

**Comments on two models for operating
two-warehouse inventory systems with deteriorating
items and inflationary effects**

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Abstract

This paper deals with the two-warehouse partial backlogging inventory problem under inflation for a deteriorating product with a constant demand rate over an infinite horizon. In contrast to the traditional model in which each replenishment cycle starts with an instant replenishment and ends with shortages, an alternative model has been proposed in recent literature in which each cycle starts with shortages, and it is proven to be less expensive to operate than the traditional model in terms of the present value of the cost per unit time. The present paper points out that the criteria of minimizing the cost per unit time is unreasonable when the inflationary effect is taken into consideration, and instead, the criteria of minimizing the present value of the total cost over the whole infinite planning horizon should be used. Along this line, we change the objective functions of these two models and prove that the model with shortages at the start of the cycle is also less expensive to operate than the traditional model in terms of the present value of the total cost, but the optimal solutions of the models minimizing the cost per unit time indicate significantly higher total costs.

Keywords: Inventory; Two-Warehouse; Partial backlogging; Deterioration; Inflation

Introduction

The inventory problem for deteriorating items with deterministic demand rates has been studied extensively in the literature. Nahmias (1982), and Goyal and Giri (2001) provided excellent reviews for research works in this field before 1980s and 2000s respectively. The problem attracts growing interests from researchers thereafter (Dye, Ouyang & Hsieh, 2007a & 2007b; Shah, Shah & Wee, 2009; Teng, Chang, Dye & Hung, 2002). In contrast to the studies that assume an organization owns only a single warehouse with unlimited capacity, the last decade sees an explosive number of studies focusing on two warehouses, i.e., an owned warehouse (OW) with limited capacity and a rented warehouse (RW) which is assumed to be available with abundant capacity. An early discussion on the effect of two warehouses can be traced back to Hartely (1976, pp.315-317), and recently the two-warehouse inventory models have been considered by many other researchers (Chung, Her & Lin, 2009; Dey, Mondal & Maiti 2008; Dye, Ouyang & Hsieh, 2007b; Gayen & Pal, 2009; Hsieh, Dye & Ouyang, 2008; Lee & Hsu, 2009; Niu & Xie 2008; Rong, Mahapatra & Maiti 2008; Wee, Yu & Law, 2005; Yang, 2004 & 2006; Zhou, 2003; Zhou & Yang, 2005).

In traditional inventory models, it is generally assumed that each replenishment cycle starts with an instant replenishment and ends with shortages. In a recent paper, Yang (2004) considered the two-warehouse inventory problem for deteriorating items with shortages under inflation, and proposed an alternative model in which each cycle begins with shortages and ends without shortages. Under some assumptions, he proved that the model with shortages at the start of the cycle is less expensive to

operate than the traditional models under the objective of minimizing the cost per unit time. More recently, Yang (2006) extended the completely backlogging model to incorporate partial backlogging.

Among all literature on inventory management, the models provided by Yang (2004 & 2006) are the only models where each cycle begins with shortages and ends without shortages. These models are unique in literature and the practical managers might have a special interest in them if they are really less expensive to operate than traditional models. However, careful examinations of these models reveal that there are some serious shortcomings in the objective functions used in these models. First, the objective function to be minimized in these models is the present value of the cost per unit time. When the inflationary effect is taken into consideration and the planning horizon is infinite, the objective function to be minimized should be the present value of the total cost over the whole planning horizon (Dye, Ouyang & Hsieh, 2007a; Hsieh, Dye & Ouyang, 2008; Kim & Philippatos, 1986). Second, these models ignore the purchasing cost in the objective function, just as the models without the inflationary effect usually do since the purchasing cost is a constant. But when the inflationary effect exists and the shortages are partially or completely backlogged, the total purchasing cost is not a constant and thus should be taken into consideration explicitly (Teng, Chang, Dye & Hung, 2002; Wee, Yu & Lau, 2005). Finally, the present value of opportunity cost due to lost sales is evaluated at the time when the next replenishment occurs in Yang (2004 & 2006). However, when the inflationary effect is considered, this cost should be evaluated just at the time when the actual lost

sales occur. Owing to these shortcomings, these models and the resulting optimal replenishment cycles derived from them are suspicious. In fact, the optimal replenishment schedule obtained in Theorem 1 of the paper by Yang (2006) is only a local minimum solution, and the global minimum solution does not exist for the models (the present value of the cost per unit time converges to zero when the replenishment cycle goes to infinity). In order to avoid misleading the practical managers, it is important to correct these models and conduct a fair comparison of these models with the traditional models.

In this paper, we reformulate and compare the two models in Yang (2006) under the objective of minimizing the present value of the total cost over the whole planning horizon. In the next two sections, the above-mentioned shortcomings in Yang (2006) will be discussed in details, and the modified models with the objective of minimizing the present value of the total cost over the whole planning horizon will be developed to overcome these shortcomings. Then we characterize the optimal solutions for the modified models, and draw a conclusion that the model with shortages at the start of the cycle is also less expensive to operate in terms of the present value of the total cost. Finally, some numerical examples are presented to show that the optimal solutions of the models minimizing the cost per unit time lead to significantly higher total costs than the modified models.

Analysis of the existing models

For simplicity, we refer to the model that starts with an instant replenishment and

ends with shortages as Model 1, and refer to the model that starts with a shortage and ends without shortage as Model 2. Throughout this paper, we follow the notation and assumptions used in Yang (2006) except that the present value of the cost per unit time for Model i ($i = 1, 2$) in Yang (2006) is denoted by TC_i^{Yang} . Besides, we introduce CC_i , TC_i , TC_i^1 , $i = 1, 2$ (please see below).

$f(t)$ = the demand rate at time t , which is assumed to be deterministic at a constant rate of D units per unit time, i.e., $f(t) = D$.

W = the capacity of OW (the capacity of RW is assumed to be unlimited).

$\delta(t) = e^{-\sigma t}$, the backlogging rate where $\sigma \geq 0$ is a constant and t is the waiting time.

α = the deterioration rate in OW, where $0 < \alpha < 1$.

β = the deterioration rate in RW, where $0 < \beta < 1$.

r = the inflation rate.

c_o = the replenishment cost per order.

c_p = the purchasing cost per unit.

c_{h1} = the holding cost per unit per unit time in OW.

c_{h2} = the holding cost per unit per unit time in RW. $c_{h1} + \alpha c_p < c_{h2} + \beta c_p$.

c_b = the backlogging cost per unit per unit time, if the shortage is backlogged.

c_l = the unit opportunity cost due to lost sale, if the shortage is lost. Specifically,

$c_l = p + c_g$, where p is the selling price of the product, and c_g is the cost of lost goodwill. Clearly, $c_l \geq p > c_p$.

t_r , t_r' = the time at which the inventory level reaches zero in RW.

t_o , t_o' = the time at which the inventory level reaches zero in OW.

t_s, t'_s = the time at which the shortage level reaches the lowest point in the replenishment cycle.

TC_i^{Yang} = the present value of the relevant cost per unit time for Model i in Yang (2006), $i=1,2$.

CC_i = the present value of the cost for the first cycle for the modified Model $i, i=1, 2$.

TC_i = the present value of the total cost over the whole planning horizon for the modified Model $i, i=1, 2$.

TC_i^1 = the present value of the total cost over the whole planning horizon for the modified Model $i, i=1, 2$, when the capacity of OW is so abundant that the RW is not to be used any more.

Note that the inventory costs (including holding cost and deterioration cost) in RW are assumed to be higher than those in OW (i.e., $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$). Thus, the inventories are first stored in OW with overflows going to RW. But when retrieving for consumption, it is always from RW when available before retrieving from OW. Please also note that it is more reasonable to assume $t_s \geq t_o \geq t_r \geq 0$ and $t'_o \geq t'_r \geq t'_s$, other than $t_s > t_o > t_r > 0$ and $t'_o > t'_r > t'_s$ as in Yang (2006).

The inventory situation for Model 1 and Model 2 can be depicted graphically in Figures 1 and 2 (adopted from Yang (2006)), respectively.

