A composite contract based on buy back and quantity flexibility contracts

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A composite (CP) contract for a two-stage supply chain by organically combining two component contracts: a buy back (BB) contract and a quantity flexibility (QF) contract. The CP contract is shown to have advantages over both component contracts in terms of supply chain coordination, profit allocation, and risk allocation. In particular, we obtain the following results: (a) as long as one of the component contracts is able to coordinate the supply chain, so is the CP contract. Moreover, when contract parameters are constrained, we find situations where the CP contract coordinates the supply chain when neither of the component contracts coordinates. (b) When contract parameters are constrained, the CP contract is more flexible in terms of profit allocation among supply chain members than the component contracts. (c) The CP contract is more flexible in terms of risk allocation than the component contracts.

1. Introduction

It’s well known that double marginalization is a prevailing phenomenon in supply chain management (Spengler, 1950). For example, in a two-stage supply chain consisting of a manufacturer and a retailer, the retailer typically orders fewer products than the optimal order quantity for the integrated system, which negatively affects the supply chain performance. To resolve this problem, various contracts have been proposed to coordinate the supply chain by aligning objectives of the supply chain members. Majority of supply chain literature focuses on a few popular contracts such as buy back (e.g., Pasternack, 1985), quantity flexibility (e.g., Tsay, 1999), revenue sharing (e.g., Cachon and Lariviere, 2005), quantity discount (e.g., Tomlin, 2003), sales rebate (e.g., Taylor, 2002). To evaluate effectiveness of a supply chain contract, Cachon (2003) suggests three criteria: (1) Is the contract capable of coordinating the supply chain? “A contract is said to coordinate the supply chain if the set of supply chain optimal actions is a Nash equilibrium i.e., no firm has a profitable unilateral deviation from the set of supply chain optimal actions.” (Cachon, 2003); (2) Is the contract flexible enough to allow for any division of the supply chain’s profit among supply chain members? (3) Can the contract be implemented economically and conveniently in practice? A natural question is: can we identify other supply chain contracts that have advantages over the popular contracts mentioned above based on the three criteria?

To answer this question, we consider a classical two-stage supply chain with a manufacturer and a newsvendor retailer. We propose a composite (CP) contract by organically combining two popular contracts: a buy back (BB) contract and a quantity flexibility (QF) contract. In the remaining paper, we call these two contracts component contracts of the CP contract. In comparison with the component contracts, the proposed CP contract is capable of coordinating the supply chain whenever the component contracts coordinate. It can also be conveniently and economically implemented.

More importantly, we show that the CP contract has several advantages over the two popular component contracts when contract parameters are restricted to certain ranges. Although contract parameters are not restricted in most supply chain contract literature, they are often time constrained in reality. For example, in a competitive market, the supplier has very limited flexibility in determining its wholesale price. It is in general set exogenously (e.g., Wang and Webster, 2007). In publishing industry, it has also been reported that even though booksellers are allowed to return unsold books to their publishers, the percentage of return is usually capped at 35% (Cachon and...
According to Lariviere (1999), supply chain contract literature can be classified into two main categories. The first focuses on replenishment policies and detailed contract parameters for a given type of contract. Examples include Anupindi and Bassok (1999), Huang and Graves (2006), Martinez-de-Albéniz and Simchi-Levi (2005, 2006), Spinler and Huchzermeier (2002), Wu et al. (2002) and Yazlali and Erhun (2006). In particular, Martinez-de-Albéniz and Simchi-Levi (2005) propose a portfolio contract based on option contracts within a general multi-period framework. Optimal portfolios are obtained in terms of buyer’s expected profit. Martinez-de-Albéniz and Simchi-Levi (2006) extend the results of Martinez-de-Albéniz and Simchi-Levi (2005) and introduce risk as one of the objectives in designing effective portfolios. Typically, literature in the first category mainly stands at the point of buyers; more specifically, optimizes the buyer’s procurement strategy, however, with very little regard to objectives of sellers. The second category concentrates on choosing the terms of the contract so as to achieve supply chain coordination. Examples include Barnes-Schuster et al. (2002), Cachon and Lariviere (2005), Pasternack (1985), Taylor (2002), Tomlin (2003) and Tsay (1999). Unlike literature in the first category, the objective of literature in this category is to choose contracts and their parameters so as to allow each party’s optimal decisions (for itself) to lead to the whole supply chain’s optimal performance. Our paper falls into the second category. Next we will review related literature in detail in the second category.

Pasternack (1985) is the first to analyze BB and QF contracts. He shows that neither full returns with full buy back credit nor no returns is system optimal. But the supply chain can be coordinated by an intermediate buy back contract policy (e.g., partial returns with full buy back credit or full returns with partial buy back credit). Since then, numerous papers on BB and QF contracts have appeared. Padmanabhan and Png (1995) investigate advantages and disadvantages of some return policies as well as the motivations why the return policies are carried out. Tsay (1999) studies a QF contract in a more complicated model, which incorporates the issues of capacity planning, information updating, and supply chain coordination. For a comprehensive survey on BB and QF contracts, see Cachon (2003).

