting, for any Bayesian Nash equilibrium of a given mechanism \( m \), there is a Bayesian incentive compatible mechanism \( m' \) with a truth telling Bayesian Nash equilibrium, such that the expected payoff of all types of players are equivalent in the two Bayesian Nash equilibria [8]. Hence, Myerson restricts the arbitrator’s choice to the Bayesian incentive compatible mechanisms. In our setting, the similar result also holds, and the proof is also similar.

**Theorem 2.2.** *For any Bayesian Nash equilibrium \( \{s_i\}_{i=1}^n \) of a mechanism \( m \), there is a Bayesian incentive compatible mechanism \( m' \), s.t., for all \( 1 \leq i \leq n \),

\[
E_{\mu} \left[ \int_{R_i} \int_{A} \pi_i(a, t)m(r)(da)s_1(t_1)(dr_1) \cdots s_n(t_n)(dr_n)|t_1 \right] = E_{\mu} \left[ \int_{A} \pi_i(a, t)m'(t)(da)|t_1 \right] \text{ almost surely.} \tag{4}
\]

**Proof.** Let \( S \) be any Borel set of \( A \). For any \( t \in T \), define a measure:

\[
m'(t)(S) = \int_{R_1 \times \cdots \times R_n} m(r_1, \ldots, r_n)(S)s_1(t_1)(dr_1) \cdots s_n(t_n)(dr_n). \tag{5}
\]

Then it is easy to check that the equality \( (4) \) holds almost surely. Hence we only need to show that \( m' \) is Bayesian incentive compatible. That is, we need to check that truth telling strategy forms a Bayesian Nash equilibrium.

For type-\( t_i \) player \( i \) under mechanism \( m' \), if all other players report their true types, then the expected payoff of player \( i \) when he reports \( \tilde{t}_i \) is

\[
E_{\mu} \left[ \int_{A} \pi_i(a, t)m'(t_1, \ldots, \tilde{t}_i, \ldots, t_n)(da)|t_1 \right] = E_{\mu} \left[ \int_{R_i} \int_{A} \pi_i(a, t)m(r)(da)s_1(t_1)(dr_1) \cdots s_i(t_i)(dr_i) \cdots s_n(t_n)(dr_n)|t_1 \right] \leq E_{\mu} \left[ \int_{R_i} \int_{A} \pi_i(a, t)m(r)(da)s_1(t_1)(dr_1) \cdots s_i(t_i)(dr_i) \cdots s_n(t_n)(dr_n)|t_1 \right] \tag{6}
\]

\[
= E_{\mu} \left[ \int_{A} \pi_i(a, t)m'(t)(da)|t_1 \right] \text{ almost surely.}
\]

The reason of the inequality in \( (6) \) is that, under mechanism \( m \), since \( \{s_i\} \) is a Bayesian Nash equilibrium, type-\( t_i \) player \( i \) who pretends to be type \( \tilde{t}_i \) and reports with respect to \( s_i(\tilde{t}_i) \) will gain no more than he reports with respect to \( s_i(t_i) \). The above calculation further indicates that, for almost all types \( t_i \), reporting with respect to any other mixed strategy will make no more profit than reporting true type for player \( i \) under mechanism \( m' \). Thus we have proved that truth telling strategy forms a Bayesian Nash equilibrium under \( m' \). \( \square \)

With the above theorem, it suffices to just study the Bayesian incentive compatible mechanisms. Hence the choice problem of the arbitrator reduces to the determination of such a mechanism. If a Bayesian incentive compatible mechanisms \( m \) is chosen, then we can calculate the expected payoff of play \( i \) as follows:

\[
f_i^m(t_i) = E_{\mu} \left[ \int_{A} \pi_i(a, t)m(t)(da)|t_1 \right]. \tag{7}
\]

Here the function \( f_i^m \) can be regarded as a map from \( T_i \) to \( \mathbb{R} \), hence we can define the following set,

\[
F^* = \{ f^m = (f_1^m, \ldots, f_n^m) | m \text{ is Bayesian incentive compatible} \}, \tag{8}
\]

and the determination of a Bayesian incentive compatible mechanism \( m \) is reduced to choose an element in \( F^* \). It should be noted that choosing \( m \) and choosing \( f^m \) are not equivalent, since different \( m \) may generate the same \( f^m \). However, we assume all players are only interested in their expected payoffs, thus different \( m \) can be treated equally if they generate the same expected payoffs for all players. Theorem 2.3 identifies a number of properties of \( F^* \). These properties are now shown to characterize \( F^* \).
Theorem 2.3. $F^*$ is a non-empty convex set. Further, for any $f^m \in F^*$ we have $|f^m_i| \leq K$ which means $F^*$ is bounded.

Proof. For any $a \in A$, if a mechanism $m_a$ is always choosing $a$ regardless of the players’ responses, then it is easy to check that $m_a$ is Bayesian incentive compatible, which means $F^*$ is non-empty and we denote $f^{m_a}$ to be the element generated by $m_a$. Now for any $f^{m_1} \in F^*(i = 1, 2)$ and $\forall A \in [0, 1]$, we know that the mechanism $\lambda f^{m_1} + (1 - \lambda) m_2$ is also Bayesian incentive compatible, hence $\lambda f^{m_1} + (1 - \lambda) f^{m_2} = \lambda f^{m_1}(1 - \lambda) m_2 \in F^*$, which indicates that $F^*$ is convex. Finally, since $|r_i| \leq K$ is bounded, $\forall f^m \in F^*$ we have $|f^m_i| \leq K$. 