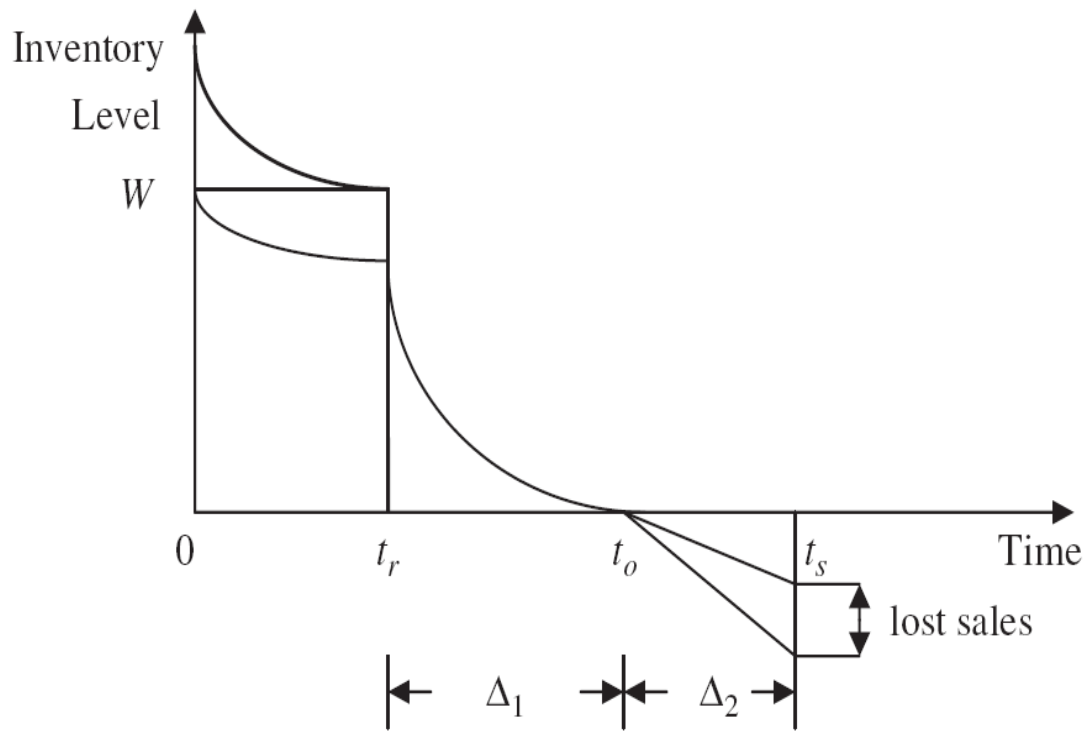


Figure 1. Graphical representation of a two-warehouse inventory system for Model 1.

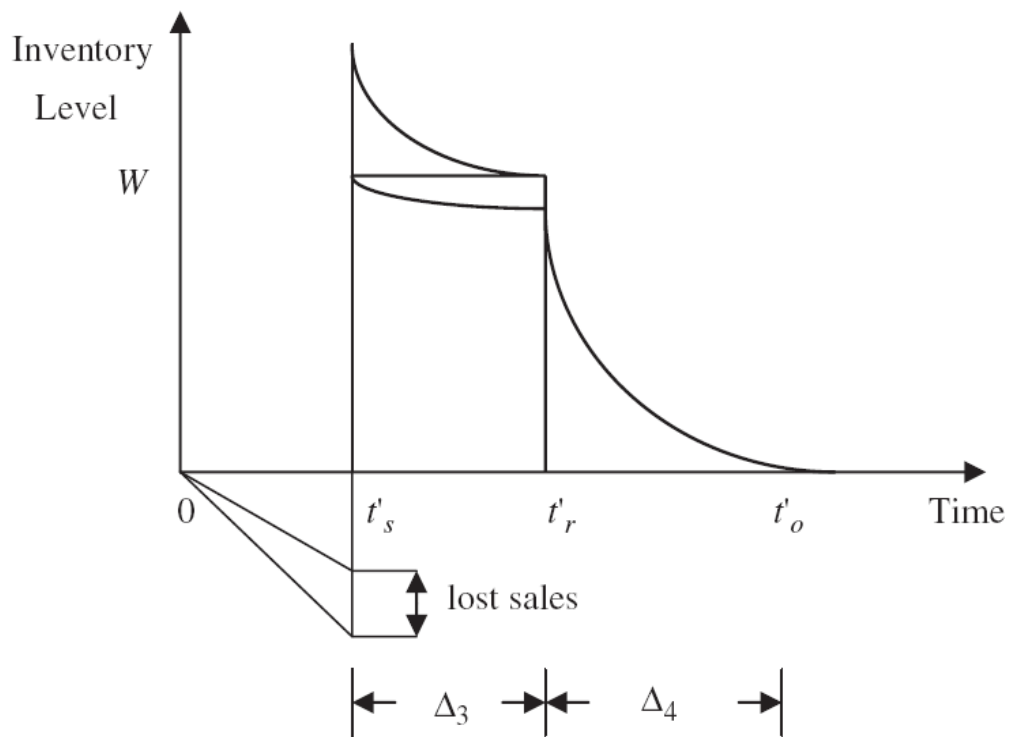


Figure 2. Graphical representation of a two-warehouse inventory system for Model 2.

In Yang (2006), the objective functions to be minimized are the present values of the relevant cost per unit time during the first cycle. Specifically, for Models 1 and 2, the objective functions are expressed as follows:

$$\begin{aligned}
& TC_1^{Yang} \\
&= \left\{ c_o + \frac{c_{h2} + \beta c_p}{\beta + r} \left[\frac{1}{\beta} (e^{\beta t_r} - 1) - \frac{1}{r} (1 - e^{-r t_r}) \right] D + \frac{c_{h1} + \alpha c_p}{\alpha + r} \left[W - \frac{D}{r} e^{-r t_r} (1 - e^{-r \Delta_1}) \right] \right. \\
&\quad \left. + e^{-r(t_r + \Delta_1 + \Delta_2)} D \int_{t_o}^{t_s} \left[\frac{c_b}{r} (e^{r(t_s - t)} - 1) \delta(t_s - t) + c_l (1 - \delta(t_s - t)) \right] dt \right\} / (t_r + \Delta_1 + \Delta_2)
\end{aligned} \quad (1)$$

where

$$\Delta_1 = t_o - t_r = \ln(1 + \alpha W e^{-\alpha t_r} / D) / \alpha, \Delta_2 = t_s - t_o, \quad (2)$$

$$\begin{aligned}
& TC_2^{Yang} \\
&= e^{-r t'_s} \left\{ c_o + \frac{c_{h2} + \beta c_p}{\beta + r} \left[\frac{1}{\beta} (e^{\beta \Delta_3} - 1) - \frac{1}{r} (1 - e^{-r \Delta_3}) \right] D \right. \\
&\quad \left. + \frac{c_{h1} + \alpha c_p}{\alpha + r} \left[W - \frac{D}{r} e^{-r \Delta_3} (1 - e^{-r \Delta_4}) \right] \right. \\
&\quad \left. + D \int_0^{t'_s} \left[\frac{c_b}{r} (e^{r(t'_s - t)} - 1) \delta(t'_s - t) + c_l (1 - \delta(t'_s - t)) \right] dt \right\} / (t'_s + \Delta_3 + \Delta_4)
\end{aligned} \quad (3)$$

where

$$\Delta_3 = t'_r - t'_s, \Delta_4 = t'_o - t'_r = \ln(1 + \alpha W e^{-\alpha \Delta_3} / D) / \alpha. \quad (4)$$

From Eq. (2) (or correspondingly, Eqn. (4)), it is obvious that TC_1^{Yang} (TC_2^{Yang}) can be explicitly expressed as a function with only two decision variables t_r and Δ_2 (Δ_3 and t'_s).

Theorem 1. (a) For fixed $t_r \geq 0$, $\lim_{\Delta_2 \rightarrow \infty} TC_1^{Yang} = 0$.

(b) For fixed $\Delta_3 \geq 0$, $\lim_{t'_s \rightarrow \infty} TC_2^{Yang} = 0$.

Proof. See Appendix 1.

Therefore, the solutions obtained from Yang (2006) are only local minimum solutions. In fact, neither TC_1^{Yang} nor TC_2^{Yang} has a global minimum. For example,

using the data of Example 1 of Yang (2006) and fixing $t_r(\Delta_3)=0.4903$ (the local minimum point), we can plot the objective functions TC_1^{Yang} and TC_2^{Yang} in terms of Δ_2 and t'_s respectively (see Figures 3 and 4). It's easy to see that $\Delta_2(t'_s)=0.0542$ is only a local minimum, and TC_1^{Yang} (TC_2^{Yang}) convergences to zero when $\Delta_2(t'_s)$ goes to infinity.

From above observations, the models in Yang (2006) and the conclusions drawn by them are suspicious. The critical point is that, when the inflationary effect is taken into consideration and the planning horizon is infinite, the objective of minimizing the present value of the cost per unit time is unreasonable. In the next section, we will reformulate the models in terms of the present value of the total cost over the whole planning horizon.