Literature on composite contracts is sparse. Taylor (2002) studies a supply chain where demand is influenced by retailer’s sales effort. In this case, popular contracts, such as linear rebate and returns or target rebate alone, could not achieve coordination in a implementable way. He designs a contract by combining a target rebate contract and a returns contract to achieve supply chain coordination and a win–win outcome. Wang and Webster (2007) consider a supply chain with a risk-neutral manufacturer and a loss-averse retailer. They identify a class of distribution free coordinating contracts by combining gain/loss-sharing and BB contracts. Gan et al. (2005) propose a risk sharing contract to coordinate a supply chain with a downside-risk-averse retailer. Interestingly, the contract they propose is also a composite contract based on a BB contract and a QF contract. However, their contract design is quite different from ours (please refer to Section 5 for details). While none of the three papers mentioned above investigates the issue of risk allocation as an advantage of the composite contract, we explicitly explore this issue in addition to supply coordination and profit allocation.

Our paper is also related to the literature on supply chain risk management. Currently, risk consideration is often incorporated into objective functions for both newsvendor models (see Eeckhoudt et al., 1995; Schweitzer and Cachon, 2000 and Chen and Federgruen, 2000 for example) and supply chain contract models (Agrawal and Seshadri, 2000; Buzacott et al., 2001; Tsay, 2002; Gan et al., 2004; Xiao and Yang, 2009 and Wei and Choi, 2010). Our paper, however, focuses on comparing risk allocation among supply chain members for several coordinating contracts.

3. The composite contract

Consider a two-stage supply chain with an upstream manufacturer and a downstream retailer. The retailer is a newsvendor who faces an uncertain customer demand $D$, which is a non-negative random variable with PDF $f(x)$ and corresponding CDF $F(x)$. We assume that $F(x)$ is a differentiable and strictly increasing function on its support. The retailer orders $q$ units of products from the manufacturer before the selling season at a wholesale price $w$ and sells the products to its customers at a fixed retail price $p$ during the selling season. The manufacturer adopts a make-to-order policy and produces the products at a cost $c$. Without loss of generality, we assume that the salvage value for the unsold product is 0. To avoid trivial cases, we assume $p > c > 0$. It is well known that the optimal production (order) quantity for the integrated system is $q^*=F^{-1}(p-c)/(p)$ and we denote the corresponding system optimal expected profit as $\pi^*$.

We now describe in details the contracts considered in this paper. A BB contract, specified by contract parameters $(w,a)$, allows the retailer to return the unsold products to the manufacturer at the buy back price $b$ (clearly, to coordinate the supply chain, the buy back price $b$ should be less than the wholesale price $w$, since the retailer will order as much as possible when $b > w$). A QF contract, specified by contract parameters $(w,a)$, allows the retailer to return up to $a$ units of unsold products with full credit $w$. The CP contract we propose is specified by three contract parameters $(w,b,a)$. It allows the retailer to return up to $a$ units of unsold products to the manufacturer with full credit $w$ and the remaining unsold products, if any, at the buy back price $b$. Clearly, both BB and QF contracts are special cases of the CP contract. When $a = 0$, the CP contract reduces to the BB contract and when $b = 0$, the CP contract reduces to the QF contract.
Under the CP contract, it is easy to verify that the retailer’s and the manufacturer’s profits are, respectively,

\[
\Pi_{C}(w, b, x, q) = \begin{cases} 
    pD - w(1 - x)q & \text{if } D \leq (1 - x)q, \\
    +b[(1 - x)q - D] & \text{if } (1 - x)q < D \leq q, \\
    pg - wq & \text{if } D > q,
\end{cases}
\]

(1)

\[
\Pi_{CM}(w, b, x, q) = \begin{cases} 
    w(1 - x)q - cq & \text{if } D \leq (1 - x)q, \\
    -b[(1 - x)q - D] & \text{if } (1 - x)q < D \leq q, \\
    wD - cq & \text{if } D > q.
\end{cases}
\]

(2)

Here, \(\Pi_{ij}\) represents the profit (a random variable) of member \(j\) under contract \(i\), where \(i = B, Q, C\) denotes the BB, QF and CP contracts, respectively and \(j = R, M\) denotes the retailer and the manufacturer, respectively. By definition, we have:

\[
\Pi_{BM}(w, b, q) = \Pi_{C}(w, b, 0, q), \quad \Pi_{QM}(w, x, q) = \Pi_{CM}(w, 0, x, q), \quad \Pi_{QM}(w, b, q) = \Pi_{CM}(w, b, 0, q),
\]

and \(\Pi_{QM}(w, x, q) = \Pi_{CM}(w, 0, x, q)\).

Before providing a condition for the CP contract to coordinate the supply chain, we make the following technical assumption: For the trivial case \(w = p\), where the retailer gains 0 profit regardless of its order quantity, while the manufacturer extracts all the profit of the supply chain, we assume that the retailer will order \(q^*\) units of products. This assumption is made for the ease of mathematical representations and the managerial implications of this paper will not change without this assumption. Denote:

\[
w^*_C(b, x) = b + \frac{pc - bc}{c + p[(1 - x)qF(1 - x)q]^{1 - \alpha}},
\]

(3)

It can be verified that \(c < w^*_C(b, x) \leq p\) for all \(0 \leq b \leq p\) and \(0 \leq x \leq 1\). Therefore, it is a valid wholesale price. In fact, as stated in the next result, for any given \(b\) and \(x\), \(w^*_C(b, x)\) is the unique wholesale price that leads to the supply chain coordination.

