HAJLASZ-SOBOLEV TYPE SPACES AND $p$-ENERGY ON THE SIERPINSKI GASKET

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ABSTRACT. We study Hajłasz-Sobolev type spaces on metric spaces that depend on quasi-distances; in particular, we may take the quasi-distance to be the power $\sigma$ of the metric with $\sigma > 1$, if the metric space is highly irregular or porous. We take the Sierpinski gasket in $\mathbb{R}^2$ as an example, and show that the Hajłasz-Sobolev type space is non-trivial for $1 < \sigma < \beta_p/p$ with $\beta_p$ characterizing the intrinsic property of the Sierpinski gasket. This work was strongly motivated by [8], and generalizes the result in [9] to any $1 < p < \infty$.

1. Hajłasz-Sobolev type spaces

Let $F$ be a non-empty set and $d$ be a metric on $F$. Let $q(x, y)$ be a quasi-distance on $F$ (cf [14]), that is $q : F \times F \rightarrow [0, \infty]$ satisfies

1. $q(x, y) = 0$ if and only if $x = y$;
2. $q(x, y) = q(y, x)$ for all $x, y \in F$;
3. there exists a constant $1 \leq c_1 < \infty$ such that, for all $x, y, z \in F$,
   
   $q(x, y) \leq c_1(q(x, z) + q(z, y)).$

Let $\mu$ be a Borel measure on the metric space $(F, d)$. Let $1 \leq p \leq \infty$. We denote by $L^p(\mu) := L^p(F, \mu)$ the usual space of all $p$-integrable real-valued functions on $F$ with respect to $\mu$, with the norm

$\|f\|_p := \left(\int_F |f(x)|^p d\mu(x)\right)^{1/p}$

(with the obvious modification when $p = +\infty$). Motivated by [5], we say that a function $f \in L^p(\mu)$ belongs to a Hajłasz-Sobolev type space $M^p(\mu)$, if there exists a non-negative function $g \in L^p(\mu)$, termed an upper gradient of $f$, such that

$|f(x) - f(y)| \leq q(x, y)(g(x) + g(y))$

for $\mu$-almost all $x, y \in F$ with $0 < q(x, y) < r_0$ and some $r_0 \in (0, \infty]$. The norm of $f \in M^p(\mu)$ is defined by

$\|f\|_{M^p(\mu)} := \|f\|_p + \inf_g \|g\|_p$

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where the infimum is taken for all \( g \) satisfying (1.1). It is not hard to see that \( M^p(\mu) \) is a Banach space for \( 1 \leq p < \infty \) (the proof is similar to that in [5] or [7]). Observe that different values on \( r_0 \) for (1.1) holding give equivalent spaces.

Note that \( q(x, y) = d(x, y)^\sigma \) is a quasi-distance on \( F \) for any \( 0 < \sigma < \infty \). The case \( \sigma = 1 \) was addressed in [5], and it was shown that \( M^p(\mu) \) is the usual Sobolev space \( W^{1,p}(F) \) if \( F \) is an open domain with Lipschitz boundary in \( \mathbb{R}^n \) and \( \mu \) is the Lebesgue measure. In [9], it was extended to the case \( \sigma > 1 \) when \( F \) is a fractal in the Euclidean setting, and was demonstrated that for \( p = 2 \), \( M^p(\mu) \) is non-trivial when \( 1 < \sigma < \beta/2 \) and is trivial when \( \sigma > \beta/2 \), if \( F \) is the Sierpinski gasket in \( \mathbb{R}^n \), where \( \beta = \log(n + 3)/\log 2 \) is the walk dimension of \( F \) (for Hajlasz-Sobolev spaces on fractals, see also [6, 16]). (We say that \( M^p(\mu) \) is trivial if \( M^p(\mu) \) contains only constant functions. In this connection, see [1, 3, 2]. Note that \( M^p(\mu) \) is always trivial if \( F \) is an open set in \( \mathbb{R}^n \) and \( q(x, y) = |x - y|^\sigma \) with \( \sigma > 1 \), and nothing needs to be discussed under this circumstance. But if \( F \) is irregular (eg. highly porous), the situation is considerably different, and \( M^p(\mu) \) may be non-trivial, see [9] and below.) Whilst in this paper we will generalize the result in [9] to the non-Euclidean setting on one hand, we mainly give an example, on the other hand, that \( M^p(\mu) \) is non-trivial for any \( 1 < p < \infty \) and \( q(x, y) = d(x, y)^\sigma \) with \( \sigma > 1 \) in a certain range. We take \( F \) to be the Sierpinski gasket in \( \mathbb{R}^2 \). Our example is motivated by [8]. As a by-product, we also answer the question raised in [8] of what is the domain of the \( p \)-energy. We thank R.S. Strichartz for sending [8] to our attention.