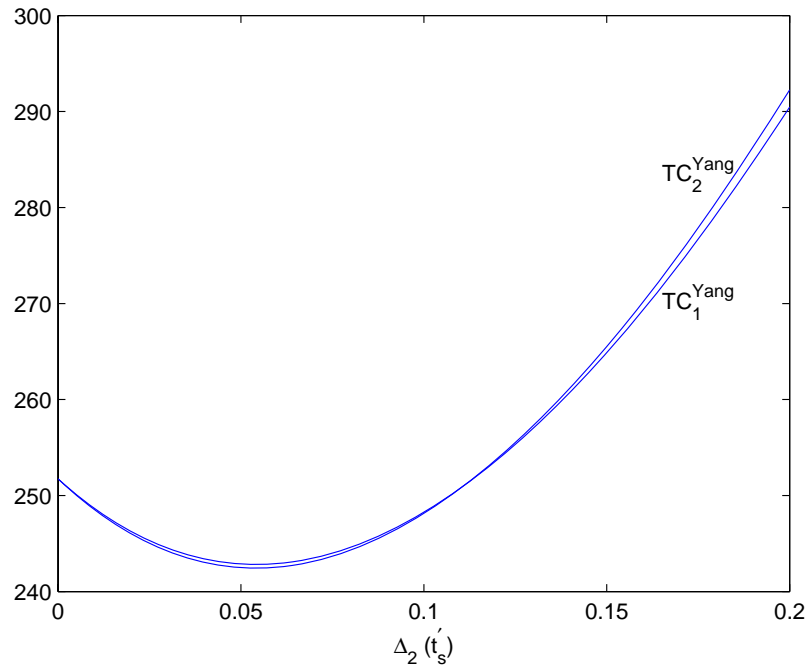


Figure 3. TC_1^{Yang} (TC_2^{Yang}) as a function of $\Delta_2(t'_s)$ when $\Delta_2(t'_s)$ is small

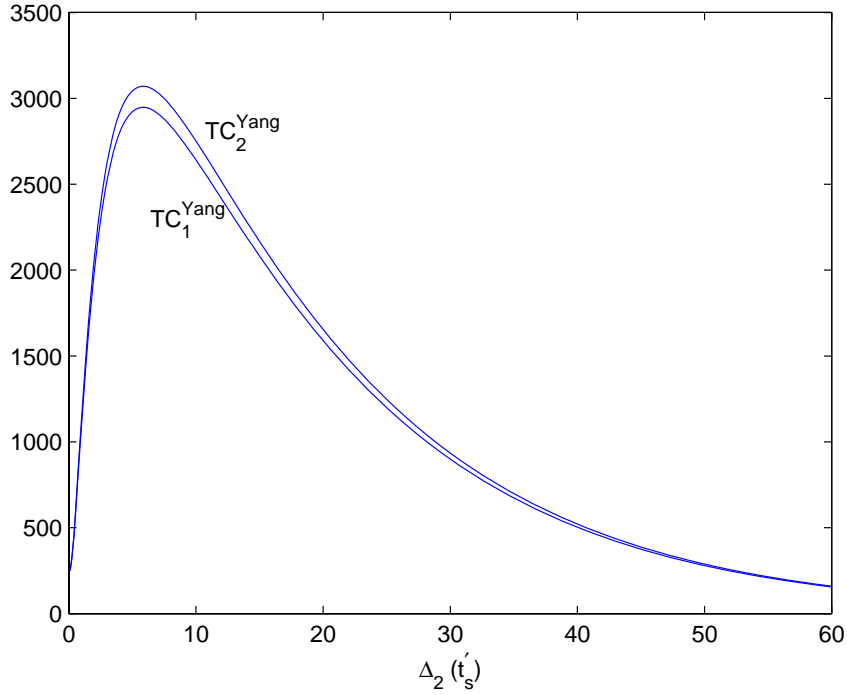


Figure 4. TC_1^{Yang} (TC_2^{Yang}) as a function of $\Delta_2(t'_s)$ when $\Delta_2(t'_s)$ is large

Formulation of the modified models

In this section, we propose modified models for the same problems as in Yang (2006). Let us firstly consider the present value of the relevant cost for the first cycle (one-cycle cost) for Model 1. We make two modifications of the one-cycle cost in Yang (2006) as follows.

First, the present value of the opportunity cost due to lost sales is expressed as $c_l e^{-rt_s} \int_{t_o}^{t_s} [1 - \delta(t_s - t)] f(t) dt$ (A.18) in Appendix A in Yang (2006), which implies that the cost is evaluated at time t_s (i.e., the end of the cycle). This is unreasonable because of that the cost should be evaluated just at the time the actual lost sales occur. Therefore, this cost should be expressed as $c_l \int_{t_o}^{t_s} e^{-rt} [1 - \delta(t_s - t)] f(t) dt$.

Second, when the inflationary effect is considered and the shortages are partially or

completely backlogged, the total purchasing cost is not a constant and should be taken into consideration explicitly (Teng, Chang, Dye & Hung, 2002; Wee, Yu & Lau, 2005). Besides, when the purchase cost is added to the relevant cost, the cost for the deteriorated items ((A.19) in Appendix A in Yang (2006)) does not need to explicitly add in any more. Notice that the items consumed and deteriorated from time 0 to t_o are purchased at the beginning of the cycle (time 0), and the items used to fulfill the backordered demand during time t_o and t_s are purchased at the end of the cycle (t_s). Therefore, the present value of the purchasing cost in the first cycle can be expressed as $c_p[W + D(e^{\beta t_r} - 1) / \beta + e^{-rt_s} D \int_{t_o}^{t_s} \delta(t_s - t) dt]$.

By the above analyses and with a similar method as in Appendix A in Yang (2006), we can come to our present value of the total cost of the first cycle for Model 1:

$$\begin{aligned}
CC_1 = & \{c_o + c_p[W + \frac{D}{\beta}(e^{\beta t_r} - 1)] + c_p e^{-r(t_r + \Delta_1 + \Delta_2)} D \int_{t_o}^{t_s} \delta(t_s - t) dt \\
& + \frac{c_{h2}}{\beta + r} [\frac{1}{\beta}(e^{\beta t_r} - 1) - \frac{1}{r}(1 - e^{-rt_r})] D + \frac{c_{h1}}{\alpha + r} [W - \frac{D}{r} e^{-rt_r} (1 - e^{-r\Delta_1})] \\
& + e^{-r(t_r + \Delta_1 + \Delta_2)} D \int_{t_o}^{t_s} [\frac{c_b}{r}(e^{r(t_s - t)} - 1) \delta(t_s - t) + c_i e^{r(t_s - t)} (1 - \delta(t_s - t))] dt\}. \quad (5)
\end{aligned}$$

Since the planning horizon of the inventory problem is infinite and the inflationary effect is considered, we should use the present value of the total cost for the infinite planning horizon as our objective function to be minimized. The similar approach has been used to inventory models under inflation (Dye, Ouyang & Hsieh, 2007a; Hsieh, Dye & Ouyang, 2008; Kim & Philippatos, 1986). Making use of the one-cycle cost in Eqn. (5), the present value of the total cost over the infinite planning horizon for Model 1 is

$$TC_1 = CC_1 \sum_{n=0}^{\infty} e^{-nr(t_r+\Delta_1+\Delta_2)}. \quad (6)$$

Noticing

$$\sum_{n=0}^{\infty} e^{-nr(t_r+\Delta_1+\Delta_2)} = \frac{1}{1 - e^{-r(t_r+\Delta_1+\Delta_2)}}, \quad (7)$$

and substituting Eqn. (5) into Eqn. (6), we obtain

$$\begin{aligned} TC_1 = & \frac{1}{(1 - e^{-r(t_r+\Delta_1+\Delta_2)})} \{c_o + c_p [W + \frac{D}{\beta}(e^{\beta t_r} - 1)] + c_p e^{-r(t_r+\Delta_1+\Delta_2)} D \int_{t_o}^{t_s} \delta(t_s - t) dt \\ & + \frac{c_{h2}}{\beta + r} [\frac{1}{\beta}(e^{\beta t_r} - 1) - \frac{1}{r}(1 - e^{-rt_r})] D + \frac{c_{h1}}{\alpha + r} [W - \frac{D}{r} e^{-rt_r} (1 - e^{-r\Delta_1})] \\ & + e^{-r(t_r+\Delta_1+\Delta_2)} D \int_{t_o}^{t_s} [\frac{C_b}{r}(e^{r(t_s-t)} - 1)\delta(t_s - t) + c_i e^{r(t_s-t)} (1 - \delta(t_s - t))] dt\}, \end{aligned} \quad (8)$$

where

$$\Delta_1 = t_o - t_r = \ln(1 + \alpha W e^{-\alpha t_r} / D) / \alpha, \Delta_2 = t_s - t_o. \quad (9)$$