**Proposition 1.** Suppose \(0 \leq b \leq p\) and \(0 \leq x \leq 1\). The CP contract \((w, b, x)\) coordinates the supply chain if and only if \(w = w^*_C(b, x)\).

**Proof.** Given a CP contract \((w, b, x)\), the retailer will choose an order quantity \(q\) to maximize its expected profit

\[
E_q[\Pi_C(w, b, x, q)] = (p - b) \int_{0}^{q} xf(x)dx - (w - b)[1 - x]qF[(1 - x)q] + (p - w)q[1 - F(q)].
\]

(4)

Taking the first and second derivatives of \(E_q[\Pi_C(w, b, x, q)]\) with respect to \(q\), one has,

\[
\frac{\partial E_q[\Pi_C(w, b, x, q)]}{\partial q} = -(w - b)[1 - x]f[(1 - x)q] + (p - w)[1 - F(q)],
\]

(5)

\[
\frac{\partial^2 E_q[\Pi_C(w, b, x, q)]}{\partial q^2} = -(w - b)[1 - x]f^2[(1 - x)q] - (p - w)f(q) < 0.
\]

(6)

Thus, \(E_q[\Pi_C(w, b, x, q)]\) is a concave function of \(q\) and the first order condition coincides the optimality condition of the profit maximization:

\[
(p - w)[1 - F(q)] - (w - b)[1 - x]f[(1 - x)q] = 0.
\]

(7)

Since the supply chain is coordinated if and only if the retailer chooses the system optimal production (order) quantity \(q^*\), we only need to prove that \(q^*\) is the unique solution of (7) if and only if \(w = w^*_C(b, x)\).

*Only if:* Suppose \(q^*\) satisfies Eq. (7). By substituting \(F(q^*) = (p - c)/p\) into Eq. (7) and solving for \(w\), we obtain \(w = w^*_C(b, x)\).

If \(w = w^*_C(b, x)\), then it is straightforward to verify that \(q^*\) is a solution of Eq. (7). Notice that the solution is also unique if \(w < p\). This is because \(F(x)\) is strictly increasing, which implies that the left hand side of Eq. (7) is a strictly decreasing function of \(q\). In case \(w = p\), the solution is also \(q^*\) by our technical assumption just before Eq. (3). Therefore, \(q^*\) is the unique solution of Eq. (7). \(\Box\)

**Remark 1.** If \(x = 0\), the BB contract coordinates the supply chain if and only if \(w = w^*_C(b) \equiv w^*_C(b, 0)\). Similarly, if \(b = 0\), the QF contract coordinates the supply chain if and only if \(w = w^*_C(x) \equiv w^*_C(0, x)\).

We now provide a characterization of the coordinating wholesale price \(w^*_C(b, x)\) for the CP contract.

**Proposition 2.** If \(0 \leq b \leq p\), \(0 \leq x \leq 1\), then \(w^*_C(b, x)\) is differentiable and strictly increasing in \(b\) and \(x\). In addition, we have:

\[
w^*_C(b, x) \geq \max\left\{w^*_C(b), w^*_C(x)\right\}.
\]

(8)

**Proof.** Since \(F(x)\) is differentiable and strictly increasing, in view of Eq. (3), \(w^*_C(b, x)\) is also differentiable and strictly increasing in \(b\) and \(x\). Since \(w^*_C(0, 0) = w^*_C(b)\) and \(w^*_C(0, x) = w^*_C(x)\), inequality (8) follows immediately. \(\Box\)

**Remark 2.** Inequality (8) states that for the CP contract to coordinate the supply chain, it has to charge a higher wholesale price than its component contracts.
We next investigate the expected profits of supply chain members for a coordinated supply chain under the CP contract and its component contracts: their relationships with each other as well as their dependence on corresponding contract parameters. Denote \( \pi_{ij}^c \) the expected profit of \( P_i \) for member \( j \) under coordinating contract \( i \), where \( i = B, Q, C \), and \( j = R, M \). For example,

\[
\pi_{12}^c(b, x) = E_{[H(x), \pi_{01}^c(\theta, \alpha, q)]} = (p - b) \int_0^{1 - x \theta} xf(x) dx - [\pi_{01}^c(b, x) - b] (1 - x \theta) q' [1 - F((1 - x) \theta)] + [p - \pi_{01}^c(b, x)] \int_0^\theta xf(x) dx + [p - \pi_{01}^c(b, x)] q' [1 - F((1 - x) \theta)].
\]

(9)

Apparently, we have \( \pi_{12}^c(b) = \pi_{12}^c(b, 0) \), \( \pi_{12}^c(x) = \pi_{12}^c(0, x) \). In addition, since \( F(x) \) is differentiable, \( \pi_{01}^c(\cdot) \) is differentiable for all \( i = B, Q, C \), and \( j = R, M \).