If \( q(x, y) = d(x, y)^\sigma (0 < \sigma < \infty) \) and \( \mu \) is a doubling measure, that is \( \mu \) satisfies, for some \( c_2 > 0 \),

\[
\mu(B(x, 2r)) \leq c_2 \mu(B(x, r))
\]

for all \( x \in F \) and all \( 0 < r < \infty \), where \( B(x, r) = \{ y \in F : d(y, x) < r \} \) is a ball in \( F \), then \( M^p(\mu) \) may be characterized as follows: for \( f \in L^p(\mu) \) with \( 1 < p < \infty \), we have that \( f \in M^p(\mu) \) if and only if \( \tilde{f} \in L^p(\mu) \), where

\[
\tilde{f}(x) := \sup_{0<r<r_0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} \frac{|f(x) - f(y)|}{q(x, y)} d\mu(y), \quad x \in F,
\]

see also [9] (we always assume that \( |f(x) - f(y)|/q(x, y) = 0 \) if \( x = y \)). To see this, let \( f \in M^p(\mu) \). Then, we have from (1.1) that

\[
\tilde{f}(x) \leq \sup_{0<r<r_0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} (g(x) + g(y)) d\mu(y)
\]

\[
= g(x) + \sup_{0<r<r_0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} g(y) d\mu(y) \in L^p(\mu),
\]

since

\[
M(g)(x) := \sup_{0<r<r_0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} g(y) d\mu(y) \in L^p(\mu),
\]
due to the doubling condition (1.2) (see for example [7]). Conversely, let \( \tilde{f} \in L^p(\mu) \). Fix \( x, y \in F \) such that \( 0 < r := d(x, y) < r_0/2 \). Then we see that, using (1.2),

\[
|f(x) - f(y)| \leq \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(z)| + |f(z) - f(y)| \, d\mu(z)
\]

\[
\leq \frac{1}{\mu(B(x, r))} \int_{B(x, r)} r^\sigma \left( |f(x) - f(z)| + |f(z) - f(y)| \right) \, d\mu(z)
\]

\[
= C r^\sigma (\tilde{f}(x) + \tilde{f}(y))
\]

proving that \( f \in M^p(\mu) \) if \( \tilde{f} \in L^p(\mu) \). Here and in the sequel, we denote by \( C \) the general constant whose value may be different at a different occurrence. The function \( \tilde{f} \) defined as in (1.3) is the upper gradient of \( f \) (multiple a constant). In what follows we will focus on a class of Hajlasz-Sobolev type spaces where \( q(x, y) = d(x, y)^\sigma \) and \( 1 < \sigma < \infty \), and we denote this space by \( M^p(\mu) \).

For \( 1 \leq p < \infty \) and \( 0 < \sigma < \infty \), we say that \( f \in \text{Lip}(\sigma, p, \infty)(\mu) \) if \( f \in L^p(\mu) \) and

\[
W_{\sigma, p}(f)^p := \sup_{0 < r < r_0} r^{-\sigma p} \left( \int_F \left( \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(y)| \, d\mu(y) \right) \mu(x) < \infty. \right.
\]

The norm of \( f \in \text{Lip}(\sigma, p, \infty)(\mu) \) is defined by

\[
\|f\|_{\text{Lip}(\sigma, p, \infty)(\mu)} = \|f\|_p + W_{\sigma, p}(f).
\]

It is easy to see that \( \text{Lip}(\sigma, p, \infty)(\mu) \) is a Banach space for \( 1 \leq p < \infty \) and \( 0 < \sigma < \infty \) (cf [10, 11]). By (1.1), we see that

\[
M^p(\mu) \subset \text{Lip}(\sigma, p, \infty)(\mu)
\]

if \( \mu \) is a doubling measure. The converse is also true if \( F \) is a smooth domain in \( \mathbb{R}^n \) and \( \mu \) is the Lebesgue measure, see [9]. However, if \( F \) is irregular, the converse may not be true. But for an \( \alpha \)-regular measure \( \mu \), the space \( M^p(\mu) \) is arbitrarily close to \( \text{Lip}(\sigma, p, \infty)(\mu) \). We say that a measure is \( \alpha \)-regular if there exists a constant \( c_3 > 0 \) such that

