Nonlinear Diffusion Equations on Unbounded Fractal Domains

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We investigate the nonlinear diffusion equation
\[ \frac{\partial u}{\partial t} = \Delta u + u^p, \quad p > 1, \quad x \in G, \quad t > 0, \]
with non-negative initial values
\[ u|_{t=0} = \phi(x), \quad x \in G. \]

There is considerable interest in PDEs on fractal domains both from the points of view of theory and application. Linear diffusion equations have been investigated on certain specific fractals, see [1–5], but little has been done in the nonlinear situation. While there has been an extensive study of (1.1) on smooth domains, see, for example, [9, 10, 14, 16, 17], the fractal case is more awkward for several reasons. There is the problem of what (1.1) means on a fractal, and a variety of definitions of the “Laplacian” have been proposed depending on the type of fractal domain. Our approach

1. INTRODUCTION

Let \( G \) be an unbounded domain in \( \mathbb{R}^N (N \geq 2) \) which will generally be a fractal. We consider the nonlinear diffusion equation
\[ \frac{\partial u}{\partial t} = \Delta u + u^p, \quad t > 0, \quad x \in G, \quad p > 1, \]
with non-negative initial values
\[ u|_{t=0} = \phi(x), \quad x \in G. \]
here is through a family of integral operators defined in terms of a heat kernel. We first consider “weak solutions” to \((1.1)-(1.2)\), that is, solutions of a corresponding integral equation involving a heat kernel \(k\) on \(G\). The Laplacian may then be defined as the infinitesimal generator of the associated semigroup, to enable us to investigate “strong solutions.” However, the existence of a heat kernel with suitable properties on a fractal is a non-trivial question, and in general such heat kernels cannot be expressed explicitly.

We consider the existence and non-existence of non-negative global solutions to \((1.1)-(1.2)\), and the regularity properties of non-negative bounded global solutions when they exist. The term “global” implies a solution \(u \in G \times (0, \infty)\), that is, existing for all \(t > 0\).

Let \(G\) be a locally compact subset of \(\mathbb{R}^n\) and let \(\mu\) be a locally finite Borel measure on \(G\). We are particularly interested in the case where \(G\) is an unbounded fractal (such as the unbounded Sierpinski triangle) when \(\mu\) will normally be a \(d_f\)-dimensional Hausdorff measure on a set \(G\) of Hausdorff dimension \(d_f\); see [6].

We term a continuous \(k : (0, \infty) \times G \times G \to \mathbb{R}\) a heat kernel if it satisfies

\[(K_1)\) (Positivity) \(k(t, x, y) > 0\) for all \(t > 0\) and all \((x, y) \in G \times G\);

\[(K_2)\) (Symmetry) \(k(t, x, y) = k(t, y, x)\) for all \(t > 0\) and all \((x, y) \in G \times G\);

\[(K_3)\) (Normalization) \(\int_G k(t, x, y) d\mu(y) = 1\) for all \(t > 0\) and all \(x \in G\);

\[(K_4)\) (Semigroup property) \(k(s + t, x, y) = \int_G k(t, z, y) k(s, z, y) d\mu(z)\) for all \(t, s > 0\) and all \((x, y) \in G \times G\); and

\[(K_5)\) (Approximate identity) \(\lim_{t \downarrow 0} \int_G k(t, x, y) f(y) d\mu(y) = f(x)\) in the \(L^2\)-norm for all \(f \in L^2(G)\).

(Note that integration spaces always refer to the measure \(\mu\).) We shall always assume that the heat kernel \(k\) satisfies \((K_1)-(K_5)\), but several of our results depend on further estimates on \(k\), which reflect the fractal structure and the heat diffusion properties of \(G\).

Typically a heat kernel has an inverse power law behaviour in \(t\) and decays exponentially with the separation of the spatial arguments. In the following estimate, the exponent \(d_w\) is the walk dimension, which reflects the rate of transport of heat through \(G\), and \(d_s\) is the spectral dimension which turns out to give the asymptotic distribution of the eigenvalues of the associated Laplacian, so that

\[d_s = 2 \lim_{\lambda \to \infty} \frac{\log \# \{\text{eigenvalues of } \Delta \text{ less than } \lambda\}}{\log \lambda}.
\]
In general
\[
\frac{d_s}{2} = \frac{d_f}{d_w},
\]
where \(d_f\) is the fractal dimension, that is, the Hausdorff dimension, of \(G\); see [3, 4, 7]. We will need estimates:

\((K_6)\) (Bounds) there exist constants \(0 < c_2 \leq c_1\) and \(a_1, a_2 > 0\) such that
\[
a_1 t^{-\frac{d_s}{2}} \exp\left(-c_1 \left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w}}\right) \leq k(t, x, y) \\
\leq a_2 t^{-\frac{d_s}{2}} \exp\left(-c_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w}}\right),
\]
for all \(t > 0\) and \((x, y) \in G \times G\).

Note that for the classical case where \(G = \mathbb{R}^N\) and \(\mu\) is \(N\)-dimensional Lebesgue measure, there is the usual Gaussian heat kernel
\[
k(t, x, y) = (2\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x-y|^2}{t}\right)
\]
with \(d_s = d_v = N\) and \(d_w = 2\).

For later estimates we will require a Hölder condition on the heat kernel:

\((K_7)\) (Hölder condition) there exist \(\nu \geq 1\) and \(0 < \sigma \leq 1\) such that
\[
|k(t, x_2, y) - k(t, x_1, y)| \leq M_0 t^{\sigma} |x_2 - x_1|^\nu \text{ for all } t > 0 \text{ and } x_1, x_2 \in G.
\]

When we come to consider strong solutions, we need to control the derivatives of \(k\) with respect to \(t\):

\((K_8)\) \(\frac{\partial k}{\partial t}(t, x, y)\) exists with \(|\frac{\partial k}{\partial t}(t, x, y)| \leq ct^{-1+d_s/2}\) for all \(t > 0\) and \(x, y \in G\) for some \(c > 0\).