Recall that in the problem formulation, there is a disagreement point $a^* \in A$, hence there is a constrain on the choice of $f^m$, that is $f^m \geq f^{m_a}$ almost surely (in the following we write $f^*$ instead of $f^{m_a}$, and $m_a$ is the mechanism that always chooses $a^*$ regardless of the players’ responses).

Now we are able to reformulate the choice problem of the arbitrator by the bargaining basis $B = \{(F^*, f^*), (T, F, \mu)\}$. It is assumed that $f^* \in F^*$, and $F^*$ is non-empty, bounded and convex.

In the following section, we will follow the procedures in Harsanyi and Selten’s work [3] to find the bargaining solution.

3. Axioms and the bargaining solution. It is pointed out by Weidner [15] that, the approach used in Myerson’s work [8] bears some problems when types of players are not independently distributed. Hence, in this paper, we assume that the types of players are independently distributed, and we follow the procedures in Harsanyi and Selten’s work [3] to give the solution. The basic structure is as follows: firstly, some axioms that are consistent with the axioms proposed by Harsanyi and Selten [3] for finite type spaces are proposed; then a solution that satisfies the axioms is given; finally, the uniqueness of the solution that satisfies all the axioms is shown.

3.1. Axioms. Before we give the axioms, a restriction on the type space should be introduced. This is because in Harsanyi and Selten’s work [3], there is an axiom that represents the symmetric property of types. That is, when type space $T$ is a finite set with $n$ elements, the symmetric group $S_n$ can act on $T$ in a nature way, and the bargaining solution should be invariant under the action of $S_n$. In mathematics, an action of a group is a way of interpreting the symmetry of elements. Hence, when type space $T$ is not finite, a straightforward idea is to introduce a group $S$ together with group action on $T$, and require the bargaining solution to be invariant under the group action. However, such procedure would exclude some commonly used type spaces in literature, e.g., Borel set of $R$, since there does not exist a commonly known group action on a Borel set. However, if $T$ is a compact group, then we know that $T$ can act on itself in a nature way. And for any Borel set $U$ of $R$, there exists a measurable map from compact group $S^1$ (1-dim circle) to $U$. Based on these observations, we propose the following requirement of type spaces.

Definition 3.1. A probability space $(T, \mathcal{F}, \mu)$ is proper if there is a compact group $T'$ with a left invariant Borel probability measure $\mu'$ and a measurable function $\varphi : T' \rightarrow T$ such that, $\mu = \mu' \circ \varphi^{-1}$. For a bargaining basis $B = \{(F^*, f^*), (T, F, \mu)\}$, we say $(T, F, \mu)$ is proper if $(T', \mathcal{F}', \mu')$ is proper for any $i$.

In the above definition, any compact group has a left invariant Borel probability measure [14]. The following examples are given to show that some commonly used probability spaces are proper.

Remark 2. Examples of proper probability space: $(I, \text{Borel}(I), \mu)$ is proper, where $I$ is a closed interval and $\mu$ is any Borel probability measure on $I$. As a consequence, any discrete probability space is proper.

Proof. It suffices to consider $I = [0, 1]$. Define the following Borel measurable function:

$$\varphi(t) = \inf\{s | \mu([0, s]) \geq t\}. \quad (9)$$

Let $u$ be the uniform distribution on $[0, 1]$, then we first show that $\mu = u \circ \varphi^{-1}$. For any $s \in [0, 1]$ and $t \leq u([0, s])$, we have $\varphi(t) \in [0, s]$, hence $t \in \varphi^{-1}([0, s])$. This means $u \circ \varphi^{-1}([0, s]) \geq u([0, \mu([0, s])) = \mu([0, s])$. On the other hand, for any $t \in \varphi^{-1}([0, s])$, we have $\varphi(t) = s_0 \in [0, s]$, and for any $N > 0$, $t \leq \mu([0, s_0 + 1/N])$. Since $[0, 1]$ is compact, then $\mu$ is regular [14], hence $\mu([0, s_0 + 1/N]) = \lim_{N \rightarrow \infty} \mu([0, s_0 + 1/N]) \geq t$. Thus $t \leq \mu([0, s])$, which means $\varphi^{-1}([0, s]) \subseteq [0, \mu([0, s])]$ and $u \circ \varphi^{-1}([0, s]) \leq \mu([0, s])$. Combining the above two inequalities, we have $u \circ \varphi^{-1}([0, s]) = \mu([0, s])$ for any $s \in [0, 1]$, which means $u \circ \varphi^{-1} = \mu$. 


Now we take a measurable function \( \phi \) from \( S^1 = [0,2\pi) \) to \([0,1]\), such that \( \phi(\theta) = \theta/\pi \) if \( 0 \leq \pi \) and \( \phi(\theta) = 2 - \theta/\pi \) if \( \theta > \pi \). Denote by \( v \) the uniform distribution on \( S^1 \), then \( u = u \circ \phi^{-1} \), thus \( \mu = v \circ \phi^{-1} \circ \varphi^{-1} \). Note that \( S^1 \) is a compact group and \( v \) is the invariant Borel probability measure, therefore \(([0,1],\text{Borel}([0,1]),\mu)\) is proper.