**Proposition 3.** If \( 0 \leq b < p, 0 \leq x < 1 \), then

\[
\begin{align*}
(a) & \quad \frac{\pi_{12}^c(b)}{\pi_{12}^c} = \frac{p - b}{p}, \\
(b) & \quad \frac{d\pi_{12}^c(x)}{dx} < 0, \quad \pi_{12}^c(0) = \pi_{12}^c(1) = 0, \\
(c) & \quad \frac{\pi_{12}^c(b, x)}{\pi_{12}^c} = \frac{\pi_{12}^c(b)}{\pi_{12}^c(x)}, \\
(d) & \quad \frac{\partial \pi_{12}^c(b, x)}{\partial b} < 0, \quad \frac{\partial \pi_{12}^c(b, x)}{\partial x} < 0.
\end{align*}
\]

(10)-(13)

**Proof.** Part (a) is implied by a more general result stated in Cachon (2003). Part (b) has been proven by Tsay (1999). We now prove Part (c). According to expression (9),

\[
\pi_{12}^c(x) = \pi_{12}^c(0, x) = p \int_0^{1 - x \theta} xf(x) dx - (1 - x \theta) q' W_0(x) F((1 - x) \theta) + [p - W_0(x)] \int_0^\theta xf(x) dx + [p - W_0(x)] q' [1 - F((1 - x) \theta)].
\]

(14)

Based on definition (3) and direct calculations, we obtain:

\[
\frac{p - W_0(x)}{p - W_0(0, x)} = \frac{p - b}{p - W_0(b, x)} = \frac{p - b}{p - W_0(b, 0, x)} \quad \text{and} \quad \frac{W_0(b, x) - b}{W_0(0, x)} = \frac{W_0(b, x) - b}{W_0(b, 0, x)} = \frac{p - b}{p}.
\]

(15)

Comparing Eqs. (9) and (14), we have:

\[
\frac{\pi_{12}^c(b, x)}{\pi_{12}^c(0, x)} = \frac{p - b}{p},
\]

(16)

which, together with Part (a), leads to Part (c).

Finally, we prove Part (d). From Part (a), \( \pi_{12}^c(b) \) is a decreasing function of \( b \). Then based on Part (c), for any fixed \( x \), \( \pi_{12}^c(b, x) \) is also a decreasing function of \( b \). The second portion of Part (d) can be proved similarly. \( \square \)

**Remark 3.** Parts (a) and (b) of Proposition 3 assure that both BB and QF contracts can allocate profits arbitrarily among supply chain members when the supply chain is coordinated.

**Remark 4.** Part (c) implies that for any given \((b, x)\), the CP contract allocates less profit to the retailer than its component contracts when the supply chain is coordinated.

**Remark 5.** Intuitively, one may think that the retailer should prefer a higher \( b \) and a larger \( x \). This is certainly true for a fixed wholesale price. However, Part (d) of Proposition 3 reveals that the opposite is true when the wholesale price is simultaneously adjusted to coordinate the supply chain. In fact, the manufacturer gains more of the supply chain’s profit by providing a higher \( b \) and a larger \( x \).

4. The advantage of the composite contract

In this section, we evaluate the performance of the CP contract based on the criteria proposed by Cachon (2003). In particular, we show that when the contract parameters are restricted to certain ranges, the CP contract has some advantages over its corresponding component contracts in terms of supply chain coordination (in Section 4.1) and profit allocation among supply chain members (in Section 4.2). Furthermore, in Section 4.3 we show that the CP contract is also more flexible than its corresponding component contracts in terms of risk allocation among supply chain members, even if the contract parameters are not restricted.

4.1. Supply chain coordination

Suppose now that supply chain contract parameters \( w, b, x \) are restricted to ranges \([w, \bar{w}], [0, \bar{b}], [0, \bar{x}]\), respectively. A contract is called a feasible contract if its parameters are within the respective ranges. If, in addition, the contract coordinates the supply chain, we call it a...
feasible coordinating contract. Since BB and QF contracts are special cases of the CP contract, if there exists a feasible coordinating BB or QF contract, there also exists a feasible coordinating CP contract. A more interesting question is if there exists a feasible coordinating CP contract when neither feasible coordinating BB nor QF contract exists. The following theorem provides a positive answer to this question.

**Theorem 1.** If
\[
\max \left\{ w_g(b), w_q(x) \right\} < w < w_c(b, x),
\]
then neither feasible coordinating BB nor QF contract exists. But there exists a feasible coordinating CP contract.

**Proof.** In order for a CP contract \((w, b, x)\) to be a feasible coordinating contract, its contract parameters need to satisfy the following conditions:

\[
w = w_c(b, x) \in \left[ w, w_p \right], \quad b \in [0, b], \quad x \in [0, x].
\]

Since \(w_c(b, x)\) is continuous and strictly increasing in \(b\) and \(x\) by Proposition 2, the existence of a feasible coordinating CP contract follows from the second inequality of (17). Based on the same logic, the first inequality of (17) implies that neither feasible coordinating BB nor QF contract exists. \(\square\)

**Remark 6.** It turns out that condition (17) is also a necessary condition for the existence of a feasible coordinating CP contract and the non-existence of feasible coordinating BB and QF contracts. In fact, the first inequality of condition (17) is equivalent to the non-existence of feasible coordinating BB and QF contracts, while the second inequality is equivalent to the existence of a feasible coordinating CP contract.