Heat kernels have been constructed on several classes of fractals. The best-known instance is the Sierpinski gasket in \(\mathbb{R}^N\), see Barlow and Perkins [4], where heat kernels are termed transition densities and studied from the probabilistic point of view. In this case \((K_1)-(K_5)\) hold, together with \((K_6)\) for \(d_s = \frac{\log(N+1)}{2\log(N+3)} < 2\), \(d_f = \frac{\log(N+1)}{\log(N+2)}\), and \(d_w = \frac{\log(N+2)}{\log(N+3)}\) (see [12]), and \((K_7)\) with \(\nu = 1\) and \(\sigma = d_w - d_s\); see [1]. Estimates of the form \((K_8)\) hold for affine nested fractals; see [8]. For more general post-critically finite self-similar sets, only a weaker version of \((K_6)\) is known, which holds for a bounded range of \(t\); see [13].

Heat kernels exist for the Sierpinski carpet in \(\mathbb{R}^N\), see [1, 2, 5], and for generalized Sierpinski carpets (where different patterns of squares or cubes are selected in the construction) for which, in particular, \((K_6)\) holds [3]. For the Sierpinski carpet in \(\mathbb{R}^2\), we have \(d_s \approx 1.80\). For the Sierpinski carpet
in $\mathbb{R}^N$ the heat kernel $k(t, x, y)$ is smooth in $t > 0$ for all $x, y \in G$ and satisfies ($K_8$); see [3]. The existence and properties of heat kernels on other fractals are a topic of active research.

We assume that the initial data $\phi : G \to \mathbb{R}$ is measurable; in what follows $\phi$ will generally be non-negative, sometimes satisfying further integrability conditions. By a (weak) solution to (1.1)–(1.2) we mean a measurable $u : (0, \infty) \times G \to \mathbb{R}$ satisfying the integral equation

$$u(t, x) = \int_G k(t, x, y)\phi(y)d\mu(y) + \int_0^t d\tau\int_G k(t-\tau, x, y)u(\tau, y)\rho d\mu(y). \quad (1.3)$$

Let $\{P_t, t > 0\}$ be the family of linear operators associated with $k$, that is,

$$P_tf(x) = \int_G k(t, x, y)f(y)d\mu(y); \quad (1.4)$$

thus $P_t\phi$ may be thought of as the solution of the linear equation $\partial u / \partial t = \Delta u$. From ($K_1$)–($K_3$), $P_t : L^q(G) \to L^q(G)$ for all $1 \leq q \leq \infty$. In particular, $\{P_t, t > 0\}$ is a family of symmetric operators on $L^2(G)$ which, by ($K_4$), possesses the semigroup property

$$P_{s+t} = P_sP_t. \quad (1.5)$$

The contraction property

$$\|P_tf\|_q \leq \|f\|_q \quad \text{for all } f \in L^q(G) \text{ and all } t > 0 \quad (1.6)$$

follows for all $1 \leq q \leq \infty$, using the weighted Hölder inequality and ($K_5$). Property ($K_5$) states that the family of operators $\{P_t\}$ is strongly continuous, that is,

$$\lim_{t \to 0} \|P_tf - f\|_2 = 0.$$

In particular, this implies that there exists an infinitesimal generator $\Delta$

$$\Delta f = \lim_{t \downarrow 0} \frac{P_tf - f}{t}, \quad f \in \mathcal{D}(\Delta), \quad (1.7)$$

where $\mathcal{D}(\Delta)$ is the space of all functions $f \in L^2(G)$ such that the limit in (1.7) exists in the $L^2$ norm and is finite; see [11]. This definition of the Laplacian enables us to consider strong solutions of (1.1)–(1.2).

In Section 2 we consider the non-existence problem and show that, provided that the heat kernel satisfies ($K_8$), there are no non-negative global solutions to (1.3) if $p \leq 1 + 2/d_x$, however small the initial data $\phi \not= 0$. In Section 3 we show that non-negative global solutions exist if $p > 1 + 2/d_x$ and the initial data are small enough. In the classical situation, bounded solutions (1.3) are smooth in both $x$ and $t$; see [9]. We cannot expect such smoothness on fractals, but in Section 4 we prove that solutions in $L^\infty(G) \cap L^1(G)$ are Hölder continuous in $x$ if the initial data are Hölder continuous. Moreover, given ($K_8$) the solution is differentiable with respect to $t$ for almost all $t > 0$ for all $x \in G$, and so satisfies (1.1) at such points.
2. NON-EXISTENCE OF GLOBAL SOLUTIONS

Our aim in this section is to show using \((K_6)\) that there are no non-negative global solutions to \((1.3)\) if \(p \leq 1 + 2/d_\varepsilon\) for non-negative initial data \(\phi \neq 0\), however small. The estimate \((2.1)\) plays a major role in the non-existence proof. Note that \((K_5)\) and \((K_6)\) are not required at this stage.

**Proposition 2.1.** Assume that the heat kernel \(k\) satisfies \((K_1)-(K_4)\). Let \(u(t, x)\) be a non-negative essentially bounded solution of \((1.3)\) in \((0, T) \times G\) where \(\phi\) is essentially bounded. Then

\[
\int_G k(t, x, y)\phi(y)d\mu(y) \leq M_1
\]

for all \(t \in (0, T)\) and all \(x \in G\), where \(M_1\) is independent of \(T\) and \(\phi\).

**Proof.** We sketch the proof, following [9, 16]. Since \(u(t, x)\) satisfies \((1.3)\), we get that

\[
\int_G k(t, x, y)\phi(y)d\mu(y) \leq P_t\phi(x)
\]

with \(P_t\), given by \((1.4)\). Using \((1.3)\) again, it follows that

\[
\begin{align*}
\int_G k(t, x, y)\phi(y)d\mu(y) &\geq \int_0^t d\tau \int_G k(t - \tau, x, y)[P_\tau\phi(y)]\phi(y)d\mu(y) \\
&\geq \int_0^t d\tau \left\{ \int_G k(t - \tau, x, y)P_\tau\phi(y)d\mu(y) \right\}^p \\
&= \int_0^t d\tau [P_{t-\tau}(P_\tau\phi)(x)]^p \\
&= \int_0^t d\tau [P_t\phi(x)]^p \\
&= t[P_t\phi(x)]^p,
\end{align*}
\]

where we have used the weighted Hölder inequality, \((K_4)\), and \((1.5)\). Repeating this procedure of substitution in \((1.3)\) we obtain by induction that

\[
u(t, x) \geq \frac{t^{1+p+\cdots+p-1}[P_t\phi(x)]^p}{(1+p)^{p-2}(1+2p^2)^{p-3}\cdots(1+p+\cdots+p^{n-1})}.
\]

Therefore, for all \(n \geq 1\),

\[
t^{(p^n-1)/(p-1)p}P_t\phi(x) \leq u(t, x)^{1/p} \prod_{j=2}^{\infty} (1+p+\cdots+p^{j-1})^{1/p'}.
\]

\[(2.2)\]
Since
\[
\log \prod_{j=2}^{\infty} (1 + p + \cdots + p^{j-1})^{1/p^j} \leq \sum_{j=2}^{\infty} \frac{1}{p^j} \log(jp^j) < \infty,
\]
the estimate (2.1) follows on letting \( n \to \infty \) in (2.2).