Similarly, the present value of the total relevant cost over the infinite planning horizon for Model 2 can be expressed as

$$\begin{aligned} TC_2 = & \frac{e^{-rt'_s}}{(1 - e^{-r(t'_s+\Delta_3+\Delta_4)})} \{c_o + c_p [W + \frac{D}{\beta}(e^{\beta \Delta_3} - 1)] + c_p D \int_0^{t'_s} \delta(t'_s - t) dt \\ & + \frac{c_{h2}}{\beta + r} [\frac{1}{\beta}(e^{\beta \Delta_3} - 1) - \frac{1}{r}(1 - e^{-r\Delta_3})] D + \frac{c_{h1}}{\alpha + r} [W - \frac{D}{r} e^{-r\Delta_3} (1 - e^{-r\Delta_4})] \\ & + D \int_0^{t'_s} [\frac{C_b}{r}(e^{r(t'_s-t)} - 1)\delta(t'_s - t) + c_i e^{r(t'_s-t)} (1 - \delta(t'_s - t))] dt\}, \end{aligned} \quad (10)$$

where

$$\Delta_3 = t'_r - t'_s, \Delta_4 = t'_o - t'_r = \ln(1 + \alpha W e^{-\alpha \Delta_3} / D) / \alpha. \quad (11)$$

Solutions to the inventory models

From Eqn. (9), we know Δ_1 is a function of t_r , i.e., $\Delta_1 = \Delta_1(t_r) = \ln(1 + \alpha W e^{-\alpha t_r} / D) / \alpha$ (in the rest of the paper, we denote $\Delta_1(t_r)$ by Δ_1 if no

ambiguity results from it), thus TC_1 can be explicitly expressed as a function with only two decision variables t_r and Δ_2 . Similarly, TC_2 can be explicitly expressed as a function of Δ_3 and t'_s . For convenience, denote

$$A(t_r) = c_0 + c_p \left[W + \frac{D}{\beta} (e^{\beta t_r} - 1) \right] + \frac{c_{h2}}{\beta + r} \left[\frac{1}{\beta} (e^{\beta t_r} - 1) - \frac{1}{r} (1 - e^{-r t_r}) \right] D + \frac{c_{h1}}{\alpha + r} \left[W - \frac{D}{r} e^{-r t_r} (1 - e^{-r \Delta_1}) \right], \quad (12)$$

$$B(\Delta_2) = D \int_0^{\Delta_2} \left[c_p \delta(y) + \frac{c_b}{r} (e^{r y} - 1) \delta(y) + c_l e^{r y} (1 - \delta(y)) \right] dy. \quad (13)$$

It's easy to check

$$TC_1 = TC_1(t_r, \Delta_2) = \frac{1}{1 - e^{-r(t_r + \Delta_1 + \Delta_2)}} \left[A(t_r) + e^{-r(t_r + \Delta_1 + \Delta_2)} B(\Delta_2) \right], \quad (14)$$

$$TC_2 = TC_2(\Delta_3, t'_s) = \frac{e^{-r t'_s}}{1 - e^{-r(\Delta_3 + \Delta_4 + t'_s)}} \left[A(\Delta_3) + B(t'_s) \right]. \quad (15)$$

For Model 1, from Eqn. (9), we have

$$\frac{d\Delta_1}{dt_r} = \frac{-\alpha W e^{-\alpha t_r}}{D + \alpha W e^{-\alpha t_r}} = e^{-\alpha \Delta_1} - 1. \quad (16)$$

The first order conditions for TC_1 in Eqn. (14) to be minimized are

$$\begin{aligned} & \frac{\partial TC_1(t_r, \Delta_2)}{\partial t_r} \\ &= \frac{1}{\left(1 - e^{-r(t_r + \Delta_1 + \Delta_2)}\right)^2} \left\{ -r e^{-r(t_r + \Delta_1 + \Delta_2)} e^{-\alpha \Delta_1} \left(A(t_r) + B(\Delta_2) \right) + \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2)}\right) A'(t_r) \right\} \\ &= 0, \end{aligned} \quad (17)$$

$$\begin{aligned} & \frac{\partial TC_1(t_r, \Delta_2)}{\partial \Delta_2} \\ &= \frac{1}{\left(1 - e^{-r(t_r + \Delta_1 + \Delta_2)}\right)^2} \left\{ -r \left(A(t_r) + B(\Delta_2) \right) + \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2)}\right) B'(\Delta_2) \right\} \\ &= 0. \end{aligned} \quad (18)$$

By Eqns. (12) and (13), we have

$$A'(t_r) = c_p D e^{\beta t_r} + \left[\frac{c_{h2}}{\beta + r} (e^{\beta t_r} - e^{-r t_r}) + \frac{c_{h1}}{\alpha + r} e^{-r t_r} (1 - e^{-(r+\alpha)\Delta_1}) \right] D, \quad (19)$$

$$B'(\Delta_2) = D \left\{ c_p \delta(\Delta_2) + \frac{c_b}{r} (e^{r\Delta_2} - 1) \delta(\Delta_2) + c_l e^{r\Delta_2} [1 - \delta(\Delta_2)] \right\}. \quad (20)$$

Denote

$$K_1(t_r) = e^{(\alpha+r)\Delta_1} e^{r t_r} A'(t_r), \quad (21)$$

$$K_2(\Delta_2) = e^{-r\Delta_2} B'(\Delta_2). \quad (22)$$

Then we have the following equation by Eqns. (17) and (18):

$$K_1(t_r) = K_2(\Delta_2). \quad (23)$$

Lemma 1. (a) Suppose that $c_p \geq (\alpha - \beta)c_{h2}/(\beta + r)^2$ and $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$.

$K_1(t_r)$ is a strictly increasing function of t_r .

(b) Suppose $c_b > r c_l$. $K_2(\Delta_2)$ is increasing on $[0, \bar{\Delta}_2]$ and decreasing on $[\bar{\Delta}_2, +\infty)$, where

$$\bar{\Delta}_2 = \frac{1}{r} \ln \frac{(r + \sigma)(c_b - r c_p)}{r(c_b - r c_l)}.$$

Proof. See Appendix 2.

In Part (a) of Lemma 1, the condition $c_p > (\alpha - \beta)c_{h2}/(\beta + r)^2$ is not a rigorous condition and can be satisfied in reality for many cases, e.g., for the case of $\alpha \leq \beta$, which means the deterioration rate in RW is not less than that in OW. In Part (b), the condition $c_b > r c_l$ can be also easily satisfied since the inflationary rate is usually very low. After obtaining the results in Lemma 1, we can come to the following lemma, which characterizes the solutions to Eqns. (17) and (18).

Lemma 2. Under the assumptions $c_p > (\alpha - \beta)c_{h2}/(\beta + r)^2$, $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$ and $c_b > r c_l$, there are at most two positive solutions for Eqns. (17) and (18):

$(t_r^{(1)}, \Delta_2^{(1)})$ and $(t_r^{(2)}, \Delta_2^{(2)})$, where $\Delta_2^{(1)} \in (0, \bar{\Delta}_2]$ and $\Delta_2^{(2)} \in [\bar{\Delta}_2, +\infty)$.

Proof. See Appendix 3.

By Lemma 2, we can characterize the minimum points of TC_1 in the following theorem.

Theorem 2. Assume $c_p > (\alpha - \beta)c_{h2}/(\beta + r)^2$, $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$, and $c_b > rc_1$.

Then $TC_1(t_r, \Delta_2)$ is minimized at $(t_r^*, \Delta_2^*) = (t_r^{(1)}, \Delta_2^{(1)})$, where $\Delta_2^{(1)} \in (0, \bar{\Delta}_2]$, or on the boundary of the domain: $\Delta_2^* = 0$ or $t_r^* = 0$.

Proof. See Appendix 4.