\[
c_3^{-1} r^\alpha \leq \mu(B(x, r)) \leq c_3 r^\alpha
\]

for all \( x \in F \) and all \( 0 < r < r_0 \) (some \( r_0 > 0 \)). It is not hard to see that if \( \mu \) is \( \alpha \)-regular, then

\[
W_{\sigma, p}(f)^p \simeq \sup_{m \geq 1} 2^{m(\alpha + \sigma p)} \int_F \int_{B(x, c_0 2^{-m})} |f(x) - f(y)| \, d\mu(y) \, d\mu(x),
\]

for any fixed \( c_0 > 0 \).
Proposition 1.1. Let $1 < p < \infty$ and $0 < \sigma < \infty$, and let $0 < \alpha < \infty$. Assume that $\mu$ is $\alpha$-regular. Then

$$\text{Lip}(\sigma, p, \infty)(\mu) \subset M^p_{\sigma-\theta}(\mu)$$

for any $0 < \theta < \sigma$.

Proof. See [9]. \qed

Proposition 1.1 says that $M^p_{\sigma}(\mu)$ is slightly smaller than the Besov space $\text{Lip}(\sigma, p, \infty)(\mu)$ if $\mu$ is $\alpha$-regular.

2. Examples

In this section we show that $M^p_{\sigma}(\mu)$ is non-trivial for any $1 < p < \infty$ and $\sigma > 1$ in a certain range, if the underlying metric space is irregular. We take the Sierpinski gasket in $\mathbb{R}^2$ for an example. The proof is quite technical.

Let $F$ be the Sierpinski gasket in $\mathbb{R}^2$, that is, $F$ is the unique non-empty compact subset of $\mathbb{R}^2$ determined by

$$F = \bigcup_{i=1}^{3} \phi_i(F),$$

where $\phi_i(x) = (q_i + x)/2, x \in \mathbb{R}^2$ (1 \leq i \leq 3), and $q_1,q_2,q_3$ are the three vertices of an equilateral triangle in $\mathbb{R}^2$. Alternatively, we may view the Sierpinski gasket $F$ as the closure of $V_{\ast} = \bigcup_{m=1}^{\infty} V_m$ under the Euclidean metric, where $V_m = \bigcup_{i=1}^{3} \phi_i(V_{m-1}), m \geq 1,$ and $V_0 = \{q_1,q_2,q_3\}$, see Figure 1. For $p = 2$, Kigami [12] constructed a local regular Dirichlet form on $F$ by using the difference scheme. Jonsson [10] identified the domain of this Dirichlet form with a Besov space. Recently, Herman, Peirone and Strichartz [8] have extended Kigami’s result to the case $1 < p < \infty$. Here we briefly describe the main result in [8] that will lead to our example. For $1 < p < \infty$, let $E_p : \mathbb{R}^3 \to [0,\infty]$ be given by

$$E_p(a_1,a_2,a_3) = |a_1 - a_2|^p + |a_2 - a_3|^p + |a_3 - a_1|^p, \quad a_1, a_2, a_3 \in \mathbb{R},$$

and define

$$E_p^{(m)}(f) := \sum_{|\omega|=m} E_p(f(\phi_\omega(q_1)), f(\phi_\omega(q_2)), f(\phi_\omega(q_3))), \quad m \geq 1$$

FIGURE 1
for any $f : F \to \mathbb{R}$, where the summation is taken over all words $\omega$ of length $m$, and 
\[ \phi_\omega(q_1) = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_m}(q_1) \]
for the word $\omega = i_1i_2 \cdots i_m(i_k \in \{1, 2, 3\}$ for $1 \leq k \leq m$.

Let $A_p : \mathbb{R}^3 \to [0, \infty]$ be a function satisfying (among other properties)
\[ c_1^{-1} E_p(a) \leq A_p(a) \leq c_4 E_p(a) \]
for some positive constant $c_1$ and for all $a := (a_1, a_2, a_3) \in \mathbb{R}^3$; in particular, $A_p$ solves the
renormalization problem: Given $a \in \mathbb{R}^3$ and letting
\[ A^{(2)}_p(a, b) := A_p(a_1, b_2, b_3) + A_p(b_1, a_2, b_3) + A_p(b_1, b_2, a_3) \]
for $b = (b_1, b_2, b_3) \in \mathbb{R}^3$, we have that there exists a number $r_p$ such that
\[ \min_{b \in \mathbb{R}^3} A^{(2)}_p(a, b) = r_p A_p(a) \]
for all $a \in \mathbb{R}^3$.