We now prove the non-existence of global solutions if \( p \leq 1 + 2/d_s \), that is, any solutions “blow up” or become unbounded in a finite time. In the case of \( p < 1 + 2/d_s \), this is an easy consequence of (2.1) and only requires the left hand inequality of \((K_6)\).

**Theorem 2.2.** Suppose that \( k \) satisfies \((K_1)-(K_4)\) and \((K_6)\). If \( p \leq 1 + 2/d_s \), then (1.3) has no non-negative essentially bounded global solutions if \( \phi(x) \geq 0 \) and \( \phi(x) \not\equiv 0 \).

**Proof.** By \((K_6)\), we have
\[
\liminf_{t \to \infty} t^{d/2} \int_G k(t, x, y) \phi(y) \, d\mu(y) \geq a_1 \liminf_{t \to \infty} \int_G \exp \left( -c_1 \frac{|x - y|_{d_s}}{t} \right) \frac{1}{t} \phi(y) \, d\mu(y) \geq M_2,
\]
where \( M_2 = 1 \) if \( \| \phi \|_1 = \int_G \phi(y) \, d\mu(y) = +\infty \) and \( M_2 = a_1 \| \phi \|_1 \) if \( \| \phi \|_1 < \infty \). Combining with (2.1) and (2.3), this requires that, for some \( M_3 > 0 \), we have \( t^{d/2} - (1/(p-1)) \geq M_3 \) for all large \( t \), which is impossible if \( p < 1 + 2/d_s \).

We now consider the more delicate case of \( p = 1 + 2/d_s \). Then (2.1) becomes
\[
t^{d/2} \int_G k(t, x, y) \phi(y) \, d\mu(y) \leq M_1
\]
which using the left hand side of \((K6)\) gives
\[
\int_G \phi(y) \, d\mu(y) \leq M_4 < \infty,
\]
where \( M_4 = M_1/a_1 \). Observe that for any \( t_0 > 0 \), \( v(t, x) \equiv u(t + t_0, x) \) is a solution to (1.3) with initial data \( \psi(x) = u(t_0, x) \). Repeating the above procedure we have that
\[
\int_G u(t, y) \, d\mu(y) \leq M_5, \quad \text{for all } t > 0,
\]
for some \( M_5 > 0 \).
Next we claim that for all \( t_0 > 0 \), there exists \( \epsilon > 0 \) and \( b_1 > 0 \), depending on \( t_0 \) and \( \phi \), such that
\[
u(t_0, x) \geq b_1 k(\epsilon, x, 0)
\]
for all \( x \in G \). To see this, let \( \alpha = 1/(d_w - 1) \) and \( \beta = d_w/(d_w - 1) \). By \( (K_b) \),
\[
k(\epsilon, x, 0) a_2^{-1} e^{d/2} \exp \left( c_2 \frac{|x|^{\beta}}{e^{\alpha}} \right) \leq 1
\]
and thus, using \( (1.3) \) and \( (K_a) \),
\[
u(t_0, x) \geq \int_G k(t_0, x, y) \phi(y) d\mu(y)
\]
\[
\geq a_1 t_0^{-d/2} \int_G \exp \left( -c_1 \frac{|x - y|^{\beta}}{t_0^\alpha} \right) \phi(y) d\mu(y)
\]
\[
\geq a_1 \left( \frac{\epsilon}{t_0} \right)^{d/2} k(\epsilon, x, 0) \int_G \exp \left( \frac{c_2}{e^{\alpha}} |x|^{\beta} - \frac{c_1}{t_0^\alpha} |x - y|^{\beta} \right)
\]
\[
\times \phi(y) d\mu(y).
\]
(2.7)

Note that for \( \beta \geq 1 \)
\[
|x - y|^{\beta} \leq 2^{\beta-1} (|x|^{\beta} + |y|^{\beta})
\]
for all \( x, y \in \mathbb{R} \), so
\[
\frac{c_2}{e^{\alpha}} |x|^{\beta} - \frac{c_1}{t_0^\alpha} |x - y|^{\beta} \geq \left( \frac{c_2}{e^{\alpha}} - 2^{\beta-1} \frac{c_1}{t_0^\alpha} \right) |x|^{\beta} - 2^{\beta-1} \frac{c_1}{t_0^\alpha} |y|^{\beta}
\]
\[
= -2^{\beta-1} \frac{c_1}{t_0^\alpha} |y|^{\beta}
\]
for all \( x, y \), if \( \epsilon \) is chosen to satisfy \( c_2 \epsilon^{-\alpha} = 2^{\beta-1} c_1 t_0^{-\alpha} \). Hence \( (2.6) \) follows on taking
\[
b_1 = \frac{a_1}{a_2} \left( \frac{\epsilon}{t_0} \right)^{d/2} \int_G \exp \left( -2^{\beta-1} \frac{c_1}{t_0^\alpha} |y|^{\beta} \right) \phi(y) d\mu(y)
\]
in \( (2.7) \).