**Theorem 1** clearly shows that when contract parameters are restricted, one has more opportunities to coordinate the supply chain by choosing the CP contract instead of its component contracts.

### 4.2. Profit allocation

According to Cachon (2003), a contract is more desirable if it allows more flexibility in allocating supply chain profit among the supply chain members. In this section, we investigate the profit allocation flexibility for the CP contract. In particular, when the contract parameters \(w, b, x\) are restricted to certain ranges \([w, w_p], [0, b], [0, x]\), we show that the CP contract is more flexible in allocating supply chain profit than its component contracts.

To measure profit allocation flexibility for a type of contract, we introduce the concept of Percentage Profit Allocation Range (PPAR): a set of retailer’s shares of supply chain profit resulting from all feasible coordinating contracts within this type of contract. For example, more precise definitions of PPAR for BB, QF and CP contracts are given as follows, respectively:

\[
\begin{align*}
I_b &= \left\{ \frac{\pi_{bb}(b)}{\pi^*} \middle| w_g(b) \in [w, w_p], b \in [0, b] \right\}, \\
I_q &= \left\{ \frac{\pi_{qq}(x)}{\pi^*} \middle| w_q(x) \in [w, w_p], x \in [0, x] \right\}, \\
I_c &= \left\{ \frac{\pi_{c}(b, x)}{\pi^*} \middle| w_c(b, x) \in [w, w_p], b \in [0, b], x \in [0, x] \right\}.
\end{align*}
\]

By definition, PPAR is a subset of \([0, 1]\). Since the manufacturer’s and the retailer’s profit shares add up to 1 (e.g., under the coordinating CP contracts, \(\pi_{c}(b, x)/\pi^* + \pi_{C}(b, x)/\pi^* = 1\)), it suffices to use only retailer’s profit shares to characterize the profit allocation flexibility for a given type of contract. In addition, since the retailer’s expected profit is a continuous function of contract parameters, it can be shown that all PPARs \(I_b, I_q\) and \(I_c\) are closed intervals. Furthermore, since BB and QF contracts are special cases of the CP contract, it is always true that \(I_b \cup I_q \subseteq I_c\), i.e., the CP contract is at least as flexible as its component contracts in allocating supply chain profit among the supply chain members. A more challenging question is if we can identify situations where the CP contract is more flexible for profit allocation. **Theorem 2** answers this question affirmatively.

**Theorem 2.** If the CP contract parameter ranges \(w, w_p, b\) meet the following conditions: \(c \leq w \leq \max \left\{ w_q(b), w_g(x) \right\} < w < p \) and \(\min(b, x) > 0\), then \(I_b \cup I_q \subseteq I_c\).

**Proof.** Since \(w \leq \max \left\{ w_g(b), w_q(x) \right\} \leq w\), then there exist three cases:

1. \(w < w_g(b) < w\) and \(w < w_q(x) < w\);  
2. \(w_q(x) < w \leq w_g(b) < w\);  
3. \(w_g(b) < w \leq w_q(x) < w\).

For simplicity, we only prove the theorem under (more complicated) Case 1. The proofs for the other two cases are similar.

Based on the similar arguments used in the proof of **Theorem 1**, there exist feasible coordinating BB, QF, and CP contracts respectively for the supply chain under Case 1. Denote \(I_b = [x_1, x_2]\) and \(I_q = [y_1, y_2]\). Since both \(\pi_{bb}(b)\) and \(\pi_{qq}(x)\) are decreasing functions by Proposition 3, \(x_1 = \pi_{bb}(b)/\pi^*\). Without loss of generality, we assume \(x_1 < y_1\). Now consider a CP contract \((w_c(b, c), b, c)\) with \(c > 0\). Since \(a > 0\) and \(w < w_c(b, 0) = w_g(b) < w\), we may choose \(c\) sufficiently small such that \(c \in (0, x)\) and \(w \leq w_c(b, 0) \leq w_c(b, c) < w\). By definition, the CP contract \((w_c(b, c), b, c)\) is a feasible coordinating contract. Thus, \(\pi_{C}(b, c)/\pi^* \) belongs to \(I_c\). In addition, from Part (d) of Proposition 3, we have:
\[ \pi_{CB}(b, e) < \pi_{CF}(b, 0) < \pi_{BF}(b) \quad \text{for} \quad x_1 \leq y_1. \] (22)

Hence, \( \pi_{CB}(b, e) / \pi^{*} \) does not belong to \( I_B \cup I_Q \) and interval \( I_C \) contains \( I_B \cup I_Q \) properly. \( \square \)

We next quantify the extra flexibility gained from the CP contract. To this end, let \( m(l) \) stand for the length of interval \( l \). Clearly, \( m[I_C \setminus (I_B \cup I_Q)] \) represents the extra flexibility gained from the CP contract over its component contracts.

**Theorem 3.** If the CP contract parameter ranges \( w, w, z \) meet the following conditions: \( b = p/2, \pi_{QF}(\bar{x}) = p^{*}/2, c \leq w \leq \max \{ w_{SP}(b), w_{Q}(z) \} \), and \( w_{C}(b, z) \leq w \leq p \), then \( m[I_C \setminus (I_B \cup I_Q)] \geq 1/4 \).