Such a function $A_p$ was shown to exist in [8]. Moreover, the number $r_p$ is unique (independent of the choice of $A_p$) and satisfies
\[ 2^{1-p} \leq r_p \leq 2^{p-1} \left( 1 + \sqrt{1 + 2^{3-\frac{1}{p}}}) \right)^{1-p} < 3 \cdot 2^{-p}, \]
see Lemma 3.8 in [8]. We mention that $r_p = \frac{3}{2}$ for $p = 2$. The function $A_p$ may or may not be unique on $F$; what is important is that the renormalization factor $r_p$ is unique which reflects the intrinsic properties of the Sierpinski gasket $F$. Now, for any $f : F \to \mathbb{R}$, we define the $p$-energy $\mathcal{E}(f)$ of $f$ on $F$ as the limit of
\[ \mathcal{E}_m(f) = r_p^{-m} \sum_{|\omega|=m} A_p(f(\phi_\omega(q_1)), f(\phi_\omega(q_2)), f(\phi_\omega(q_3))), \quad m \geq 1, \]
that is,
\[ \mathcal{E}(f) = \lim_{m \to \infty} \mathcal{E}_m(f). \]
Note that (2.7) makes sense since
\[ \{\mathcal{E}_m(f)\}_m \]
is increasing in $m$ for any function $f$, due to the renormalization problem. Note that
\[ c_5^{-1} \mathcal{E}_m(f) \leq r_p^{-m} E_p^{(m)} \leq c_5 \mathcal{E}_m(f) \]
for all $m \geq 1$ and all $f : F \to \mathbb{R}$, where $c_5 > 0$. Let
\[ \mathcal{D}(\mathcal{E}) = \{ f \in C(F) : \mathcal{E}(f) < \infty \}, \]
termed the domain of the $p$-energy, where $C(F)$ denotes the space of all continuous functions on $F$ with the usual supremum norm. It was shown that $\mathcal{D}(\mathcal{E})$ is dense in $C(F)$, see [8]. The space $\mathcal{D}(\mathcal{E})$ will provide a critical exponent $\beta_p := \log_2(3r_p^{-1})$ (some $r_p > 0$) that determines whether or not a Hajlasz-Sobolev type space $M_p^\mu(1 < p < \infty)$ is non-trivial. To see this, we first identify $\mathcal{D}(\mathcal{E})$ with a Besov space.

**Theorem 2.1.** Let $\mu$ be the $\alpha := \log_2 3$-dimensional Hausdorff measure on $F$. Then
\[ W_{\beta_p/p,p}(f)^p \simeq \mathcal{E}(f) \]
for all $f \in C(F)$, where $W_{\beta_p/p,p}(f)$ is defined as in (1.5) and $\beta_p = \log_2(3r_p^{-1})$. Thus
\[ \mathcal{D}(\mathcal{E}) = \text{Lip} \left( \beta_p/p, p, \infty \right) (\mu), \]
where $\mathcal{D}(\mathcal{E})$ is defined as in (2.9).

**Remarks 1.** When $p = 2$, we have that $r_p = 3/5$ and so $\beta_p = \log_2 5$, the *walk dimension* of the Sierpinski gasket.

2. If $\mu$ is $\alpha$-regular and $\sigma > \alpha/p$, then $\text{Lip}(\sigma, p, \infty)(\mu)$ is embedded into the Hölder space with exponent $\sigma - \alpha/p$ on $F$, that is,

$$
(2.12) \quad |f(x) - f(y)| \leq C |x - y|^\sigma - \frac{\alpha}{p} W_{\sigma, p}(f)
$$

for all $f \in \text{Lip}(\sigma, p, \infty)(\mu)$, where $C$ is independent of $x, y$ and $f$, see for example a direct proof in [4]. Thus

$$
\text{Lip}(\beta_p/p, p, \infty)(\mu) \subset C(F),
$$

since $\beta_p/p > \alpha/p$ (due to $r_p < 1$).

3. By Theorem 2.1, the domain of the $p$-energy coincides with $\text{Lip}(\beta_p/p, p, \infty)(\mu)$ if $\mu$ is the Hausdorff measure. For other measures this may not be true.