To complete the proof when \( p = 1 + 2/d_w \), we consider \( \nu(t, x) \equiv u(t + 1, x) \). Clearly \( \nu(t, x) \) satisfies \( (1.3) \) with the initial data \( u(1, x) \) in place of \( \phi(x) \). By \( (1.3) \), \( (2.6) \), and \( (K_4) \),
\[
\nu(t, x) \geq \int_G k(t, x, y) u(1, y) d\mu(y)
\]
\[
\geq b_1 \int_G k(t, x, y) k(\epsilon, y, 0) d\mu(y)
\]
\[
= b_1 k(t + \epsilon, x, 0)
\]
and so by (1.3)
\begin{align*}
\int_G v(t, x) \, d\mu(x) &\geq \int_G d\mu(x) \int_0^t \int_G k(t - \tau, x, y)v(\tau, y)^p \, d\mu(y) \\
&= \int_0^t \int_G v(\tau, y)^p \, d\mu(y) \\
&\geq b_i^p \int_0^t \int_G k(\tau + \epsilon, y, 0)^p \, d\mu(y).
\end{align*}
(2.8)

Using \((K_6)\) twice with \(p = 1 + 2/d_s\), it follows that
\begin{align*}
k(\tau + \epsilon, y, 0)^p &\geq a_1^p (\tau + \epsilon)^{-pd_s/2} \exp(-pc_1|y|^\beta/(\tau + \epsilon)^\alpha) \\
&= a_1^p \left(\frac{c_2}{pc_1}\right)^{d_s/2\alpha} (\tau + \epsilon)^{-1} \left[\frac{\tau + \epsilon}{(pc_1c_2^{-1})^{1/\alpha}}\right]^{d_s/2} \\
&\quad \times \exp\left(-\frac{c_2|y|^\beta}{c_2(\tau + \epsilon)^\beta/pc_1}\right) \\
&\geq a_1^p \left(\frac{c_2}{pc_1}\right)^{d_s/2\alpha} (\tau + \epsilon)^{-1} k((\tau + \epsilon)(pc_1c_2^{-1})^{-1/\alpha}, y, 0)
\end{align*}
which combines with (2.8) to yield
\begin{equation*}
\int_G v(t, x) \, d\mu(x) \geq M_6 \int_0^t (\tau + \epsilon)^{-1} \, d\tau \to \infty
\end{equation*}
as \(t \to \infty\), where \(M_6 > 0\). This contradicts (2.5), proving the theorem for \(p = 1 + 2/d_s\).

The above theorem applies, for example, to the unbounded Sierpinski gasket and to generalised Sierpinski carpets.

3. EXISTENCE OF NON-NEGATIVE GLOBAL SOLUTIONS

We now consider the situation where \(p > 1 + 2/d_s\) and show that if the initial data are sufficiently small, then (1.3) possesses a non-negative global solution.

We prove the following general existence result which we then show applies in our situation, again adapting [9, 17].

**Theorem 3.1.** Suppose \(k\) satisfies \((K_1)-(K_3)\). Let \(p > 1\) and \(1 \leq q < \infty\). Let \(\phi \geq 0\) with \(\phi \in L^q(G)\) and satisfying
\begin{equation*}
\int_0^\infty \|P_x \phi\|_{p-1} \, d\tau \leq (p - 1)^{-1},
\end{equation*}
where \(P_x\) is given by (1.4). Then (1.3) possesses a non-negative global solution \(u\) with \(u \in L^\infty((0, T), L^q(G))\) for all \(T > 0\).
Proof. The proof follows [17]. We define \( b : [0, \infty) \to \mathbb{R} \) by

\[
b(t)^{-p-1} = 1 - (p - 1) \int_0^t \| P_\tau \phi \|_\infty^{p-1} \, d\tau.
\]

Using (3.1), we get that \( b(0) = 1 \) and \( b'(t) = b(t)P_t \phi \|_\infty^{p-1} \). Thus \( b \) satisfies the integral equation

\[
b(t) = 1 + \int_0^t b(\tau)P_\tau \phi \|_\infty^{p-1} \, d\tau. \tag{3.2}
\]

Now let \( u : [0, \infty) \times G \to \mathbb{R} \) be any continuous function such that

\[
P_t \phi(x) \leq u(t, x) \leq b(t)P_t \phi(x) \quad \text{for all } t \geq 0 \text{ and } x \in G,
\]

and define

\[
\mathcal{F} u(t, x) = P_t \phi(x) + \int_0^t (P_{t-\tau}u^p)(\tau, x) \, d\tau. \tag{3.3}
\]

Then for \( t \geq 0 \) and \( x \in G \), using (1.4), (1.5), and (3.2),

\[
\mathcal{F} u(t, x) \leq P_t \phi(x) + \int_0^t d\tau \int_G k(t - \tau, x, y) b(\tau)P_\tau \phi(y)^p \, d\mu(y)
\]

\[
\leq P_t \phi(x) + \int_0^t b(\tau)P_\tau \phi \|_\infty^{p-1} \, d\tau \int_G k(t - \tau, x, y)P_\tau \phi(y) \, d\mu(y)
\]

\[
= b(t)P_t \phi(x).
\]

Therefore,

\[
P_t \phi(x) \leq \mathcal{F} u(t, x) \leq b(t)P_t \phi(x), \quad \text{for all } t \geq 0 \text{ and } x \in G.
\]

Define \( u_0(t, x) = P_t \phi(x) \) and \( u_{m+1}(t, x) = \mathcal{F} u_m(t, x) \) for \( m = 0, 1, 2, \ldots \). Using (3.3) and induction, for all \( t > 0 \) and \( x \in G \), the sequence \( \{u_m(t, x)\} \) is non-decreasing in \( m \), and

\[
P_t \phi(x) \leq u_m(t, x) \leq b(t)P_t \phi(x) \quad \text{for all } m \geq 0. \tag{3.4}
\]

Thus there exists a measurable function \( u(t, x) \) such that for all \( t > 0 \) and \( x \in G \)

\[
\lim_{m \to \infty} u_m(t, x) = u(t, x) \in [0, \infty],
\]

with

\[
P_t \phi(x) \leq u(t, x) \leq b(t)P_t \phi(x), \quad t > 0, \ x \in G. \tag{3.5}
\]

Using the monotone convergence theorem, we have

\[
\lim_{m \to \infty} \int_0^t d\tau \int_G k(t - \tau, x, y) u_m(\tau, y)^p \, d\mu(y)
\]

\[
= \int_0^t d\tau \int_G k(t - \tau, x, y) u(\tau, y)^p \, d\mu(y)
\]
for all $t \geq 0$ and $x \in G$, and thus $u(t, x)$ satisfies (1.3) on taking the limit as $m \to \infty$ in

$$u_{m+1}(t, x) = \mathcal{F} u_m(t, x)$$

$$= \mathcal{P}_t \phi(x) + \int_0^t d\tau \int_G k(t - \tau, x, y) u_m(\tau, y) d\mu(y).$$

From (3.5)

$$\|u(t, \cdot)\|_q \leq b(t) \|\mathcal{P}_t \phi\|_q \leq b(t) \|\phi\|_q$$

by (1.6), so since $b(t)$ is bounded on $[0, T]$ we get $u \in L^\infty([0, T])$ for all $T > 0$.