With the above definition, we now propose the axioms that determine the solution for bargaining bases. The solution that we are considering is a function denoted by \( L \) which maps a bargaining basis \( B \) to an element in \( F^* \). The first five axioms in the following are generalized from Harsanyi and Selten’s work \cite{3} in a straightforward way, while a little bit more efforts are needed to generalize the last two axioms.

**Axiom 1** (Profitability) The solution \( L \) should satisfy \( L(B) > f^* \) almost surely.

**Axiom 2** (Player Symmetry) For any \( \sigma \in S_n \) (\( S_n \) is the symmetric group with \( n \) elements), define \( \sigma(f^m) = \sigma((f_{t(1)}^m, \ldots, f_{t(n)}^m)) = (f_{t(\sigma(1))}^m, \ldots, f_{t(\sigma(n))}^m) \), \( \sigma(F^*) = \{\sigma(f^m) | f^m \in F^*\} \) and \( \forall A \in F \), define \( \sigma(A) = \{t_{\sigma(1)}, \ldots, t_{\sigma(n)} | t \in A\} \), \( \sigma(F) = F \otimes \cdots \otimes F_{t(n)} \). \( \sigma(\mu)(\sigma(A)) = \mu(A) \). Then \( L(\sigma(B)) = \sigma(L(B)) \), where \( \sigma(B) = ((\sigma(F^*), \sigma(f^*)), (\sigma(T), \sigma(F), \sigma(\mu)) \).

**Axiom 3** (Efficiency) There does not exist an \( f^m \in F^* \) such that \( f^m \geq L(B) \) almost surely with respect to \( \mu \) and \( \mu(f^m = L(B)) \neq 1 \) (i.e., \( f^m \) is not equal to \( L(B) \) under almost sure equivalence with respect to \( \mu \)). Here \( (x_1, \ldots, x_n) \geq (y_1, \ldots, y_n) \) means \( x_i \geq y_i \) for all \( 1 \leq i \leq n \).

**Axiom 4** (Positive Affine Invariance) Let \( g, h : T \to \mathbb{R}^n \) be any two measurable functions such that, \( g = (g_1, \ldots, g_n) \) with \( g_1 : T \to \mathbb{R} \) and \( g > 0 \) almost surely, \( h = (h_1, \ldots, h_n) \). Then \( L \) should satisfy \( L(g \circ F^* + h, \sigma(f^* + h), \sigma(T), \sigma(\mu)) = g \circ L(B) + h \), where \( g \circ f^m = (g_1 \circ f_{t(1)}, \ldots, g_n \circ f_{t(n)} \) and \( g \circ F^* = \{g \circ f^m | f^m \in F^*\} \).

**Axiom 5** (Irrelevant Alternatives) Let \( B = ((F^*, f^*), (T, F, \mu)) \) and \( B' = ((F^*, f^*), (T, F, \mu)) \) be two bargaining bases, such that \( F^* \subseteq F^* \) and \( L(B) \subseteq F^* \), then \( L(B') = L(B) \).

**Axiom 6** (Type Symmetry) If for any \( i, T_i \) is a compact group, \( \mu_i \) is left invariant, and for any \( t \in T, F^* = \{f \circ t = (f_1 \circ t_1, \ldots, f_n \circ t_n) | f \in F^*\} \) (where \( f_i \circ t_i = f_i(t_1(t_i)) \) and \( f^* = f^* \circ t \). Then \( L(B) \) should be a constant vector in \( \mathbb{R}^n \) almost surely.

**Remark 3.** This axiom significantly different from that in Harsanyi and Selten’s work \cite{3}. In their work, “Type Symmetry” axiom indicates that, if a player’s two types are interchanged, then the bargaining result is obtained by interchanging the two types. However, such axiom is not suitable for infinite type spaces. The reason is that, two types could be zero measure set, hence interchanging them is meaningless. Our definition of the “Type Symmetry” axiom is based on the following observation. For finite type spaces, according to Harsanyi and Selten’s “Type Symmetry” axiom, if the prior distribution of the type spaces are uniform and the bargaining set is invariant under the operation of changing two types of a player, then the bargaining solution should be invariant under the operation of changing two types of a player. That is, the bargaining solution should be a constant vector in \( \mathbb{R}^n \). Following this logic, our axiom of “Type Symmetry” is proposed. At the end of this subsection, we will give an example to show that, simply adopting Harsanyi and Selten’s “Type Symmetry” axiom may bear some problems.

**Axiom 7** (Type Transformation) For \( B = ((F^*, f^*), (T, F, \mu)) \), and \( n \) measurable functions \( \varphi_i : (\bar{T}_i, \bar{F}_i, \bar{\mu}) \to (T_i, F_i, \mu), i = 1, \ldots, n \), define \( f^* \circ \varphi = (f_{t(1)}^* \circ \varphi_1, \ldots, f_{t(n)}^* \circ \varphi_n) \) and \( F^* \circ \varphi = \{f^m \circ \varphi | f^m \in F^*\} \). If \( \mu = \mu_1 = \varphi_1^{-1} \times \cdots \times \mu_n = \varphi_n^{-1} \), then \( L \) should satisfy \( L((F^* \circ \varphi, f^* \circ \varphi), (T, F, \bar{\mu})), (\sigma(T), \sigma(F), \sigma(\mu)) \) = \( L((F^*, f^*), (T, F, \mu)) \circ \varphi \).