**Proof.** Since \( \max \{ w_{SP}(b), w_{Q}(z) \} \leq w_{C}(b, z) \), there are three cases:

1. \( w \leq w_{SP}(b) \leq w_{C}(b, z) \leq w \) and \( w \leq w_{Q}(z) \leq w_{C}(b, z) \leq w_{C}(b, z) \leq w \);
2. \( w_{SP}(b) \leq w \leq w_{C}(b, z) \leq w_{C}(b, z) \leq w_{Q}(z) \leq w_{C}(b, z) \leq w \);
3. \( w_{Q}(z) \leq w \leq w_{C}(b, z) \leq w_{C}(b, z) \leq w_{Q}(z) \leq w_{C}(b, z) \leq w \);

Based on similar arguments used in the proof of Theorem 1 and Remark 6, feasible coordinating BB, QF, and CP contracts exist respectively for the supply chain under Case 1. Let \( I_B = [x_1, x_2] \), \( I_Q = [y_1, y_2] \), \( I_C = [z_1, z_2] \) be the respective PPARs for these contracts. Since \( w \leq w_{SP}(b) \leq w \) and \( b = p/2 \), then \( w_{SP}(b, b) \) is a feasible coordinating BB contract. By Part (a) of Proposition 3, \( \pi_{BB}(b) / \pi^{*} = 1/2 \) exists for the manufacturer.

Furthermore, \( x_1 = 1/2 \) since \( \pi_{BB}(b) \) is a strictly decreasing function of \( z \). Following the similar argument, we also have \( y_1 = 1/2 \). By Part (c) of Proposition 3,

\[ \pi_{CB}(b, z) = \pi_{BB}(b) = \pi_{QF}(z) = 1/4. \] (23)

which, together with condition \( w \leq w_{C}(b, z) \leq w \), implies that \( 1/4 \leq I_C \). In fact, \( z_1 = 1/4 \) by Part (d) of Proposition 3. Finally, we have \( z_2 \geq \max \{ x_2, y_2 \} \) by both BB and QF contracts are special cases of the CP contract. Therefore, \( m[I_C \setminus (I_B \cup I_Q)] \geq 1/4 \).

Under Case 2, based on similar arguments used in the proof of Theorem 1 and Remark 6, feasible coordinating QF and CP contracts exist for the supply chain while feasible coordinating BB contract does not exist. Let \( I_B = [y_1, y_2] \), \( I_C = [z_1, z_2] \) be the respective PPARs for QF and CP contracts. Since \( w \leq w_{QF}(z) \leq w \), \( w_{QF}(z, z) \) is a feasible coordinating QF contract, which indicates \( \pi_{BB}(z) / \pi^{*} = 1/2 \) exists for the manufacturer. Furthermore, \( y_1 = 1/2 \) since \( \pi_{BB}(b) \) is a strictly decreasing function of \( z \). By Part (c) of Proposition 3,

\[ \pi_{CB}(b, z) = \pi_{BB}(b) = \pi_{QF}(z) = 1/4. \] (24)

which, together with condition \( w \leq w_{C}(b, z) \leq w \), implies that \( 1/4 \leq I_C \). In fact, \( z_1 = 1/4 \) by Part (d) of Proposition 3. Finally, we have \( z_2 \geq y_2 \), since the QF contract is the special case of the CP contract. Therefore, \( m[I_C \setminus (I_B \cup I_Q)] = m[I_C \setminus I_Q] \geq 1/4 \).

Under Case 3, based on similar arguments used in the proof of Theorem 1 and Remark 6, feasible coordinating BB, CP, and QF contracts exist while feasible coordinating QF contract does not exist. Let \( I_B = [x_1, x_2] \), \( I_C = [z_1, z_2] \) be the respective PPARs for BB and CP contracts. Since \( w \leq w_{SP}(b) \leq w \), \( w_{SP}(b, b) \) is a feasible coordinating BB contract. By Part (a) of Proposition 3, \( \pi_{BB}(b) / \pi^{*} = 1/2 \) exists for the manufacturer. Furthermore, \( x_1 = 1/2 \) since \( \pi_{BB}(b) \) is a strictly decreasing function of \( b \). By Part (c) of Proposition 3,

\[ \pi_{CB}(b, z) = \pi_{BB}(b) = \pi_{QF}(z) = 1/4. \] (25)

which, together with condition \( w \leq w_{C}(b, z) \leq w \), implies that \( 1/4 \leq I_C \). In fact, \( z_1 = 1/4 \) by Part (d) of Proposition 3. Finally, we have \( z_2 \geq y_2 \), since the CP contract is the special case of the CP contract. Therefore, \( m[I_C \setminus (I_B \cup I_Q)] = m[I_C \setminus I_B] \geq 1/4 \). \( \square \)

**Remark 7.** As long as the CDF \( F(x) \) is differentiable and strictly increasing on its support, there always exist contract parameter ranges \( w, w, z \) that satisfy the conditions of Theorem 3. Notice that the existence of \( z \) is guaranteed by the arbitrary profit allocation property of the QF contract. The existence of \( w, w \) is ensured by the following inequalities:

\[ c = w_{Q}(0) < \min \left\{ w_{Q}(p/2), w_{Q}(z) \right\} < \pi_{C}(p/2, z) < \pi_{C}(p, z) = p. \] (26)

We next present a numerical example to demonstrate Theorem 3.