We can immediately apply this result to bounded data.

**Corollary 3.2.** Let $k$ satisfy $(K_1)-(K_6)$, and let $p > 1 + 2/d_s$. Given $\gamma > 0$ there exists $\delta > 0$ such that if

$$0 \leq \phi(x) \leq \delta k(\gamma, x, 0)$$

for all $x \in G$, then (1.3) has a non-negative bounded global solution. In particular, this will be the case if $\phi$ has compact support and $\sup_{x \in G} \phi(x)$ is sufficiently small.

**Proof.** Using $(K_4)$ and $(K_6)$,

$$\|\mathcal{P}_t \phi\|_\infty \leq \sup_{x \in G} \int_G k(t, x, y) \phi(y) d\mu(y)$$

$$\leq \delta \sup_{x \in G} k(t + \gamma, x, 0)$$

$$\leq a_2 \delta (t + \gamma)^{-d_s/2},$$

and so

$$\int_0^\infty \|\mathcal{P}_t \phi\|_\infty^{-p-1} dt \leq (a_2 \delta)^{-p-1} \int_0^\infty (t + \gamma)^{-(p-1)d_s/2} dt \leq (p - 1)^{-1}$$

for $\delta$ small enough, since $p > 1 + 2/d_s$. Therefore, Theorem 3.1 implies the existence of a non-negative global solution.

We use the Marcinkiewicz interpolation theorem to apply Theorem 3.1 with an alternative condition on the initial data. Let $D_1$ and $D_2$ be linear spaces of measurable functions on two $\sigma$-finite measure spaces with measures $\mu_1$ and $\mu_2$, respectively. We say that a mapping $H : D_1 \to D_2$ is subadditive if

$$|H(f_1 + f_2)(x)| \leq |H(f_1)(x)| + |H(f_2)(x)|$$
for all \( f_1 \) and \( f_2 \) in \( D_1 \) and \( \mu_2 \)-almost all \( x \). For \( 1 \leq r \leq \infty \) and \( 1 \leq s < \infty \), we say that \( H \) is of weak type \( (r, s) \) if there exists a constant \( M \) such that

\[
\lambda(\alpha) \leq \left( \frac{M \|f\|_r}{\alpha} \right)^s
\]

for all \( f \in L^r(\mu_1) \cap D_1 \), where \( \lambda(\alpha) = \mu_2 \{ x : |H(f)(x)| > \alpha \} \) is the distribution function of \( Hf \), and \( \|f\|_r = (\int |f(x)|^r d\mu_1(x))^{1/r} \).

**Proposition 3.3** (Marcinkiewicz Interpolation Theorem). Suppose that \( H \) is a subadditive map of weak type \( (r_i, s_i) \) where \( 1 \leq r_i \leq s_i \leq \infty \) for both \( i = 1, 2 \), and with \( s_1 \neq s_2 \). Then for all \( \theta \in (0, 1) \), \( H \) is of strong type \( (r_\theta, s_\theta) \), where

\[
1 / r_\theta = 1 - \theta / r_0 + \theta / r_1 \quad \text{and} \quad 1 / s_\theta = 1 - \theta / s_0 + \theta / s_1;
\]

that is, there exists a constant \( M_0 \) such that

\[
\|H(f)\|_{s_\theta} \leq M_0 \|f\|_{r_\theta} \quad \text{for all } f \in D_1.
\]

**Proof.** See [15, p. 184].

**Corollary 3.4.** Let \( k \) satisfy (K_1)-(K_\delta), and let \( d_s \leq 2 \) and \( p > 1 + 2/d_s \). If \( \|\phi\|_{d_s(p-1)/2} \) is sufficiently small then (1.3) has a non-negative global solution in \( L^\infty((0, T), L^{d_s(p-1)/2}(G)) \) for all \( T > 0 \).

**Proof.** Fix \( 1 \leq r \leq \infty \). We consider maps \( H : L^r(G) \to C(\mathbb{R}) \) defined by

\[
H\phi(t) = \|P_t\phi\|_\infty \quad \text{for } t > 0, \text{ where } \phi \in L^r(G).
\]

Clearly \( H \) is subadditive. Moreover,

\[
H\phi(t) = \sup_{x \in G} \left| \int_G k(t, x, y)\phi(y) d\mu(y) \right|
\]

\[
\leq \sup_{x \in G} \left( \int_G k(t, x, y)\phi(y)' d\mu(y) \right)^{1/r}
\]

\[
\leq d_2^{1/r} t^{-d_s/2r} \|\phi\|_r,
\]

by virtue of the weighted Hölder inequality and \( (K_\delta) \). Thus \( H\phi(t) > \alpha \) implies that

\[
t \leq \left( \frac{d_2^{1/r} \|\phi\|_r}{\alpha} \right)^{2r/d_s}.
\]
Thus $H$ is of weak-type $(r, s)$ whenever $1 \leq r \leq \infty$ and $s = 2r/d_s$. By Proposition 3.3, $H$ is of strong-type $(r, s)$ whenever $1 < r < \infty$ and $s = 2r/d_s$; that is, there is an $M_7 > 0$ such that

\[
\left( \int_0^\infty \| P_t \phi \|_s^r \, dt \right)^{1/s} \leq M_7 \| \phi \|_r \quad \text{for all } \phi \in L^r.
\]

Letting $r = d_s(p - 1)/2 > 1$ we have $s = 2r/d_s > 1$ since $d_s \leq 2$, so

\[
\int_0^\infty \| P_t \phi \|_s^{p-1} \, dt \leq M_7^{p-1} \| \phi \|_{d_s(p-1)/2}^{p-1} \leq (p - 1)^{-1}
\]

if $\| \phi \|_{d_s(p-1)/2}$ is sufficiently small, giving (3.1). The conclusion follows from Theorem 3.1.