**Proof.** The proof given here is motivated by [10] (see also [15, 13]) but with some modifications. We first show that

$$
(2.13) \quad W_{\beta_p/p, p}(f)^p \leq C \mathcal{E}(f)
$$

for all $f \in \mathcal{D}(\mathcal{E})$. To see this, let $f \in \mathcal{D}(\mathcal{E})$. Let

$$
(2.14) \quad f_k(x) = \begin{cases} 
\frac{1}{3} \left[ f(\phi_\omega(q_1)) + f(\phi_\omega(q_2)) + f(\phi_\omega(q_3)) \right], & \text{if } x \in \phi_\omega(F) \setminus \phi_\omega(V_0), \\
0, & \text{if } x \in \phi_\omega(V_0)
\end{cases}
$$

for $|\omega| = k$ ($k \geq 1$). Since $F$ is compact and $f$ is continuous on $F$, the piecewise constant function $f_k$ converges to $f$ pointwise on $F$ as $k \to \infty$. If we can show that, for some $c_0 > 0$ (e.g. $c_0 = \sqrt{3}/2$),

$$
(2.15) \quad 2^{(\alpha + \beta_p)m} \int_{F} \int_{|y-x| < c_0^2^{-m}} |f_{m+k}(y) - f_{m+k}(x)|^p \, d\mu(y) \, d\mu(x) \leq C \mathcal{E}(f)
$$

for all integers $m, k \geq 1$ where $C$ is independent of $f$, then (2.13) follows by letting $k \to \infty$ in (2.15) and using Fatou’s lemma, and (1.7). It remains to prove (2.15). For two words $\omega$ and $\tau$ with $|\tau| = |\omega| = m$, we denote by $\tau \sim \omega$ if $\phi_\tau(F) \cap \phi_\omega(F) \neq \emptyset$ (we allow that...
\( \tau = \omega \). Note that

\[
I_{m+k}(f) := \int_F \int_{y-x < \varepsilon_0 2^{-m}} |f_{m+k}(y) - f_{m+k}(x)|^p d\mu(y) d\mu(x)
\]

\[
\leq \sum_{|\omega| = m} \sum_{\tau \sim \omega} \int_{\phi_{\omega}(F)} \int_{\phi_{\tau}(F)} |f_{m+k}(y) - f_{m+k}(x)|^p d\mu(y) d\mu(x)
\]

\[
\leq \sum_{|\omega| = m} \sum_{\tau \sim \omega} \int_{\phi_{\omega}(F)} \int_{\phi_{\tau}(F)} \frac{2^{p-1}}{p} \left( |f_{m+k}(y) - f(x_{\omega, \tau})|^p + |f(x_{\omega, \tau}) - f_{m+k}(x)|^p \right) d\mu(y) d\mu(x)
\]

(2.16)

\[
\leq 8 \cdot 2^{p-1} \cdot 3^{-m} \sum_{|\omega| = m} \int_{\phi_{\omega}(F)} |f_{m+k}(x) - f(x_{\omega, \tau})|^p d\mu(x),
\]

where we have used the fact that \( \mu(\phi_{\omega}(F)) = 3^{-m} \) and \( \# \{ \tau : \tau \sim \omega \} \leq 4 \) for \( |\omega| = m, m \geq 1 \), and where \( x_{\omega, \tau} \) is some point in \( \phi_{\omega}(V_0) \) (in fact, \( x_{\omega, \tau} \) is the unique intersection point of two sets \( \phi_{\omega}(V_0) \) and \( \phi_{\tau}(V_0) \)). Noting that \( \mu(x) = 0 \) for any single point \( x \in F \), it follows from (2.14) that

\[
\int_{\phi_{\omega}(F)} |f_{m+k}(x) - f(x_{\omega, \tau})|^p d\mu(x) = \sum_{|\tau| = k} \int_{\phi_{\omega}(F)} \frac{1}{3} \sum_{j=1}^{3} \left| f(\phi_{\omega, \tau}(q_j)) - f(x_{\omega, \tau}) \right|^p d\mu(x)
\]

\[
= 3^{-(m+k)} \sum_{|\tau| = k} \left| \frac{1}{3} \sum_{j=1}^{3} (f(\phi_{\omega, \tau}(q_j)) - f(x_{\omega, \tau})) \right|^p
\]

\[
\leq 3^{-(m+k)-1} \sum_{j=1}^{3} \sum_{|\tau| = k} |f(\phi_{\omega, \tau}(q_j)) - f(x_{\omega, \tau})|^p.
\]

which combines with (2.16) to give that

(2.17)

\[
I_{m+k}(f) \leq 2^{p+2} \cdot 3^{-(2m+k)-1} \sum_{j=1}^{3} \sum_{x_{\omega, \tau} \in \phi_{\omega}(V_0)} \sum_{|\tau| = k} |f(\phi_{\omega, \tau}(q_j)) - f(x_{\omega, \tau})|^p.
\]