For Model 2, the first order conditions for TC_2 in Eqn. (15) to be minimized are

$$\begin{aligned} & \frac{\partial TC_2(\Delta_3, t_s')}{\partial \Delta_3} \\ &= \frac{e^{-rt_s'}}{\left(1 - e^{-r(\Delta_3 + \Delta_4 + t_s')}\right)^2} \left\{ -re^{-r(\Delta_3 + \Delta_4 + t_s')} e^{-\alpha \Delta_4} \left(A(\Delta_3) + B(t_s') \right) \right. \\ & \quad \left. + \left(1 - e^{-r(\Delta_3 + \Delta_4 + t_s')}\right) A'(\Delta_3) \right\} \\ &= 0, \end{aligned} \tag{24}$$

$$\begin{aligned} & \frac{\partial TC_2(\Delta_3, t_s')}{\partial t_s'^2} \\ &= \frac{e^{-rt_s'}}{\left(1 - e^{-r(\Delta_3 + \Delta_4 + t_s')}\right)^2} \left\{ -r \left(A(\Delta_3) + B(t_s') \right) + \left(1 - e^{-r(\Delta_3 + \Delta_4 + t_s')}\right) B'(t_s') \right\} \\ &= 0. \end{aligned} \tag{25}$$

Following the similar approaches analyzing Model 1, we have the following results for Model 2 (the proofs are omitted to save space since they are similar to those used for Model 1).

Lemma 3. Under the assumptions $c_p > (\alpha - \beta)c_{h2}/(\beta + r)^2$, $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$ and $c_b > rc_1$, there are at most two positive solutions for Eqns. (24) and (25):

$(\Delta_3^{(1)}, t_s^{(1)})$ and $(\Delta_3^{(2)}, t_s^{(2)})$. Furthermore, $(\Delta_3^{(1)}, t_s^{(1)}) = (t_r^{(1)}, \Delta_2^{(1)})$ and $(\Delta_3^{(2)}, t_s^{(2)}) = (t_r^{(2)}, \Delta_2^{(2)})$.

Theorem 3. Assume $c_p > (\alpha - \beta)c_{h2}/(\beta + r)^2$, $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$ and $c_b > rc_1$.

Then $TC_2(\Delta_3, t_s')$ is minimized at $(\Delta_3^*, t_s^*) = (\Delta_3^{(1)}, t_s^{(1)})$, where $t_s^{(1)} \in (0, \bar{\Delta}_2]$, or on the boundary of the domain: $\Delta_3^* = 0$ or $t_s^* = 0$.

Comparison between the models

In this section, we discuss which model is less expensive to operate. For convenience and without loss of generality, we can assume that the variables t_s' , Δ_3 in Model 2 are the same as the optimal solutions Δ_2^* , t_r^* in Model 1. In other words, we compare the values of $TC_2(t_r^*, \Delta_2^*)$ and $TC_1(t_r^*, \Delta_2^*)$.

First, if $\Delta_2^* = 0$, then there is no shortage in the whole planning horizon. In this case, it is clear that $TC_2(t_r^*, \Delta_2^*) = TC_1(t_r^*, \Delta_2^*)$.

Second, we consider the case that $(t_r^*, \Delta_2^*) = (t_r^{(1)}, \Delta_2^{(1)})$. By the assumptions mentioned above, and making use of Enqs. (9), (17) and (18), we have

$$\frac{TC_1(t_r^{(1)}, \Delta_2^{(1)})}{TC_2(t_r^{(1)}, \Delta_2^{(1)})} = e^{r\Delta_2^{(1)}} \frac{A(t_r^{(1)}) + e^{-rT} B(\Delta_2^{(1)})}{A(t_r^{(1)}) + B(\Delta_2^{(1)})} = e^{r\Delta_2^{(1)}} \frac{B'(\Delta_2^{(1)}) - rB(\Delta_2^{(1)})}{B'(\Delta_2^{(1)})}, \quad (26)$$

where $T = t_r^{(1)} + \Delta_1(t_r^{(1)}) + \Delta_2^{(1)}$. By Part (b) of Lemma 1, we have $e^{-ry} B'(y)$ is positive and increasing on $y \in [0, \Delta_2^{(1)}] \subseteq [0, \bar{\Delta}_2]$, which implies that

$$\begin{aligned} B(\Delta_2^{(1)}) &= \int_0^{\Delta_2^{(1)}} B'(y) dy = \int_0^{\Delta_2^{(1)}} e^{ry} e^{-ry} B'(y) dy \\ &< e^{-r\Delta_2^{(1)}} B'(\Delta_2^{(1)}) \int_0^{\Delta_2^{(1)}} e^{ry} dy = B'(\Delta_2^{(1)}) \frac{1 - e^{-r\Delta_2^{(1)}}}{r}. \end{aligned} \quad (27)$$

From (26) and (27), we have

$$\frac{TC_1(t_r^{(1)}, \Delta_2^{(1)})}{TC_2(t_r^{(1)}, \Delta_2^{(1)})} = e^{r\Delta_2^{(1)}} \frac{B'(\Delta_2^{(1)}) - rB(\Delta_2^{(1)})}{B'(\Delta_2^{(1)})} > e^{r\Delta_2^{(1)}} \frac{(1 - (1 - e^{-r\Delta_2^{(1)}}))B'(\Delta_2^{(1)})}{B'(\Delta_2^{(1)})} = 1.$$

Therefore, the following theorem is true.

Theorem 4. *If $(t_r^*, \Delta_2^*) = (t_r^{(1)}, \Delta_2^{(1)})$, then $TC_1(t_r^*, \Delta_2^*) > TC_2(\Delta_3^*, t_s^*)$.*

Theorem 4 indicates that when the capacity of OW is limited and RW should be used, Model 2 is less expensive to operate than Model 1 in terms of the present value of the total cost over the whole planning horizon, unless there are no shortages in both models.

Finally, consider the case of $t_r^* = 0$, which means the capacity of OW is so abundant that there is no need to use RW (i.e., one-warehouse case). In this case, the two-warehouse inventory problem reduces to the one-warehouse (i.e., only OW is available) inventory problem. Recently, Dye, Ouyang and Hsieh (2007b) studied the one-warehouse (i.e., only OW is available) case for the traditional inventory model (like Model 1 in our paper) under the objective of maximizing the discounted profit. They do not consider the inflationary effect (i.e., the inflation rate $r = 0$), but the pricing decisions are included. When the price is not a decision variable but a fixed constant, the similar approach used in our paper can also be extended to the one-warehouse case with inflationary considerations and with shortages at the start of the replenishment cycle (like Model 2 in our paper). Under the two-warehouse case, we always assume that the capacity W of the OW is fully utilized. However, under the one-warehouse case, even the capacity of OW might not be fully utilized in order to minimize the present value of the total cost, thus we should remove the capacity constraints W . Along this line, the objective functions TC_i^1 can be derived for

Models i ($i=1,2$) respectively for the one-warehouse inventory problem (see Appendix 5 for details).

Using the similar approaches as two-warehouse case, one can obtain the similar result as Theorem 4: When the capacity of OW is so abundant that there is no need to use RW, the one-warehouse Model 2 is less expensive to operate than one-warehouse Model 1 in terms of the present value of the total cost over the whole planning horizon, unless there are no shortages in both models.

Numerical examples

The comparison between the two models is illustrated by the following numerical examples.

Example 1 (Yang, 2006). Let $D = 400, W = 100, \delta(t) = e^{-0.6t}, c_o = 100, c_{h1} = 0.2, c_{h2} = 0.6, c_b = 3, c_l = 15, c_p = 10, \alpha = 0.05, \beta = 0.03,$ and $r=0.06$ in appropriate units.

The computational results for the two models are shown in Table 1. As comparisons to Figures 3 and 4, the cost functions TC_1 and TC_2 are plotted in Figures 5 and 6, as a function of $\Delta_2(t'_s)$ for fixed t_r (or Δ_3)=0.2527 (the global optimizer).