By our introductory remarks, Corollaries 3.2 and 3.4 imply that there exist non-negative solutions to (1.3) on the Sierpinski gasket in $\mathbb{R}^N$ and the Sierpinski carpet in $\mathbb{R}^2$ if $p > 1 + 2/d_s$ and the initial data are small in an appropriate sense.

4. REGULARITY PROPERTIES OF GLOBAL SOLUTIONS

In this section we discuss regularity properties of global solutions. Let $u \in [0, \infty) \times G$ be a non-negative global solution to (1.3). We show that if the initial data are Hölder continuous, then $u(t, x)$ is Hölder continuous in $x$ uniformly for $t \in (0, T)$ for all $T > 0$. We shall also show that $\Delta u(t, x)$ exists and satisfies (1.1) for almost every $t > 0$ for all $x \in G$, where the Laplacian $\Delta$ is viewed as the infinitesimal generator (1.7).

We first require an estimate for radial integrals which depends on local “fractal” properties of the measure $\mu$. We write $B_r(x)$ for the ball of centre $x$ and radius $r$.

**Proposition 4.1.** Let $G$ be a closed (not necessarily bounded) subset of $\mathbb{R}^N$ and $\mu$ a Borel measure supported by $G$ such that for some $d > 0$ and $M_8 > 0$

\[
\mu(B_r(x)) \leq M_8 r^d \quad (r > 0, x \in G). \tag{4.1}
\]

Suppose that $f : [0, \infty) \to [0, \infty)$ is $C_1$ with $f(r) = o(r^{-d})$ as $r \to \infty$. Then

\[
\int_G f(|y - x|) d\mu(y) \leq M_8 \int_0^\infty r^d |f'(r)| \, dr \quad (x \in G). \tag{4.2}
\]
Proof. Let \( m(r) = \mu(B_r(x)) \). Then \( m(r) \) is non-decreasing in \( r \) and continuous on the right. Therefore, for all \( x \in G \), using that \( m(0) = 0 \) and \( f(r) = o(r^{-d}) \),

\[
\int_G f(|y-x|)d\mu(y) \leq \int_0^\infty f(r)dm(r)
\]

\[
= m(r)f(r) \bigg|_0^\infty - \int_0^\infty f'(r)m(r)dr
\]

\[
= -\int_0^\infty f'(r)m(r)dr
\]

\[
\leq M_8 \int_0^\infty r^d|f'(r)|dr
\]

using (4.1).

Note that many measures satisfying (4.1) exist. For example, if \( \mu \) is the restriction of \( d\)-dimensional Hausdorff measure to a (bounded or unbounded) self-similar set \( G \) of Hausdorff dimension \( d \), then (4.1) is satisfied; see [6, 7]. In particular this applies to the Sierpiński gaskets and generalised carpets with \( d = d_f \).

**Corollary 4.2.** Let \( G \) be an unbounded subset of \( \mathbb{R}^N \) supporting a measure \( \mu \) which satisfies (4.1). Then given \( \alpha, \beta, \lambda, c_2 > 0 \), there exists \( M_9 \) such that

\[
\int_G |y-x|^\lambda \exp\left(-c_2 \frac{|y-x|^\beta}{t^\alpha}\right)d\mu(y) \leq M_9 t^{(\lambda+d)\alpha/\beta}
\]  

(4.3)

for all \( t > 0 \) and all \( x \in G \).

**Proof.** For \( t > 0 \), take

\[ f(r) = r^\lambda \exp\left(-c_2 \frac{r^\beta}{t^\alpha}\right). \]

Clearly \( f(r) = o(r^{-d}) \) and

\[ |f'(r)| \leq \left( \lambda + c_2 \frac{\beta r^\beta}{t^\alpha}\right) r^{\lambda-1} \exp\left(-c_2 \frac{r^\beta}{t^\alpha}\right). \]

Using (4.2), it follows that

\[
\int_G |y-x|^\lambda \exp\left(-c_2 \frac{|y-x|^\beta}{t^\alpha}\right)d\mu(y)
\]

\[
\leq M_8 \int_0^\infty r^{d+\lambda-1} \left( \lambda + c_2 \frac{\beta r^\beta}{t^\alpha}\right) \exp\left(-c_2 \frac{r^\beta}{t^\alpha}\right)dr
\]

\[
= M_8 t^{(\lambda+d)\alpha/\beta} \int_0^\infty s^{d+\lambda-1} \left( \lambda + c_2 \beta s^\beta\right) \exp(-c_2 s^\beta) ds,
\]

giving (4.3) since this integral is finite and independent of \( t \).
Theorem 4.3 (Hölder Continuity). Assume that the measure $\mu$ on $G$ satisfies (4.1), where $d_1$ is the Hausdorff dimension of $G$. Suppose that the heat kernel $k$ satisfies $(K_1)-(K_7)$ and suppose the initial data $\phi \in L^1(G)$ are Hölder continuous with exponent $\lambda \in (0, 1]$, that is,

$$ |\phi(x_2) - \phi(x_1)| \leq M_{10}|x_2 - x_1|^\lambda \quad \text{for all } x_1, x_2 \in G $$

for some $M_{10}$. Let $T > 0$ and let $u$ be a non-negative global solution to (1.3) that is bounded on $(0, T) \times G$ with $\|u(t, \cdot)\|_1$ bounded on $(0, T)$. Then $u(t,x)$ is Hölder continuous in $x$, with exponent $\gamma = \lambda \sigma/\lambda + v d_w$, uniformly for $t \in (0, T)$, where $d_w$ is the walk dimension; that is,

$$ |u(t,x_2) - u(t,x_1)| \leq M_{11}|x_2 - x_1|^\gamma $$

for all $t \in (0, T)$ and all $x_1, x_2 \in G$ where $M_{11}$ depends on $T$.

Proof. We first prove that

$$ u_0(t,x) \equiv P_t \phi(x) = \int_G k(t,x,y)\phi(y)d\mu(y) $$

is Hölder continuous in $x$ uniformly for $t > 0$. To see this, by $(K_7)$ we have that

$$ |u_0(t,x_2) - u_0(t,x_1)| = \left| \int_G (k(t,x_2,y) - k(t,x_1,y))\phi(y)d\mu(y) \right| $n

$$ \leq M_9 t^{-\nu}|x_2 - x_1|^\nu \|\phi\|_1 $n

$$ \leq M_9 \|\phi\|_1|x_2 - x_1|^\nu - \nu $$. 

if $t \geq |x_2 - x_1|^s$, for all $s > 0$.