Let \( q_j \) and \( \tau := i_1 i_2 \cdots i_k \) be fixed, and set \( x_k = \phi_{\omega, \tau}(q_j) \) and \( x_0 = x_{\omega, \tau} = \phi_{\omega}(q_0) \) for some \( q_0 \in V_0 \). We let \( x_l = \phi_{\omega, i_1 i_2 \cdots i_l}(q_0) \), \( 1 \leq l \leq k - 1 \), and obtain a sequence of points \( \{x_l\}_{l=0}^k \) (some of points may be the same). Repeatedly using the elementary inequality \( |a + b|^p \leq 2^{p-1}(|a|^p + |b|^p) \) for any \( a, b \in \mathbb{R} \) and \( 1 \leq p < \infty \), we see that

\[
|f(\phi_{\omega, \tau}(q_j)) - f(x_{\omega, \tau})|^p = |f(x_k) - f(x_0)|^p
\]

\[
\leq \sum_{l=1}^{k} 2^{p-1}|f(x_l) - f(x_{l-1})|^p
\]

\[
\leq \sum_{l=1}^{k} 2^{p-1} \hat{E}_i^{(\omega, p)}(f),
\]
Thus
\[
\sum_{|\tau|=k} |f(\bar{\omega}_{\tau}(q_j)) - f(x_{\omega,\tau})|^p \leq \sum_{i_1, i_2, \ldots, i_k} \sum_{l=1}^k 2^{(p-1)l}\hat{E}^{(\omega,p)}_{i_1, \ldots, i_k, i_{l-1}}(f)
\]
(2.18)
\[
= \sum_{l=1}^k 2^{(p-1)l} \cdot 3^{k-(l-1)} \sum_{i_1, \ldots, i_{l-1}} \hat{E}^{(\omega,p)}_{i_1, \ldots, i_{l-1}}(f).
\]
Observe that
\[
\sum_{|\omega|=m} \sum_{i_1, \ldots, i_{l-1}} \hat{E}^{(\omega,p)}_{i_1, \ldots, i_{l-1}}(f) = \sum_{|\omega|=m+l} E_p(f(\bar{\omega}(q_1)), f(\bar{\omega}(q_2)), f(\bar{\omega}(q_3)))
\]
\[
= E_p^{(m+l)}(f)
\]
\[
\leq c_5 r_p^{m+l} \mathcal{E}_{m+l}(f)
\]
(2.19)
by using (2.8) and the monotonicity of \(\mathcal{E}_m(f)\) in \(m\). Combining (2.17), (2.18) and (2.19), we have that
\[
I_{m+k}(f) \leq C 3^{-2m} \sum_{l=1}^k 2^{(p-1)l} \cdot 3^{-l} r_p^{m+l} \mathcal{E}(f)
\]
\[
\leq C r_p^m \cdot 3^{-2m} \mathcal{E}(f) \sum_{l=1}^\infty 2^{(p-1)l} 3^{-l} r_p^l
\]
\[
\leq C r_p^m \cdot 3^{-2m} \mathcal{E}(f)
\]
\[
= C 2^{-m(\alpha + \beta_p)} \mathcal{E}(f),
\]
where we have used the fact that
\[
\sum_{l=1}^\infty 2^{(p-1)l} 3^{-l} r_p^l < \infty
\]
since \(2^{p-1}3^{-1}r_p < 1\) by virtue of (2.5). Therefore, (2.15) follows.

We next show that
\[
E(f) \leq C W_{\beta_p/p,p}(f)^p
\]
(2.20)
for all \(f \in \text{Lip}(\beta_p/p, p, \infty)\)(\(\mu\)). By (2.8), it is sufficient to show that
\[
\mathcal{E}_p^{(m)}(f) := r_p^{-m} \sum_{|\omega|=m} \sum_{u, v \in \phi_\omega(V_0)} |f(u) - f(v)|^p \leq C W_{\beta_p/p,p}(f)^p
\]
(2.21)
for all \(f \in \text{Lip}(\beta_p/p, p, \infty)(\mu)\) and all \(m \geq 1\). Let \(f \in \text{Lip}(\beta_p/p, p, \infty)(\mu)\). By Remark 2 above we see that \(f\) is continuous on \(F\). Noting that
where

\[ \mu \]