**Remark 4.** This axiom is a modification of the axiom “Splitting Types” in Harsanyi and Selten’s work \cite{3}. “Splitting Types” axiom in their work indicates that, if one type of a player is split into two types, then the bargaining solution is also obtained by splitting the type into two types. Similar to Axiom 6, one type could be zero measure if type space is infinite. Hence, “Splitting Types” axiom cannot be adopted directly. However, splitting one type into other two types can be viewed in another way, which is a surjection from a larger set to a smaller set. And for probability space, the surjection is a measurable function which is measure-preserving. This observation leads to our “Type Transformation” axiom. It should be noted that, measure-preserving function needs not to be a surjection, therefore “Type Transformation” axiom is stronger than “Splitting Types” axiom. However, when \( T \) is a finite set, it is not hard to see that, “Type Transformation” axiom is equivalent to “Type Symmetry” axiom plus “Splitting Types” axiom in Harsanyi and Selten’s work \cite{3}. Harsanyi and Selten’s last axiom “Mixing Basic Probability Matrices” is not included here, since we are considering the case that the types of all players are independently distributed.

If Harsanyi and Selten’s “Type Symmetry” axiom is directly adopted into infinite type spaces, then the following axiom is obtained:
Axiom 6* (Type Symmetry*). Let \( B = (F^*, f^*), (T, F, \mu) \), \( k \in \{1, 2, \ldots, n\} \) and \( t_{k1}, t_{k2} \in T_k \), define an operator \( \sigma_{k1k2}^k \) as follows. Let \( \sigma_{k1k2}^k(f) = (f_1, \ldots, f_k, f_{k+1}, \ldots, f_n) \) for any \( f = (f_1, \ldots, f_n) \in F^* \), in which \( g_k(t_k) = f_k(t_k) \) for \( t_k \in T_k \), \( t_{k1} \) and \( t_{k2} \) = steps. Let \( (t_k, \sigma_{k1k2}^k(F), \sigma_{k1k2}^k(\mu_k)) \) be a probability space such that, for any \( A_k \subseteq T_k \), \( A_k \in \sigma_{k1k2}^k(F) \) if and only if one of the following four conditions holds:

1. \( t_{k1} \not\in A_k \), \( t_{k2} \not\in A_k \) and \( A_k \subseteq F_k \) (in this case, define \( \sigma_{k1k2}^k(\mu_k)(A_k) = \mu_k(A_k) \));
2. \( t_{k1} \not\in F_k \), \( t_{k2} \in A_k \) and \( 3B_k \in F_k \) such that \( t_{k1} \in B_k \), \( t_{k2} \in B_k \) and \( A_k / \{t_{k1}\} \) = \( B_k / \{t_{k2}\} \) (in this case, define \( \sigma_{k1k2}^k(\mu_k)(A_k) = \mu_k(B_k) \));
3. \( t_{k1} \in A_k \), \( t_{k2} \not\in A_k \) and \( 3B_k \in F_k \) such that \( t_{k1} \not\in B_k \), \( t_{k2} \in B_k \) and \( A_k / \{t_{k1}\} \) = \( B_k / \{t_{k2}\} \) (in this case, define \( \sigma_{k1k2}^k(\mu_k)(A_k) = \mu_k(B_k) \));
4. \( t_{k1} \in A_k \), \( t_{k2} \in A_k \) and \( 3B_k \in F_k \) (in this case, define \( \sigma_{k1k2}^k(\mu_k)(A_k) = \mu_k(A_k) \)).

Let \( \mathcal{F} = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_{k-1} \otimes \sigma_{k1k2}^k(\mathcal{F}_k) \otimes \mathcal{F}_{k+1} \otimes \cdots \otimes \mathcal{F}_n \), \( \mu = \mu_1 \times \cdots \times \mu_{k-1} \times \sigma_{k1k2}^k(\mu_k) \times \mu_{k+1} \times \cdots \times \mu_n \), then \( L \) should satisfy

\[
L(\sigma_{k1k2}^k(F^*), \sigma_{k1k2}^k(f^*)), (T, \mathcal{F}, \mu) = \sigma_{k1k2}^k(L(F^*, f^*), (T, F, \mu)).
\]