**Example 1.** Let the customer demand follow a truncated Normal distribution over \( [0, \infty) \) with mean 100 and variance 1600, \( p = 8, c = 3, \ w = 5.5, w = 8, b = 4, \) and \( z = 0.3422 \). It is easy to verify that the above parameters meet the conditions of Theorem 3. Through numerical calculation, we have \( I_B = [0.5, 0.5], I_Q = [0.5, 5.0405], I_C = [0.25, 0.5049], \) and thus \( m[I_C \setminus (I_B \cup I_Q)] = 0.2504 > 0.25 \).

### 4.3 Risk allocation

Most supply chain contract design literature focuses on profit allocation among supply chain members. However, risk allocation is also an important issue. In this section, we show that for any given profit allocation, while BB and QF contracts result in fixed risk allocation, the CP contract still allows for some flexibility in risk allocation, even if the contract parameters are not restricted. Following conventional literature (Chen and Federgruen, 2000; Markowitz, 1959 and Gan et al., 2004), we use variance of profit as a measurement of risk. We restrict our discussions to the flexibility of retailer's risk allocation. Similar results also hold for the manufacturer.
Let \( r^*_j \) represent the risk of supply chain member \( j \) under coordinating contract \( i \) with \( j = R, M \), and \( i = B, Q, C \). For example, the retailer’s risks are precisely defined below for the three contracts under consideration:

\[
\begin{align*}
    r^*_B(b) & = V_B \left\{ \Pi_B \left( w^*_B(b), b, q^* \right) \right\}, \\
    r^*_Q(x) & = V_D \left\{ \Pi_Q \left( w^*_Q(x), x, q^* \right) \right\}, \\
    r^*_C(b, x) & = V_D \left\{ \Pi_C \left( w^*_C(b, x), b, x, q^* \right) \right\}.
\end{align*}
\]  

(27)

(28)

(29)

For any retailer’s (expected) profit allocation \( \pi \in [0, \pi^*] \), if BB contract is applied to coordinate the supply chain, there is a unique contract parameter pair \( (w^*_B(b^*_B), b^*_C) \) such that \( \pi^*_B(b^*_B) = \pi \). Similarly, if QF contract is applied to coordinate the supply chain, there is also a unique contract parameter pair \( (w^*_Q(x^*_Q), x^*_Q) \) such that \( \pi^*_Q(x^*_Q) = \pi \). The retailer’s risks corresponding to profit allocation \( \pi \) are \( r^*_B(b^*_B), r^*_Q(x^*_Q) \), respectively. Clearly, for BB and QF contracts, once the profit allocation is fixed, so is the risk allocation. The next result shows that the CP contract allows for some risk allocation flexibility even when the profit allocation is determined.

**Theorem 4.** Let \( \pi \in [0, \pi^*] \). For any \( r \in \left[ \min \left( r^*_B(b^*_B), r^*_Q(x^*_Q) \right), \max \left( r^*_B(b^*_B), r^*_Q(x^*_Q) \right) \right] \), there exists a coordinating CP contract \( (w^*_C(b, x), b, x) \) with \( b \in [0, b^*_B] \) and \( x \in [0, x^*_Q] \) such that \( \pi^*_C(b, x) = \pi \) and \( r^*_C(b, x) = r \).

**Proof.** Based on Parts (a) and (c) of Proposition 3, we have:

\[
\frac{\pi^*_C(b, x)}{\pi} - \frac{\pi^*_B(b)}{\pi} = \frac{p - b}{p} \frac{\pi^*_Q(x)}{\pi}.
\]

(30)

Therefore, in order for the retailer to obtain any fixed profit allocation \( \pi \in [0, \pi^*] \) under the coordinating CP contract, its buy back price \( b \) has to satisfy the following functional relationship with respect to \( x \):

\[
b(x) = p \left( 1 - \frac{\pi}{\pi^*_Q(x)} \right). 
\]

(31)

It can be verified that function \( b(x) \) is a differentiable and strictly decreasing function of \( x \) defined on \([0, x^*_Q] \) with \( b(0) = \pi \) and \( b(x^*_Q) = 0 \). Denote \( r(x) = r^*_C(b^*_B, x) \). Clearly, \( r(x) \) is also a differentiable function of \( x \). In addition, we have \( r(0) = r^*_C(b(0), 0) = r^*_B(b^*_B), r(x^*_Q) = r^*_C(b(x^*_Q), x^*_Q) = r^*_Q(x^*_Q) \). Therefore, by Intermediate Value Theorem, for any

\[
r \in \left[ \min \left( r^*_B(b^*_B), r^*_Q(x^*_Q) \right), \max \left( r^*_B(b^*_B), r^*_Q(x^*_Q) \right) \right] = \left[ \min (r(0), r(x^*_Q)), \max (r(0), r(x^*_Q)) \right],
\]

(32)

there exist \( x \in [0, x^*_Q] \) and \( b = b(x') \in [0, b^*_B] \) such that \( r^*_C(b(x'), x') = r \) and \( \pi^*_C(b(x'), x') = \pi \). Obviously, the coordinating CP contract \( (w^*_C(b', x'), b', x') \) meets the requirements of this theorem. This completes the proof. □

**Remark 8.** Theorem 4 indicates that if the BB or the QF contract is used to coordinate the supply chain, the retailer’s risk is uniquely determined once the profit allocation is fixed. However, if the CP contract is chosen to coordinate the same supply chain, the retailer’s risk can be

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**Fig. 1.** Risk vs. profit allocations for the retailer and the manufacturer in a coordinated supply chain.
anywhere between the risks under the BB and QF contracts by properly adjusting the CP contract parameters. This implies that the CP contract is more flexible in terms of risk allocation than its corresponding component contracts.