Noting that

\[ (2.24) \]

\[ x \]

\( u \)

\( \leq \frac{2^{p-1}}{\mu(\omega(F))} \int_{\phi_\omega(F)} (|f(u) - f(x)|^p + |f(x) - f(v)|^p) d\mu(x). \]

It follows from (2.21) that

\[ (2.22) \]

\[ \mathcal{E}_p^{(m)}(f) \leq 6 \cdot 2^{p-1} r_p^{-m} \sum_{|\omega|=m} \sum_{u \in \phi_\omega(V_0)} \frac{1}{\mu(\phi_\omega(F))} \int_{\phi_\omega(F)} |f(u) - f(x)|^p d\mu(x). \]

Now let \( x \in \phi_\omega(F) \) and \( u \in \phi_\omega(V_0) \) be fixed. There exists a point \( p_0 \in V_0 \) such that \( u = \phi_\omega(p_0) \). We take \( i_0 \) such that \( \phi_{i_0}(p_0) = p_0 \). Set

\[ S_0 = \phi_\omega(F), \quad S_1 = \phi_\omega(i_0 \cdot i_0 \cdots i_0)(F), \quad S_2 = \phi_\omega(i_0 \cdot i_0 \cdots i_0)(F), \quad \ldots, \]

where \( k \) is an integer to be determined below. It is easy to see that \( u \in S_j \) for each \( j \geq 0 \), and the sequence of the sets \( \{S_j\} \) shrinks to the single point \( u \). For each \( x := x_{\omega,\tau} \in S_0 \), \( x_j \in S_j \) and each \( l \geq 1 \),

\[ |f(u) - f(x)|^p \leq 2^{p-1}(|f(u) - f(x_l)|^p + |f(x_l) - f(x_{\omega,\tau})|^p) \leq 2^{p-1}|f(u) - f(x_l)|^p + \sum_{j=1}^{l} 2^{(p-1)(j+1)}|f(x_j) - f(x_{j-1})|^p. \]

Integrating the above inequality with respect to each \( x_j \in S_j (0 \leq j \leq l) \) and then dividing by \( \mu(S_0)\mu(S_1) \cdots \mu(S_l) \), we obtain that

\[ (2.23) \]

\[ \frac{1}{\mu(\phi_\omega(F))} \int_{\phi_\omega(F)} |f(u) - f(x)|^p d\mu(x) \leq \frac{2^{p-1}}{\mu(S_l)} \int_{S_l} |f(u) - f(x_l)|^p d\mu(x_l) + \sum_{j=1}^{l} 2^{(p-1)(j+1)} \frac{1}{\mu(S_j-1)\mu(S_j)} \int_{S_{j-1}} \int_{S_j} |f(x_j) - f(x_{j-1})|^p d\mu(x_j) d\mu(x_{j-1}). \]

Noting that \( \mu(S_j) = 3^{-(m+kj)} \) for each \( j \geq 0 \) and

\[ \int_{S_{j-1}} \int_{S_j} |f(x_j) - f(x_{j-1})|^p d\mu(x_j) d\mu(x_{j-1}) \leq \int_{S_0} \int_{|\xi - \eta| \leq 2^{-(m+j-1)k}} |f(\xi) - f(\eta)|^p d\mu(\xi) d\mu(\eta), \]

we have from (2.22) and (2.23) that

\[ (2.24) \]

\[ \mathcal{E}_p^{(m)}(f) \leq 6 \cdot 2^{p-1} r_p^{-m} \sum_{|\omega|=m} \sum_{u \in \phi_\omega(V_0)} \left\{ \frac{2^{p-1}}{\mu(S_l)} \int_{S_l} |f(u) - f(x_l)|^p d\mu(x_l) \right\} + \sum_{j=1}^{l} 2^{(p-1)(j+1)} 3^{(2m-(j-1)k)} \int_{\phi_\omega(F)} \int_{|\xi - \eta| \leq 2^{-(m+j-1)k}} |f(\xi) - f(\eta)|^p d\mu(\xi) d\mu(\eta). \]
Letting \( l \to \infty \), we have that the first term on the right-hand side in (2.24) tends to zero since \( f \) is continuous and
\[
\frac{1}{\mu(S_l)} \int_{S_l} |f(u) - f(x_l)|^p d\mu(x_l) \to 0 \text{ as } l \to \infty,
\]
and the second term is less than
\[
\begin{align*}
C \ r_p^{-m} \sum_{j=1}^{\infty} 2^{(p-1)(j+1)} 3^{2m+(2j-1)k} \left( \int_{|\xi| \leq \epsilon 2^{-(m+jk)} |f(\xi) - f(\eta)|^p d\mu(\xi)d\mu(\eta) \right) \\
\leq C \ 3^m r_p^{-m} \sum_{j=1}^{\infty} 2^{(p-1)j} 3^{2j} 2^{-(m+jk)(\alpha+\beta_p)} W_{\beta_p/p,p}(f)^p \\
= C \ 3^m r_p^{-m} W_{\beta_p/p,p}(f)^p \sum_{j=1}^{\infty} 2^{(p-1)j} 3^{2j} (3^{-2} r_p)^{m+jk}
\end{align*}
\]
(2.25)
= \( C \ W_{\beta_p/p,p}(f)^p \sum_{j=1}^{\infty} 2^{(p-1)j} r_p^{jk} \),