Table 1. Computational results for Example 1

Model i	Objective function	Optimal solution			Cycle length	Total cost TC_i
$i=1$	This paper	$t_r=0.2527$	$\Delta_1=0.2472$	$\Delta_2=0.1434$	0.6433	71633.11
	Yang (2006)	$t_r=0.4903$	$\Delta_1=0.2425$	$\Delta_2=0.0542$	0.7870	72122.66
$i=2$	This paper	$\Delta_3=0.2527$	$\Delta_4=0.2472$	$t'_s=0.1434$	0.6433	72003.91
	Yang (2006)	$\Delta_3=0.4904$	$\Delta_4=0.2425$	$t'_s=0.0541$	0.7870	72108.75

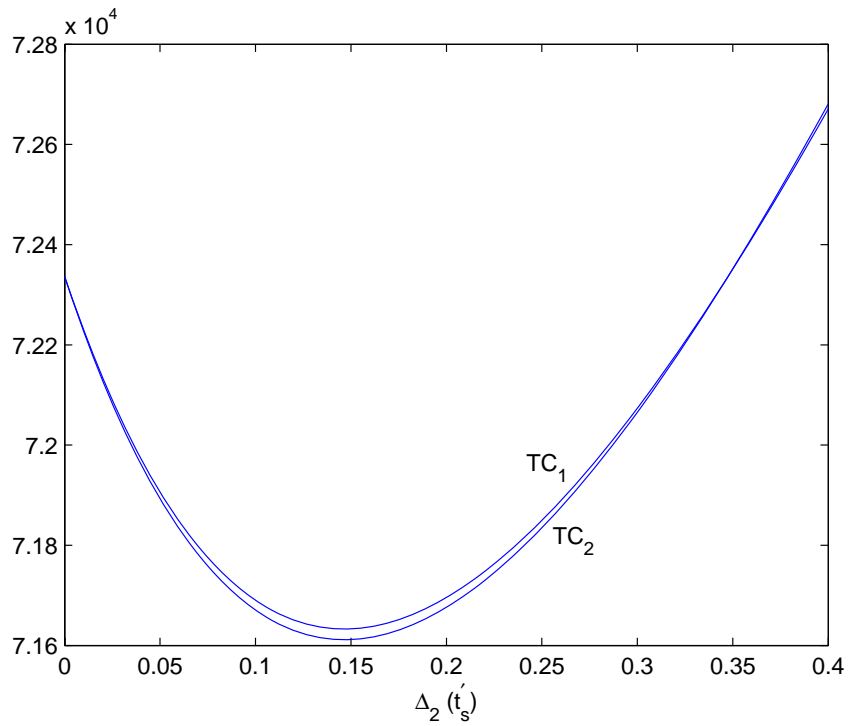


Figure 5. TC_1 (TC_2) as a function of $\Delta_2(t'_s)$ when $\Delta_2(t'_s)$ is large *small*

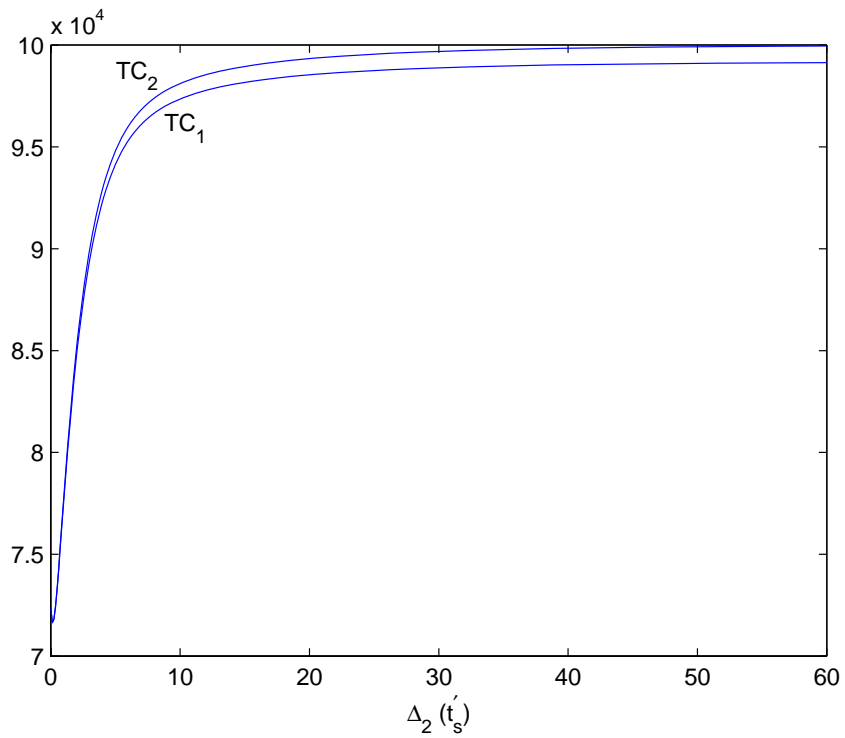


Figure 6. TC_1 and TC_2 as a function of $\Delta_2(t'_s)$ when $\Delta_2(t'_s)$ is large

Example 2(Yang, 2006). Let $D = 400, W = 100, \delta(t) = e^{-0.6t}, c_o = 100, c_{h1} = 1.0,$
 $c_{h2} = 2.5, c_b = 3, c_l = 15, c_p = 10, \alpha = 0.05, \beta = 0.03,$ and $r=0.06$ in appropriate units.

The computational results for the two models are shown in Table 2.

Table 2. Computational results for Example 2

Model	Objective	Optimal solution			Cycle	Total cost
i	function				length	TC_i
$i=1$	This paper	$t_r=0.1034$	$\Delta_1=0.2472$	$\Delta_2=0.1782$	0.5288	72612.92
	Yang (2006)	$t_r=0.1816$	$\Delta_1=0.2462$	$\Delta_2=0.0776$	0.5054	73077.57
$i=2$	This paper	$\Delta_3=0.1034$	$\Delta_4=0.2472$	$t'_s=0.1782$	0.5288	72594.46
	Yang (2006)	$\Delta_3=0.1816$	$\Delta_4=0.2462$	$t'_s=0.0776$	0.5054	73054.13

Examinations of Tables 1 and 2 reveal that the optimal cycle length minimizing the present value of the total cost over the whole planning horizon significantly differs from that minimizing the cost per unit time in Yang (2006), and the solutions derived from Yang (2006) result in significantly higher present value of the total cost. Therefore, the models under inflation should be based on minimizing the present value of the total cost other than based on minimizing the cost per unit time.

Example 3: Let $W=1000$ and the other parameters remain the same as in Example 1 in appropriate units. The computational results for the two models are shown in Table 3. We can see that the optimal $t_r=0$ (or $t'_s=0$) and our models return to one-warehouse models.

Example 4: Let $W=1000$ and the other parameters remain the same as in Example 2 in appropriate units. The computational results for the two models are shown in Table

4. We can see that the optimal $t_r=0$ (or $t'_s=0$) and our models return to one-warehouse models.

Table 3. Computational results for Example 3 for the objective function in this paper

Model 1	$t_r = 0$	$\Delta_1 = 2.3557$	$\Delta_2 = 0.3503$	$TC_1 = 77587.48$
Model 2	$\Delta_3 = 0$	$\Delta_4 = 2.3557$	$t'_s = 0.3503$	$TC_2 = 77479.17$
Model 1 (one-warehouse)	$t_o = 0.5015$		$\Delta_2 = 0.1525$	$TC_1^1 = 71820.09$
Model 2 (one-warehouse)		$\Delta'_4 = 0.5015$	$t'_s = 0.1525$	$TC_2^1 = 71797.08$

Table 4. Computational results for Example 4 for the objective function in this paper

Model 1	$t_r = 0$	$\Delta_1 = 2.3557$	$\Delta_2 = 0.5843$	$TC_1 = 83256.85$
Model 2	$\Delta_3 = 0$	$\Delta_4 = 2.3557$	$t'_s = 0.5843$	$TC_2 = 82993.12$
Model 1 (one-warehouse)	$t_o = 0.4331$		$\Delta_2 = 0.1674$	$TC_1^1 = 72288.40$
Model 2 (one-warehouse)		$\Delta'_4 = 0.4331$	$t'_s = 0.1674$	$TC_2^1 = 72260.92$

The results in Tables 3 and 4 show that Model 2 is always less expensive to operate than Model 1 in terms of the present value of the total cost, no matter it is necessary to use RW or not.

Conclusion

In this paper, we consider the two-warehouse partial backlogging inventory problem under inflation for a single deteriorating product with a constant demand rate over an infinite horizon. We point out three serious shortcomings in the models of a recent paper by Yang (2006), and change the objective from minimizing the cost per unit time to minimizing the present value of the total cost over the infinite horizon.

Under the new objective, comparisons of the two models confirm that if the inflation rate is greater than zero the model with shortages at the start of the cycle is still less expensive to operate than the traditional model without shortages at the start of the cycle. However, the solutions derived from Yang (2006) result in significantly higher present value of the total cost for the infinite horizon. Therefore, the models under inflation should be based on minimizing the present value of the total cost over the whole planning horizon other than based on minimizing the cost per unit time. Obviously, similar discussions as this paper also apply to the situation with complete backlogging (Yang, 2004), which is only a special case of partial backlogging. Future researches can be pursued to consider the situations where the demand and/or the inflation rate vary over time.