Remark 5. In the above axiom, the operation \( \sigma_{k1k2}^k \) represents interchanging two types of a player. Consider a bargaining basis \( B = ((F^*, 0), ([0, 2]^2, Borel([0, 2]^2) \otimes Borel([0, 2]), \mu)) \) in which \( \mu \) is the uniform distribution on \([0, 2]^2\) and \( F^* = \{(f_1, f_2) | 0 \leq f_1 \leq 2, f_1 f_2(x) dx \leq 1, i \in 1, 2\} \). If Axiom 6* is applied instead of Axiom 6, using Axiom 2, we know that \( L(B) = (f_0, f_0) \). For any \( 0 \leq a \leq b \leq 2 \), \( \sigma_{ab}^1(F^*) = F^* \), \( \sigma_{ab}^1(0) = 0 \), \( \sigma_{ab}^1(Borel([0, 2]^2)) = Borel([0, 2]^2) \) and \( \sigma_{ab}^1(\mu) = \mu_1 \). Hence, using Axiom 6*, we have \( \sigma_{ab}^1(L(B)) = L(\sigma_{ab}^1(F^*), \sigma_{ab}^1(0), ([0, 2]^2, Borel([0, 2]^2) \otimes \sigma_{ab}^1(Borel([0, 2]^2), \sigma_{ab}^1(\mu_1) \times \mu_2)) = L(B) \). Therefore \( f_0(a) = f_0(b) \), which means \( f_0 \) must be a constant. By the definition of \( F^* \), we know that \( f_0 \leq 1 \). However, \( g(x) = (1 + 1_{(1 \leq x \leq 2)}, 1 + 1_{(1 \leq x \leq 2)}) \in F^* \), hence \( g \geq (f_0, f_0) \) and \( \mu(g) = (f_0, f_0) \) \( \leq 0.25 \), which contradicts with Axiom 3. This example shows that, directly adopting Harsanyi and Selten’s “Type Symmetry” axiom is not appropriate.

In the next subsection, we will give a specific solution that satisfies Axioms 1 to 7.

3.2. The solution. The bargaining solution \( L^* \) that we consider in this paper is related to the maximizer of the following maximization problem:

\[
f^{m} \in F^{*}, f^{m} > f^{a.s.} \sum_{i=1}^{n} \int_{T_{ik}} \ln(f^{m}(t_{i}) - f^{*}(t_{i})) \mu_{i}(dt_{i})). \tag{10}
\]

Definition 3.2. We say a bargaining basis \( B = ((F^*, f^*), (T, F, \mu)) \) is proper, if \( (T, F, \mu) \) is proper and the maximization problem (10) has at least one solution.

Remark 6. If \( T \) is a finite set and \( (F^*, f^*) \) is regular [3], then we know that \( (T, F, \mu) \) is proper. Further, we know that \( F^* \) is a convex compact subset of the Euclidian spaces. Together with \( (F^*, f^*) \) being regular, (10) has at least one solution, which means \( B \) is proper. When \( T \) is an infinite set, it is not easy to see whether \( B \) is proper or not. Hence, in the following we restrict the discussions of the bargaining solution \( L^* \) over proper bargaining bases. And we will give an example to show that \( B \) is proper when \( T \) is an infinite set in Section 4.

Since \( \ln(x) \) is strictly concave, we know that if \( B \) is proper, then (10) has a unique maximizer up to almost sure equivalence, hence we define \( L^*(B) \) to be the unique maximizer. In the following we show that \( L^* \) satisfies Axioms 1 to 7 for proper \( B \).

Theorem 3.3. \( L^* \) satisfies Axioms 1 to 7.

Remark 7. We can solve the maximization problem (10) to obtain \( L^*(B_0) \), in which \( B_0 \) is the bargaining basis defined in Remark 5. Apply \( B_0 \) to (10), we have

\[
\max_{0 \leq t_{1}, t_{2}, f_1(x), f_2(x) \leq 1, t_{1}, t_{2} \leq 1, 2 \leq 1, 2} \int_{0}^{2} 0.5 \ln(f_1(x)) dx + \int_{0}^{2} 0.5 \ln(f_2(x)) dx. \tag{11}
\]

Note that, \( \int_{0}^{2} 0.5 \ln(f_1(x)) dx + \int_{0}^{2} 0.5 \ln(f_2(x)) dx \leq \int_{0}^{2} 0.5 \ln(f_1(x)) dx + \int_{0}^{2} 0.5 \ln(f_2(x)) dx \leq \ln 2 \) for \( f_1(x) = f_2(x) = 1 + 1_{(1 \leq x \leq 2)} \). Therefore, \( L^*(B_0) = (1 + 1_{(1 \leq x \leq 2)}, 1 + 1_{(1 \leq x \leq 2)}) \).
Proof. Axioms 1 to 5 are trivially true for $L^*$, hence we check Axioms 6 and 7 in the following.

For Axiom 6, if $\mu$, $f^*$ and $F^*$ are invariant, let $\tilde{f} = L^*(B)$, then for any $t \in T$, $\tilde{f} \circ t$ is also a maximizer. Since $F^*$ is convex, hence $\tilde{f} = f_{T_1}^*(\tilde{f} \circ t) \mu(dt) \in F^*$. Since $\ln x$ is concave, we have

\[
\sum_{i=1}^{n} \int_{T_i} \ln(f(t_i) - f^*(t)) \mu_i(dt_i) = \sum_{i=1}^{n} \int_{T_i} \ln \left( \int_{T_i} \left[ f_{T_i} \circ t_{T_i}(t_i) - f^*_i \circ t^*_i(t_i) \right] \mu_i(dt') \right) \mu_i(dt) \\
\geq \sum_{i=1}^{n} \int_{T_i} \int_{T_i} \ln \left( \int_{T_i} \left[ f_{T_i} \circ t_{T_i}(t_i) - f^*_i \circ t^*_i(t_i) \right] \mu_i(dt') \right) \mu_i(dt) \\
= \sum_{i=1}^{n} \int_{T_i} \int_{T_i} \ln \left( \int_{T_i} \left[ f_{T_i} \circ t_{T_i}(t_i) - f^*_i \circ t^*_i(t_i) \right] \mu_i(dt') \right) \mu_i(dt) \\
= \sum_{i=1}^{n} \int_{T_i} \int_{T_i} \ln \left( \int_{T_i} \left[ f_{T_i} \circ t_{T_i}(t_i) - f^*_i \circ t^*_i(t_i) \right] \mu_i(dt') \right) \mu_i(dt).
\]