since \( 2^{-(\alpha+\beta_p)} = r_p \cdot 3^{-2} \). Since \( r_p < 1 \), we take \( k \) so large that \( r_p^k < 2^{-(p-1)} \), and so
\[
\sum_{j=1}^{\infty} 2^{(p-1)j} r_p^{jk} < \infty.
\]
Therefore,
\[
\mathcal{E}^{(m)}_p(f) \leq C \ W_{\beta_p/p,p}(f)^p
\]
for all \( f \in \text{Lip}(\beta_p/p,p, \infty)(\mu) \) and all \( m \geq 1 \), proving (2.21). The other statement is obvious. \( \square \)

**Corollary 2.1.** Let \( \beta_p = \log_2(3r_p^{-1}) \) as above. Then the space \( \text{Lip}(\tilde{\beta}/p,p, \infty)(\mu) \) defined on the Sierpinski gasket in \( \mathbb{R}^2 \) contains only constant functions if \( \tilde{\beta} > \beta_p \).

**Proof.** By (2.24), (2.25), we see that
\[
\mathcal{E}^{(m)}_p(f) \leq C \ W_{\tilde{\beta}/p,p}(f)^p 2^{-m(\tilde{\beta} - \beta_p)}
\]
for all \( m \geq 1 \) and all \( f \in \text{Lip}(\tilde{\beta}/p,p, \infty)(\mu) \). Thus we have that
\[
\mathcal{E}(f) = \lim_{m \to \infty} \mathcal{E}_m(f) \leq C \lim_{m \to \infty} \mathcal{E}^{(m)}_p(f) = 0,
\]
giving that \( f = \text{const.} \) \( \square \)

**Theorem 2.2.** Let \( F \) be the Sierpinski gasket in \( \mathbb{R}^2 \) and \( \mu \) be the \( \alpha \)-dimensional Hausdorff measure on \( F \), where \( \alpha = \log_2 3 \). Let \( 1 < p < \infty \). Then there exists some \( r_p \in [2^{1-p}, 3 \cdot 2^{-p}] \) such that the Hajlasz–Sobolev space \( M_p^\sigma(\mu) \) is dense in \( C(F) \) for all \( \sigma < p^{-1} \log_2(3r_p^{-1}) \); in particular, the space \( M_p^\sigma(\mu) \) is non-trivial if \( 1 < \sigma < \log_2 5 \) when \( p = 2 \). Moreover, \( M_p^\sigma(\mu) \) is trivial if \( \sigma \geq 1 + 1/p \).
Proof. By Theorem 2.1, the space \( \text{Lip}(\sigma, p, \infty)(\mu) \) is dense in \( C(F) \) if \( 0 < \sigma \leq \frac{\beta_p}{p} = p^{-1} \log_2(3r_p^{-1}) \). Since \( \mu \) is \( \alpha \)-regular, we see from Proposition 1.1 and (2.5) that there exists some \( r_p \in [2^{1-p}, 3 \cdot 2^{-p}] \) such that \( M_p^p(\mu) \) is dense in \( C(F) \) if \( 0 < \sigma < p^{-1} \log_2(3r_p^{-1}) \). By Corollary 2.1, the space \( \text{Lip}(\sigma, p, \infty)(\mu) \) is trivial if \( \sigma \geq 1 + 1/p > \frac{\beta_p}{p} \) (due to \( r_p \geq 2^{1-p} \)). Thus the fact that \( M_p^p(\mu) \subset \text{Lip}(\sigma, p, \infty) \) implies that \( M_p^p(\mu) \) is also trivial if \( \sigma \geq 1 + 1/p \).  

References


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