On the other hand, from (4.4) and $(K_6)$, taking $\alpha = 1/(d_w - 1)$, $\beta = d_w/(d_w - 1)$ in (4.3) and recalling that $d_1/2 = d_f/d_w$,

$$ \int_G k(t,x,y)\phi(y) - \phi(x)d\mu(y) $n

$$ \leq a_2 M_{10} t^{-d_1/2} \int_G |y - x|^\alpha \exp\left(-c_2 |y - x|^\beta/\alpha\right) d\mu(y) $n

$$ \leq a_2 M_9 M_{10} t^{\lambda/d_w}. $n

Therefore, if $t \leq |x_2 - x_1|^s$, it follows from (4.4) that for all $t > 0$ and all $x_1, x_2 \in G$,

$$ |u_0(t,x_2) - u_0(t,x_1)| $n

$$ = \left| \int_G k(t,x_2,y)(\phi(y) - \phi(x_2))d\mu(y) $n

$$ - \int_G k(t,x_1,y)(\phi(y) - \phi(x_1))d\mu(y) + (\phi(x_2) - \phi(x_1)) \right| $n

$$ \leq 2a_2 M_9 M_{10} t^{\lambda/d_w} + M_{10}|x_2 - x_1|^\lambda $n

$$ \leq (M_{10} + 2a_2 M_9 M_{10})(|x_2 - x_1|^\lambda + |x_2 - x_1|^\lambda/d_w). $n
This combines with (4.7) to give

\[ |u_0(t, x_2) - u_0(t, x_1)| \leq M_{12}(|x_2 - x_1|^{\sigma - \nu} + |x_2 - x_1|^\lambda + |x_2 - x_1|^{\lambda \sigma/(\lambda + \nu d_w)}) \leq 3M_{12}|x_2 - x_1|^\lambda \sigma/(\lambda + \nu d_w) \tag{4.8} \]

for all \( t > 0 \) and all \( x_1, x_2 \in G \) with \( |x_2 - x_1| \leq 1 \), by taking \( s \) such that \( \sigma - \nu s = \lambda \sigma/(\lambda + \nu d_w) \), since \( \sigma \leq 1 \leq \nu \) and \( d_w \geq 2 \).

Now we consider

\[ w(t, x) \equiv \int_0^t d\tau \int_G k(t - \tau, x, y) u(\tau, y)^p d\mu(y). \]

Fix \( T > 0 \) and \( 0 < \eta < t < T \). Since \( u(t, x) \) is bounded in \((0, T) \times G\), it follows that

\[ \int_{t - \eta}^t d\tau \int_G k(t - \tau, x, y) u(\tau, y)^p d\mu(y) \leq M_{13} \eta \]

for some \( M_{13} > 0 \). Assume first that \( \nu > 1 \). Then for all \( t \in (0, T) \) and \( x_1, x_2 \in G \), using \((K_7)\),

\[
|w(t, x_2) - w(t, x_1)| \\
= \left| \int_{t - \eta}^t d\tau \int_G k(t - \tau, x_2, y) u(\tau, y)^p d\mu(y) \\
- \int_{t - \eta}^t d\tau \int_G k(t - \tau, x_1, y) u(\tau, y)^p d\mu(y) \\
+ \int_0^{t - \eta} d\tau \int_G (k(t - \tau, x_2, y) - k(t - \tau, x_1, y)) u(\tau, y)^p d\mu(y) \right| \\
\leq 2M_{13} \eta + M_0 \int_{t - \eta}^t \int_G |t - \tau|^{-\nu} |x_2 - x_1|^\sigma u(\tau, y)^p d\mu(y) \\
\leq M_{14}(\eta + \eta^{1-\nu}|x_2 - x_1|^\sigma), \tag{4.9}
\]

for some \( M_{14} \). Since the left-hand side of (4.9) is independent of \( \eta \), we may take \( \eta = |x_2 - x_1|^{\sigma/\nu} \) in (4.9) to get

\[
|w(t, x_2) - w(t, x_1)| \leq M_{14}(|x_2 - x_1|^{\sigma/\nu} + |x_2 - x_1|^{\sigma + (1-\nu)\sigma/\nu}) \\
\leq 2M_{14}|x_2 - x_1|^{\sigma/\nu}
\]

for all \( t \in (0, T) \) and \( x_1, x_2 \in G \). Putting this estimate and (4.8) in the integral equation (1.3), and noting \( \lambda \leq 1 < d_w \) gives the result when \( |x_1 - x_2| \leq 1 \), and this extends to all \( x_1, x_2 \in G \) since \( u \) is bounded. The case of \( \nu = 1 \) is similar, with a logarithmic integral in (4.9). \( \blacksquare \)
Next we show that, under certain conditions, the bounded solution \( u(t, x) \) of (1.3), \( \frac{\partial u}{\partial t}(t, x) \) exists for almost every \( t > 0 \) and all \( x \in G \).

**Theorem 4.4 (Lipschitz Continuity).** Suppose that \( k \) satisfies (K₁)–(K₅) and (K₆). Assume that the initial data \( \phi \) satisfy \( \phi(x) \geq 0 \), \( \|\phi\|_1 < \infty \), and

\[
|P_{t+\delta} \phi(x) - P_t \phi(x)| \leq c_0 \delta, \quad \text{for all } t > 0 \text{ and } x \in G, \quad (4.10)
\]

for some \( c_0 \), that is, the solution \( P_t \phi(x) \) of the corresponding linear equation for this initial data is Lipschitz continuous in \( t \) uniformly for \( x \). Assume further that \( u(t, x) \) is bounded; that is, there exists a positive constant \( M_{15} \) such that

\[
0 \leq u(t, x) \leq M_{15} \quad \text{for all } t > 0 \text{ and } x \in G. \quad (4.11)
\]

Then for all \( T > 0 \), the solution \( u(x, t) \) is uniformly Lipschitz on \((0, T) \times G\), that is, \( |u(t + \delta, x) - u(t, x)| \leq b \delta \) for some \( b \), so in particular \( \frac{\partial u}{\partial t}(t, x) \) exists and is bounded for almost every \( t > 0 \), for all \( x \in G \).