We now provide a numerical example to demonstrate the risk allocations for the three contracts under consideration. The customer demand is uniformly distributed on $[0, 100]$ and $p = 8, c = 3, w = c, w = p, b = p, x = 1$. Fig. 1 reveals the relationship between the profit allocation versus risk allocation for the three contracts. We have the following observations based on Fig. 1. First, high profit comes at the cost of high risk, which is consistent with our intuition. Next, for the same profit allocation, the BB contract and the QF contract lead to different risk allocations. This is because the BB and QF contracts don’t always generate the same outcome for the same realized demand, which results from interaction of several factors, such as demand distribution, cost parameters etc. Finally, for this example, the retailer prefers the BB contract, while the manufacturer prefers the QF contract. This is because for the same profit allocation, these contracts lead to lower risks for respective supply chain members. Under this situation, it will be difficult for the supply chain members to come up with a consensus on which contract to use. Clearly, the CP contract can serve as a compromise solution.

5. Conclusion and future research

In this paper, we propose a composite contract based on a BB contract and a QF contract. We discuss its advantages over the BB and QF contracts in terms of coordination, profit allocation and risk allocation. We show that: (a) as long as one of the BB and QF contracts can coordinate the supply chain, so can the CP contract. Moreover, when some contract parameters are constrained, we find situations where the composite contract coordinates the supply chain even if the two component contracts cannot. (b) When contract parameters are constrained, the CP contract is more flexible for profit allocation among supply chain members than the BB and QF contracts. In addition, we introduce a measurement to quantify the advantage of the CP contract over the BB and QF contracts in terms of profit allocation flexibility and the flexibility advantage can be more than 25% based on the measurement. (c) The CP contract is more flexible for risk allocation.

To strengthen the positioning of our paper in the contract literature, we next compare our composite idea with the existing ones. The composite contracts proposed by Taylor (2002) and Wang and Webster (2007) assume that the terms of a composite contract are the union of the terms in its component contracts. For example, in Wang and Webster (2007), the composite contract, which is based on a gain/loss-sharing contract and a buy back contract, assigns the sharing parameters corresponding to gains and losses respectively (which are the terms of the component gain/loss-sharing contract), as well as the wholesale price and the buy back price (which are the terms of the component buy back contract). However, this composite idea doesn’t apply in our model, since it may lead to contradictions by putting them together. For example, a buy back contract $(w, b), b < w$, designates that the retailer can return all unsold products with a refund of $b$ per unit. On the other hand, a quantity flexibility contract $(w, x), x < 1$, specifies that the retailer can return up to $xq$ units of products with full refund $(w$ per unit), where $q$ is the order quantity. If a composite contract is constructed by collecting the terms of the above two contracts, then the contradiction arises for the first returned $aq$ units of unsold products $(b$ per unit and $w$ per unit conflict with each other). The composite method proposed by this paper, however, can eliminate this contradiction. From this point of view, the composite idea proposed in this paper seems more natural than those proposed in Taylor (2002) and Wang and Webster (2007) when the component contracts are both returns contracts (buy back contract or quantity flexibility contract).

The composite contract in Gan et al. (2005) is characterized by five parameters $(w, b, w', q', \bar{q})$, where $(w, b)$ specify a buy back contract, $w$ is another wholesale price not necessarily equal to $w$, and $q', \bar{q}$ are two thresholds of the order quantity. This contract can be described as follows:

(i) If the retailer’s order quantity $q$ is less than or equal to $q'$, the initial buy back contract $(w, b)$ is executed.
(ii) If $q$ is greater than $q'$ but not greater than $\bar{q}$, then in addition to the initial buy back contract with an order quantity $q'$, the retailer pays the wholesale price $w'$ for each unit in excess of $q'$, and he receives a full refund of $w'$ for each unsold unit.
(iii) If $q$ is greater than $\bar{q}$, the terms of the contract are the same as those in (ii) except that the number of unsold items that are fully refundable cannot exceed $\bar{q} - q$.

Obviously, the composite idea in Gan et al. (2005) is totally different from ours. The composite contract in Gan et al. (2005), however, is more difficult to work out and implement than ours for two more contract parameters. Therefore, from the implementation perspective, the composite contract proposed in our paper seems to be more convenient than that in Gan et al. (2005).

The CP contract proposed in this paper is a combination of the BB and the QF contracts. Can we use the similar approach to generate new composite contracts based on other simple contracts? It does not seem to be an easy extension based on our early trials. Can we obtain results similar to those in this paper for other types of composite contracts? Can we identify more complicated applications where no single contract coordinates the system, while certain composite contract does? We intend to answer some of these questions in our future research.

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References