(12)

The above inequality should be equality, hence $\tilde{f}$ is also a maximizer, i.e. $\tilde{f} = L^*(B)$. Note that $\mu$ is also invariant, therefore for any $t \in T$

\[
\tilde{f}(t_i) = \int_{T_i} \tilde{f}(t_{T_i}(t_i)) \mu_i(dt) \\
= \int_{T_i} \tilde{f}(t_{T_i}(t_i)) \mu_i(dt) \\
= \tilde{f}(\bar{t}_i),
\]

(13)

which means $\tilde{f}$ is a constant vector in $\mathbb{R}^n$ almost surely. This shows that, $\tilde{f} = L^*(B)$ satisfies Axiom 6 (Type Symmetry).

For Axiom 7, let $\tilde{f} = L^*(B)$, we need to show $\tilde{f} \circ \varphi$ is the maximizer of

\[
\max_{g \in F^* \circ \varphi, g > f} \sum_{i=1}^{n} \int_{T_i} \ln \left( g_{T_i}(\bar{t}_i) - f^* \circ \varphi_{i}(\bar{t}_i) \right) \bar{\mu}_i(d\bar{t}_i).
\]

(14)

Since $\mu = \bar{\mu} \circ \varphi^{-1}$, we have for any $f \circ \varphi \in F^* \circ \varphi$

\[
\sum_{i=1}^{n} \int_{T_i} \ln \left( f(\varphi(\bar{t}_i)) - f^* \circ \varphi(\bar{t}_i) \right) \bar{\mu}_i(d\bar{t}_i) \\
= \sum_{i=1}^{n} \int_{T_i} \ln \left( \tilde{f}(\varphi(\bar{t}_i)) - f^* \circ \varphi(\bar{t}_i) \right) \bar{\mu}_i(d\bar{t}_i) \\
\geq \sum_{i=1}^{n} \int_{T_i} \ln \left( \tilde{f}(\varphi(\bar{t}_i)) - f^* \circ \varphi(\bar{t}_i) \right) \bar{\mu}_i(d\bar{t}_i) \\
= \sum_{i=1}^{n} \int_{T_i} \ln \left( \tilde{f}(\varphi(\bar{t}_i)) - f^* \circ \varphi(\bar{t}_i) \right) \bar{\mu}_i(d\bar{t}_i).
\]

(15)

Which means $\tilde{f} \circ \varphi$ is indeed the maximizer, hence $L^*$ satisfies Axiom 7 (Type Transformation).

\[\square\]

In the next subsection, we will see that $L^*$ is the unique solution that satisfies Axioms 1 to 7.

3.3. The uniqueness of the solution. Due to the Axioms 1 and 4 , it suffices to consider the case that $f^m \geq f^* \quad \text{a.s.}$, for any $f^m \in F^*$. Let $L$ be another solution that satisfies Axioms 1 to 7, and in this subsection, we aim to show $L = L^*$. In doing so, we prove a sequence of lemmas that finally deduce $L = L^*$. It should be noted that, the uniqueness holds modulo zero measure sets with respect to $\mu$.

**Lemma 3.4.** For proper $B$, if $f^* = (0, \cdots, 0)$, $T_1 = \cdots = T_n = T_0$ where $T_0$ is a compact group and $\mu_1 = \cdots = \mu_n = \eta_0$ is invariant, and $L$ satisfies Axioms 1 to 7, then $L = L^*$. 
Proof. Let \( \hat{f} = L^*(B) \) be the maximizer of

\[
\max_{f^* \in F^*} \sum_{i=1}^{n} \int_{T_0} \ln(f_i^*(t_0))\eta_0(dt_0).
\]

(16)

Then for any \( g \in F^* \) and \( \lambda \in [0, 1] \), taking \( \lambda g + (1 - \lambda)\hat{f} \) into consideration, the above formula is maximized when \( \lambda = 0 \). This gives us the first order necessary condition, i.e., for all \( g \in F^* \)

\[
\sum_{i=1}^{n} \int_{T_0} g_i(t_0)\hat{f}_i(t_0)^{-1}\eta_0(dt_0) \leq n.
\]

(17)

Define \( F^2 = \{ g \mid \sum_{i=1}^{n} \int_{T_0} g_i(t_0)\eta_0(dt_0) \leq n, g_i > 0 \text{ and } g_i \in PM(T_0, Borel(T_0)) \} \) and \( B^1 = ((F^2, 0^n), (T_0^n, F^n_\mu_0)) \). For any \( f^2 \in F^2 \), we have

\[
\sum_{i=1}^{n} \int_{T_0} \ln(f_i^2(t_0))\eta_0(dt_0) \leq \sum_{i=1}^{n} \int_{T_0} \hat{f}_i^2(t_0)\eta_0(dt_0) \leq 0.
\]

(18)

The above inequality becomes equality when \( f^2 = 1^n \), and we know that \( 1^n \in F^2 \). Hence \( B^1 \) is proper, and \( 1^n \) is one maximizer of (10).