Appendix 1. Proof of Theorem 1

Denote

$$\tilde{A}(t_r) = c_o + \frac{c_{h2} + \beta c_p}{\beta + r} \left[\frac{1}{\beta} (e^{\beta t_r} - 1) - \frac{1}{r} (1 - e^{-r t_r}) \right] D + \frac{c_{h1} + \alpha c_p}{\alpha + r} \left[W - \frac{D}{r} e^{-r t_r} (1 - e^{-r \Delta_1}) \right],$$

$$\tilde{B}(\Delta_2) = D \int_0^{\Delta_2} \left[\frac{c_b}{r} (e^{r y} - 1) \delta(y) + c_i (1 - \delta(y)) \right] dy.$$

It can be easily verified that

$$TC_1^{Yang} = \frac{\tilde{A}(t_r) + e^{-r(t_r + \Delta_1 + \Delta_2)} \tilde{B}(\Delta_2)}{t_r + \Delta_1 + \Delta_2}.$$

Since

$$\tilde{B}'(\Delta_2) = D \left[\frac{c_b}{r} (e^{r \Delta_2} - 1) \delta(\Delta_2) + c_i (1 - \delta(\Delta_2)) \right],$$

$$\lim_{\Delta_2 \rightarrow +\infty} \tilde{B}'(\Delta_2) = \begin{cases} Dc_l, & \text{if } \sigma > r, \\ D(c_l + \frac{c_b}{r}), & \text{if } \sigma = r, \\ +\infty, & \text{if } \sigma < r, \end{cases}$$

then $\lim_{\Delta_2 \rightarrow +\infty} \tilde{B}(\Delta_2) = \lim_{\Delta_2 \rightarrow +\infty} \int_0^{\Delta_2} \tilde{B}'(y) dy = +\infty$. Thus, by L'Hôpital's rule, we have

$$\begin{aligned} \lim_{\Delta_2 \rightarrow +\infty} e^{-r\Delta_2} \tilde{B}(\Delta_2) &= \lim_{\Delta_2 \rightarrow +\infty} \frac{\tilde{B}'(\Delta_2)}{re^{r\Delta_2}} \\ &= \lim_{\Delta_2 \rightarrow +\infty} \frac{1}{r} D\left[\frac{c_b}{r}(1 - e^{-r\Delta_2})\delta(\Delta_2) + c_l e^{-r\Delta_2}(1 - \delta(\Delta_2))\right] = 0. \end{aligned}$$

Therefore, for fixed $t_r \geq 0$, $\lim_{\Delta_2 \rightarrow +\infty} TC_1^{Yang} = 0$.

This completes the proof of Part (a). The proof of Part (b) is similar and thus it is omitted to save space.

Appendix 2. Proof of Lemma 1.

Part (a). Differentiating $K_1(t_r) = e^{(\alpha+r)\Delta_1} e^{rt_r} A'(t_r)$, we have

$$K_1'(t_r) = e^{(\alpha+r)\Delta_1} \left[((\beta+r)c_p + c_{h2})e^{(\beta+r)t_r} + \left((\alpha+r)c_p + \frac{\alpha+r}{\beta+r}c_{h2}(e^{(\beta+r)t_r} - 1) + c_{h1} \right) \frac{d\Delta_1}{dt_r} \right].$$

Since $d\Delta_1/dt_r = e^{-\alpha\Delta_1} - 1 > -1$ and $c_{h2} + \beta c_p > c_{h1} + \alpha c_p$, then

$$\begin{aligned} K_1'(t_r) &> e^{(\alpha+r)\Delta_1} \left[((\beta+r)c_p + c_{h2})e^{(\beta+r)t_r} - \left((\alpha+r)c_p + \frac{\alpha+r}{\beta+r}c_{h2}(e^{(\beta+r)t_r} - 1) + c_{h1} \right) \right] \\ &> e^{(\alpha+r)\Delta_1} (e^{(\beta+r)t_r} - 1) \left((\beta+r)c_p + c_{h2} - \frac{\alpha+r}{\beta+r}c_{h2} \right). \end{aligned}$$

Thus, we have $K_1'(t_r) > 0$ by the condition that $c_p \geq (\alpha - \beta)c_{h2}/(\beta + r)^2$.

Part (b). Differentiating $K_2(\Delta_2) = e^{-r\Delta_2} B'(\Delta_2)$, we have

$$K_2'(\Delta_2) = e^{-r\Delta_2} \delta(\Delta_2) \left[-\sigma e^{r\Delta_2} \left(\frac{c_b}{r} - c_l \right) + (\sigma + r) \left(\frac{c_b}{r} - c_p \right) \right].$$

Since $c_b > rc_l$ and $c_l > c_p$, it is easy to verify that $K_2'(\Delta_2) > 0$ if $\Delta_2 \in [0, \bar{\Delta}_2)$ and

$K_2'(\Delta_2) < 0$ if $\Delta_2 \in (\bar{\Delta}_2, +\infty)$.

Appendix 3. Proof of Lemma 2.

By part (b) of Lemma 1, we have that for any $t_r \geq 0$, there are at most two values of Δ_2 satisfying Eqn. (23). Denote the two values by $\Delta_2^{(1)}(t_r) \in [0, \bar{\Delta}_2]$ and $\Delta_2^{(2)}(t_r) \in [\bar{\Delta}_2, +\infty)$ if they exist. By substituting Δ_2 in Eqn. (18) by $\Delta_2^{(1)}(t_r)$, we know that Eqn. (18) is equivalent to

$$-r(A(t_r) + B(\Delta_2^{(1)}(t_r))) + \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2^{(1)}(t_r))}\right) B'(\Delta_2^{(1)}(t_r)) = 0. \quad (\text{A1})$$

The derivative of the left hand side of Eqn. (A1) with respect to t_r is

$$\begin{aligned} & -r \left(B'(\Delta_2^{(1)}(t_r)) \frac{d\Delta_2^{(1)}(t_r)}{dt_r} + A'(t_r) \right) + \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2^{(1)}(t_r))}\right) B''(\Delta_2^{(1)}(t_r)) \frac{d\Delta_2^{(1)}(t_r)}{dt_r} \\ & + r e^{-r(t_r + \Delta_1 + \Delta_2^{(1)}(t_r))} \left(e^{-\alpha_1 \Delta_1} + \frac{d\Delta_2^{(1)}(t_r)}{dt_r} \right) B'(\Delta_2^{(1)}(t_r)) \\ & = \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2^{(1)}(t_r))}\right) e^{r\Delta_2^{(1)}(t_r)} \frac{d \left(e^{-r\Delta_2^{(1)}(t_r)} B'(\Delta_2^{(1)}(t_r)) \right)}{dt_r} \\ & = \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2^{(1)}(t_r))}\right) e^{r\Delta_2^{(1)}(t_r)} \frac{dK_1(t_r)}{dt_r} > 0. \end{aligned}$$

Thus, there is at most one solution $(t_r^{(1)}, \Delta_2^{(1)})$ ($\Delta_2^{(1)} \in (0, \bar{\Delta}_2]$) for Eqn. (A1). Similarly, there is at most one solution $(t_r^{(2)}, \Delta_2^{(2)})$ ($\Delta_2^{(2)} \in [\bar{\Delta}_2, +\infty)$) for

$$-r(A(t_r) + B(\Delta_2^{(2)}(t_r))) + \left(1 - e^{-r(t_r + \Delta_1 + \Delta_2^{(2)}(t_r))}\right) B'(\Delta_2^{(2)}(t_r)) = 0. \quad (\text{A2})$$

Therefore, there are at most two positive solutions for Eqns. (17) and (18).

Appendix 4. Proof of Theorem 2.

Note that the minimum points of TC_1 are either the critical points or on the boundary of the domain. In this proof, we first argue that (a) $(t_r^{(1)}, \Delta_2^{(1)})$ is a local minimum point (if it exists and $\Delta_2^{(1)} \neq \Delta_2^{(2)}$) and $(t_r^{(2)}, \Delta_2^{(2)})$ is not a local minimum

point (if it exists and $\Delta_2^{(2)} \neq \Delta_2^{(1)}$), and then argue that (b) TC_1 will not be minimized when $t_r \rightarrow +\infty$ or $\Delta_2 \rightarrow +\infty$.