Using Axiom 2 (Player Symmetry) and Axiom 6 (Type Symmetry), we know that \( L(B^2) \) is a constant vector in \( \mathbb{R}^n \) and all coordinates are equal. Using Axiom 3 (Efficiency), we know that \( L(B^2) = 1^n \). Using Axiom 4 (Positive Affine Invariance), we know that \( L((f^* \circ f^* \circ p), (T^n_\mu_0, F^n_\mu_0)) = \hat{f} \). Noting that \( f \in F^* \subseteq f^* \circ F^2 \), and using Axiom 5 (Irrelevant Alternatives), we have \( L(B) = f = L^*(B) \). \( \square \)

Using Axiom 4 (Positive Affine Invariance) again, we know \( L = L^* \) for \( f^* \neq 0^n \) in the above lemma.

Lemma 3.5. For proper \( B \), if \( \mu_i = \eta_i \) be the invariant measure of \( T_i \) where \( T_i \) is a compact group, and \( L \) satisfies Axioms 1 to 7, then \( L = L^* \).

Proof. Let \( p_i : T \to T_i \) be the projection which is Borel measurable, and let \( p = (p_1, \cdots, p_n) \). Then it is easy to check that \( ((F^* \circ p, f^* \circ p), (T^n_\mu_0, F^n_\mu_0)) \) satisfies all the conditions in Lemma 3.4 (Lemma 3.4 also holds for \( f^* \neq 0^n \)). Using Axiom 7 (Type Transformation) and Lemma 3.4, we have

\[
L((F^* \circ p, f^* \circ p), (T^n_\mu_0, F^n_\mu_0)) = L^*((F^* \circ p, f^* \circ p), (T^n_\mu_0, F^n_\mu_0)).
\]

(19)

Since \( p \) is a projection, it is surjective, hence \( L = L^* \). \( \square \)

Finally, we are able to show that \( L = L^* \) for all proper bargaining bases.

Theorem 3.6. For any proper bargaining basis \( B \), if \( L \) satisfies Axioms 1 to 7, then \( L(B) = L^*(B) \).

Proof. Since \( (T, F, \mu) \) is proper, there exists a compact group \( T_\tilde{\mu} \) with invariant measure \( \tilde{\eta} \) and a Borel measurable \( \phi : T_i \to T_\tilde{\mu} \) such that \( \mu_i = \tilde{\eta}_i \circ \phi_i^{-1} \). Let \( \varphi = (\varphi_1, \cdots, \varphi_n) \), then it is easy to check that \( ((F^* \circ \varphi, f^* \circ \varphi), (T, F, \tilde{\eta})) \) is a proper bargaining basis with \( \tilde{\eta} \) being invariant. Hence by Lemma 3.5, we know that

\[
L((F^* \circ \varphi, f^* \circ \varphi), (T, F, \eta)) = L^*((F^* \circ \varphi, f^* \circ \varphi), (T, F, \tilde{\eta})).
\]

(20)

Using Axiom 7 (Type Transformation), we have \( L(B) \circ \varphi = L^*(B) \circ \varphi \). Note that the probability of the image of \( \varphi \) is 1 (i.e., \( \mu_i(Im(\varphi_i)) = 1 \)), hence \( L(B) = L^*(B) \) almost surely, i.e. \( L = L^* \). \( \square \)

In summary, we have shown that the solution of a proper bargaining basis uniquely exists. In the following section, we aim to find the exact solution of an example with infinite type space.
4. Example. The example we are going to introduce here comes from supply chain management. Assume that there is a supply chain which consists of one manufacturer and one retailer. The manufacturer produces one kind of goods, and sells the goods to consumers through the retailer. However, the retailer knows the exact demand of consumers while the manufacturer does not know (since the retailer is closer to the market). The manufacturer only has some prior information about the demand. In this channel, the manufacturer and the retailer decide to cooperate through bargaining. In the following, we define this bargaining game formally.

Define the manufacturer to be Player 1 and the retailer to be Player 2. Player 1 holds no private information, hence \( T_1 = \{0\} \); Player 2 holds private demand information, and we assume the demand information is supported in a closed interval \( T_2 = [a, b] \) \((b > 0)\). Assume that the prior demand information known by Player 1 is a Borel probability measure \( \mu \) which is supported on \([a, b] \). The unit production cost of Player 1 is \( c \) and the retailing price of Player 2 is \( p \) \((p > c \geq 0)\).