For simplicity, we denote $T = t_r + \Delta_1 + \Delta_2$. By Eqn. (14), we have

$$\begin{aligned} \frac{\partial^2 TC_1}{\partial t_r^2} &= \left[\frac{2r^2 e^{-2rT} e^{-2\alpha\Delta_1}}{(1-e^{-rT})^3} + \frac{r^2 e^{-rT} e^{-2\alpha\Delta_1}}{(1-e^{-rT})^2} + \frac{r e^{-rT} e^{-\alpha\Delta_1} \alpha (e^{-\alpha\Delta_1} - 1)}{(1-e^{-rT})^2} \right] \times \\ &\quad (A(t_r) + B(\Delta_2)) - \frac{2r e^{-rT} e^{-\alpha\Delta_1}}{(1-e^{-rT})^2} A'(t_r) + \frac{1}{1-e^{-rT}} A''(t_r), \\ \frac{\partial^2 TC_1}{\partial t_r \partial \Delta_2} &= \left[\frac{2r^2 e^{-2rT} e^{-\alpha\Delta_1}}{(1-e^{-rT})^3} + \frac{r^2 e^{-rT} e^{-\alpha\Delta_1}}{(1-e^{-rT})^2} \right] (A(t_r) + B(\Delta_2)) \\ &\quad - \frac{r e^{-rT} e^{-\alpha\Delta_1}}{(1-e^{-rT})^2} B'(\Delta_2) + \frac{r e^{-rT}}{(1-e^{-rT})^2} A'(t_r), \\ \frac{\partial^2 TC_1}{\partial \Delta_2^2} &= \left[\frac{2r^2 e^{-2rT}}{(1-e^{-rT})^3} + \frac{r^2 e^{-rT}}{(1-e^{-rT})^2} \right] (A(t_r) + B(\Delta_2)) \\ &\quad - \frac{2r e^{-rT}}{(1-e^{-rT})^2} B'(\Delta_2) + \frac{e^{-rT}}{1-e^{-rT}} B''(\Delta_2). \end{aligned}$$

By Eqns. (17) and (18), we have

$$\left. \frac{\partial^2 TC_1}{\partial t_r^2} \right|_{(t_r^{(1)}, \Delta_2^{(1)})} = \frac{e^{-rT} e^{-\alpha\Delta_1}}{1-e^{-rT}} \left. \frac{d(e^{rT} e^{\alpha\Delta_1} A'(t_r))}{dt_r} \right|_{t_r=t_r^{(1)}} > 0, \quad (\text{A3})$$

$$\left. \frac{\partial^2 TC_1}{\partial t_r \partial \Delta_2} \right|_{(t_r^{(1)}, \Delta_2^{(1)})} = 0, \quad (\text{A4})$$

$$\left. \frac{\partial^2 TC_1}{\partial \Delta_2^2} \right|_{(t_r^{(1)}, \Delta_2^{(1)})} = \frac{e^{-r(t_r+\Delta_1)}}{1-e^{-rT}} \left. \frac{d(e^{-r\Delta_2} B'(\Delta_2))}{d\Delta_2} \right|_{\Delta_2=\Delta_2^{(1)}} > 0, \quad (\text{A5})$$

where the inequalities in (A3) and (A5) are implied by Lemma 1. Since (A3), (A4) and (A5) indicate that the Hessian matrix of TC_1 at $(t_r^{(1)}, \Delta_2^{(1)})$ is positive definite, $(t_r^{(1)}, \Delta_2^{(1)})$ is a local minimum. Similarly, we have

$$\left. \frac{\partial^2 TC_1}{\partial \Delta_2^2} \right|_{(t_r^{(2)}, \Delta_2^{(2)})} = \frac{e^{-r(t_r+\Delta_1)}}{1-e^{-rT}} \left. \frac{d(e^{-r\Delta_2} B'(\Delta_2))}{d\Delta_2} \right|_{\Delta_2=\Delta_2^{(2)}} < 0, \quad (\text{A6})$$

which indicates that $(t_r^{(2)}, \Delta_2^{(2)})$ is not a local minimum. This ends the proof of Part (a).

It can be easily verified that $\lim_{t_r \rightarrow +\infty} TC_1(t_r, \Delta_2) = +\infty$, which indicates that TC_1 will not be minimized when $t_r \rightarrow +\infty$. It is also easy to know that

$$TC_1(t_r, +\infty) = \lim_{\Delta_2 \rightarrow +\infty} TC_1(t_r, \Delta_2) = A(t_r) + \frac{e^{-r(t_r+\Delta_1)}c_l}{r},$$

$$TC_1(t_r, 0) = \frac{1}{1 - e^{-r(t_r+\Delta_1)}} A(t_r).$$

Let p be the unit selling price of each product. Then we can express the present value of total profit over the whole planning horizon as

$$TP_1(t_r, \Delta_2) = \int_0^{+\infty} pDe^{-rt} dt - TC_1(t_r, \Delta_2) = \frac{pD}{r} - TC_1(t_r, \Delta_2).$$

To prove that TC_1 will not be minimized when $\Delta_2 \rightarrow +\infty$, we only need to prove that TP_1 will not be maximized when $\Delta_2 \rightarrow +\infty$. Since $c_l > p$, then

$$\begin{aligned} TP_1(t_r, +\infty) &= \frac{pD}{r} - TC_1(t_r, +\infty) \\ &= \frac{1 - e^{-r(t_r+\Delta_1)}}{r} pD - A(t_r) - \frac{e^{-r(t_r+\Delta_1)}}{r} (c_l - p)D \\ &> \frac{1 - e^{-r(t_r+\Delta_1)}}{r} pD - A(t_r). \end{aligned}$$

Besides, we have

$$TP_1(t_r, 0) = \frac{1}{1 - e^{-r(t_r+\Delta_1)}} \left[\frac{1 - e^{-r(t_r+\Delta_1)}}{r} pD - A(t_r) \right].$$

Suppose $(1 - e^{-r(t_r+\Delta_1)}) pD/r - A(t_r)$ is minimized at t_r^* . If $(1 - e^{-r(t_r^*+\Delta_1)}) pD/r - A(t_r^*) \leq 0$, then $TP_1(t_r, +\infty) < 0$ for all $t_r \geq 0$. Thus, TP_1 will clearly not be maximized when $\Delta_2 \rightarrow +\infty$, as long as the whole system is profitable at some $\Delta_2 < +\infty$ (it is reasonable to assume this since it is meaningless to study a

nonprofit system). If $(1 - e^{-r(t_r^* + \Delta_1(t_r^*))})pD/r - A(t_r^*) > 0$, then it is obvious that $TP_1(t_r^*, +\infty) < TP_1(t_r^*, 0)$, which indicates $\max_{t_r \geq 0} TP_1(t_r, +\infty) < \max_{t_r \geq 0} TP_1(t_r, 0)$. Therefore, TP_1 will not be maximized when $\Delta_2 \rightarrow +\infty$. This ends the proof.

Appendix 5. One-warehouse inventory models

When OW is so abundant that there is no need to use RW, Model 1 reduces to the one-warehouse inventory model. We remove the capacity constraint of OW, and rewrite the present value of the total cost for the infinite planning horizon for Model 1 in Eqn. (8) as

$$TC_1^1 = \frac{1}{(1 - e^{-r(t_o + \Delta_2)})} \left\{ c_o + c_p \frac{D}{\alpha} (e^{\alpha t_o} - 1) + c_p e^{-r(t_o + \Delta_2)} D \int_{t_o}^{t_s} \delta(t_s - t) dt + \frac{c_{h1}}{\alpha + r} \left[\frac{1}{\alpha} (e^{\alpha t_o} - 1) - \frac{1}{r} (1 - e^{-r t_o}) \right] D + e^{-r(t_o + \Delta_2)} D \int_{t_o}^{t_s} \left[\frac{c_b}{r} (e^{r(t_s - t)} - 1) \delta(t_s - t) + c_l e^{r(t_s - t)} (1 - \delta(t_s - t)) \right] dt \right\},$$

where $\Delta_2 = t_s - t_o$.

Similarly, the present value of the total cost for the infinite planning horizon for Model 2 in Eqn. (10) can be rewritten as

$$TC_2^1 = \frac{e^{-r t_s'}}{(1 - e^{-r(t_s' + \Delta_4)})} \left\{ c_o + c_p \frac{D}{\alpha} (e^{\alpha \Delta_4} - 1) + c_p D \int_0^{t_s'} \delta(t_s' - t) dt + \frac{c_{h1}}{\alpha + r} \left[\frac{1}{\alpha} (e^{\alpha \Delta_4} - 1) - \frac{1}{r} (1 - e^{-r \Delta_4}) \right] D + D \int_0^{t_s'} \left[\frac{c_b}{r} (e^{r(t_s' - t)} - 1) \delta(t_s' - t) + c_l e^{r(t_s' - t)} (1 - \delta(t_s' - t)) \right] dt \right\},$$

where $\Delta_4 = t_o' - t_s'$.

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