The action space of the players can be restricted on \( T = (\pi, m) \), where for any \((q, s) \in T\), \( q \) is the production quantity of Player 1 and \( s \) is the transfer payment from Player 2 to Player 1. If the players fail to cooperate, their disagreement point is \((0, 0)\), which means that they don’t produce goods at all and their disagreement payoff is \((0, 0)\). Given \( t_2 \) (the type of player 2), and an action \((q, s)\), the payoffs of the players are:

\[
\begin{align*}
\pi_1(q, s; t_2) &= s - cq, \\
\pi_2(q, s; t_2) &= p(q \& t_2) - s.
\end{align*}
\]

With the above description, we know that the choice problem is to solve the following maximization problem (see (10)):

\[
\begin{align*}
\max_m & \quad \ln \left\{ \int_{T_2} \left[ \int_A \pi_1(q, s; t_2)m(t_2)(dq)(ds) \right] \mu(dt_2) \right\} \\
& + \int_{T_2} \left\{ \ln \left[ \int_A \pi_2(q, s; t_2)m(t_2)(dq)(ds) \right] \right\} \mu(dt_2) \\
\text{s.t.} & \quad m : T \to PM(A, Borel(A)), \\
& \quad \int_{A} \pi_1(q, s; t_2)m(t_2)(dq)(ds) \mu(dt_2) \geq 0, \\
& \quad \int_{A} \pi_2(q, s; t_2)m(t_2)(dq)(ds) > 0 \quad \text{a.s. with respect to } \mu,
\end{align*}
\]

Since \( \ln(x) \) is concave, we have:

\[
\begin{align*}
\ln \left\{ \int_{T_2} \left[ \int_A \pi_1(q, s; t_2)m(t_2)(dq)(ds) \right] \mu(dt_2) \right\} \\
& + \int_{T_2} \left\{ \ln \left[ \int_A \pi_2(q, s; t_2)m(t_2)(dq)(ds) \right] \right\} \mu(dt_2) \\
\leq & \ln \left\{ \int_{T_2} \left[ \int_A \pi_1(q, s; t_2)m(t_2)(dq)(ds) \right] \mu(dt_2) \right\} \\
& + \ln \left\{ \int_{T_2} \left[ \int_A \pi_2(q, s; t_2)m(t_2)(dq)(ds) \right] \mu(dt_2) \right\} \\
& \leq 2\ln \left\{ \int_{T_2} \left[ \int_A \pi_1(q, s; t_2) + \pi_2(q, s; t_2)m(t_2)(dq)(ds) \right] \mu(dt_2) \right\} \\
& \leq 2\ln \left\{ \int_{T_2} \left[ \int_A \frac{(p - c)t_2}{2} \mu(dt_2) \right] \right\}.
\end{align*}
\]

Thus we know that the maximization problem (22) has an upper bound \( 2\ln\{0.5(p - c)\int_{T_2} t_2\mu(dt_2)\} \), denoted by \( 2\ln \bar{\pi} \), where \( \bar{\pi} = 0.5(p - c)\int_{T_2} t_2\mu(dt_2) > 0 \). In the following, we give a Bayesian incentive compatible mechanism, such that the upper bound is achieved.

Define \( m^*\) to be the measure that gives the point \((t_2, pt_2 - \bar{\pi})\) probability one. Then we can check that \( m^* \) is Bayesian incentive compatible: \( \forall t_2' \in T_2 \)

\[
\begin{align*}
\pi_2(t_2, pt_2 - \bar{\pi}; t_2) &= \bar{\pi} \\
\geq & \bar{\pi} - pt_2' + p(t_2 \& t_2') \\
= & \pi_2(t_2', pt_2' - \bar{\pi}; t_2).
\end{align*}
\]
And we can also check that both players gain no less than the disagreement point:
\[
\int_{T_2} \pi_1(t_2, pt_2 - \bar{\pi}; t_2) \mu(dt_2) = (p - c) \int_{T_2} t_2 \mu(dt_2) - \bar{\pi} = \bar{\pi} > 0,
\]
Finally, we can check that the upper bound is obtained:
\[
\ln \left\{ \int_{T_2} \pi_1(t_2, pt_2 - \bar{\pi}; t_2) \mu(dt_2) \right\} + \int_{T_2} \ln \bar{\pi} \mu(dt_2) = 2 \ln \bar{\pi}.
\]

5. Conclusion. In this paper, we aim to generalize the bargaining solution to the case when type space and action space are both infinite sets. The existence and the uniqueness of the bargaining solution are obtained under some constraints. However, there are still some further problems which can be studied. The first one is that, the types of players are assumed to be independently distributed in this paper. It is interesting to consider the case when the types of players are not independently distributed. Another problem is to give necessary and sufficient conditions that make a bargaining basis proper. When type and action spaces are finite sets, this problem is answered in [8]. However, when type and action spaces are infinite sets, this problem seems difficult. One way to address this problem is to make restrictions on the Bayesian incentive mechanisms that the arbitrator can use. In this paper, the Bayesian incentive compatible mechanism \( m : T \rightarrow PM(A, \text{Borel}(A)) \) is only required to be measurable. However, if all \( T, A \) are further metric spaces, and the Bayesian incentive compatible mechanism \( m : T \rightarrow PM(A, \text{Borel}(A)) \) is required to be \( \text{Lipchitz} - \kappa \) continuous for some constant \( \kappa > 0 \), then we can show that the bargaining basis is proper. But being \( \text{Lipchitz} \) continuous is too restrictive, which excludes many valuable mechanisms. Hence, it is interesting to find more reasonable restrictions on the Bayesian incentive compatible mechanisms. Finally, a challenging future work is to provide an efficient algorithm to find the bargaining solution for all proper bargaining bases.

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