**Proof.** Rewriting (1.3) as

\[
u(t, x) = \int_G k(t, x, y) \phi(y) d\mu(y) + \int_0^t \int_G k(\tau, x, y) u(\tau - t, y)^p d\mu(y),
\]

it follows from (4.10), (4.11) that

\[
|u(t + \delta, x) - u(t, x)|
\]

\[
\leq |P_{t+\delta} \phi(x) - P_t \phi(x)| + \int_t^{t+\delta} \int_G k(\tau, x, y) u(\delta + \tau - t, y)^p d\mu(y)\]

\[
+ \int_0^\tau \int_G k(\tau, x, y) u(\delta + \tau - t, y)^p - u(t - \tau, y)^p |d\mu(y)\]

\[
\leq c_0 \delta + M_{15}^p \delta + p M_{15}^{p-1} \int_0^\tau \int_G k(\tau, x, y) u(\delta + \tau - t, y)\]

\[
- u(t - \tau, y) |d\mu(y)
\]

for \( t > 0, x \in G \), where we have used the inequality

\[
|a^p - b^p| \leq p \max(a^{p-1}, b^{p-1}) |a - b|, \quad a \geq 0, b \geq 0.
\]

Therefore, letting

\[
f(t) = \sup_{x \in G} |u(t + \delta, x) - u(t, x)|, \quad t > 0,
\]

we have that

\[
f(t) \leq (c_0 + M_{15}^p) \delta + p M_{15}^{p-1} \int_0^t f(t - \tau) d\tau.
\]
Using Gronwall’s inequality

\[ f(t) \leq (c_0 + M_1^p) \delta \exp(pM_1^{p-1}t), \quad t > 0, \]

so for \( T > 0 \),

\[ |u(t + \delta, x) - u(t, x)| \leq b\delta, \quad \text{for all } t \in (0, T) \text{ and } x \in G, \quad (4.12) \]

where \( b = (c_0 + M_1^p) \exp(pM_1^{p-1}T) \). Thus \( u(t, x) \) is Lipschitz continuous in \( t \) for all \( x \in G \), and so \( \frac{\partial u}{\partial t}(t, x) \) exists for almost every \( t > 0 \) and for all \( x \in G \). Clearly, by (4.12), \( \frac{\partial u}{\partial t}(t, x) \) is bounded for \( (t, x) \in (0, T) \times G \) for \( T > 0 \). \( \blacksquare \)

Theorem 4.4 states that if, for given the initial data \( \phi \), the weak solution to the linear equation \( \partial v/\partial t = \Delta v \) satisfies a Lipschitz condition, then so does the solution of the non-linear problem (1.3) for the same initial data. However, condition (4.10) may be difficult to verify directly. For one case, if \( \phi \) is of the form

\[ \phi(x) = \int_G k(\gamma, x, y)\psi(y)d\mu(y), \quad (4.13) \]

where \( \psi(x) \geq 0, \|\psi\|_1 < \infty \), and \( \gamma > 0 \), then

\[ P_t\phi(x) = \int_G k(t + \gamma, x, y)\psi(y)d\mu(y), \]

so by (1.4),

\[ |P_{\delta+t}\phi(x) - P_t\phi(x)| = \left| \int_G \psi(y)d\mu(y) \int_{t+\gamma}^{t+\delta+\gamma} \frac{\partial k}{\partial \tau}(\tau, x, y)d\tau \right| \leq \text{const} \gamma^{-(1+d/2)}\|\psi\|_1\delta, \]

so \( \phi \) satisfies (4.10).

Recall the definition (1.7) of the infinitesimal generator \( \Delta \) of the semi-group \( \{P_t, t > 0\} \) associated with the heat kernel \( \bar{k} \), which we take to be the Laplacian in (1.1). With this definition we can show that the weak solutions of (1.3) are strong solutions to (1.1)–(1.2).

**Theorem 4.5 (Regularity).** Suppose that \( k \) satisfies satisfies \((K_1)–(K_5)\) and \((K_6)\). Let \( u(t, x) \) be a non-negative bounded continuous solution of (1.3) for \( t \in (0, T) \) and suppose that \( \frac{\partial u}{\partial \tau}(t, x) \) exists for all \( x \in G \) and \( t \in (0, T) \). Moreover, suppose that \( u(t, x), \frac{\partial u}{\partial \tau}(t, x) \) are both uniformly \( L^2(G) \)-integrable for \( t \in (0, T) \). Then

\[ \frac{\partial u}{\partial \tau}(t, x) = \Delta u(t, x) + u(t, x)^p \]

for \( t \in (0, T) \) and \( x \in G \).
Proof. Since $u(t, x)$ satisfies (1.3) we have that, for $t_0 > 0$,

$$P_t u(t_0, x) = \int_G k(t, x, y)u(t_0, y)d\mu(y)$$

$$= \int_G k(t, x, y)\left( \int_G k(t_0, y, z)\phi(z)d\mu(z) \right) d\mu(y)$$

$$+ \int_0^{t_0} d\tau \int_G k(t_0 - \tau, y, z)u(\tau, z)^p d\mu(z) \right) d\mu(y)$$

$$= \int_G k(t + t_0, x, z)\phi(z)d\mu(z)$$

$$+ \int_0^{t_0} d\tau \int_G k(t + t_0 - \tau, x, z)u(\tau, z)^p d\mu(z).$$

Using (1.3) again, it follows that

$$P_t u(t_0, x) - u(t_0, x) = \int_G (k(t + t_0, x, z) - k(t_0, x, z))\phi(z)d\mu(z)$$

$$+ \int_0^{t_0} d\tau \int_G (k(t + t_0 - \tau, x, z) - k(t_0 - \tau, x, z))$$

$$u(\tau, z)^p d\mu(z).$$

(4.14)

On the other hand, by (1.3),

$$u(t + t_0, x) - u(t_0, x)$$

$$= \int_G (k(t + t_0, x, z) - k(t_0, x, z))\phi(z)d\mu(z)$$

$$+ \int_0^{t_0} d\tau \int_G (k(t + t_0 - \tau, x, z) - k(t_0 - \tau, x, z))u(\tau, z)^p d\mu(z)$$

$$+ \int_{t_0}^{t + t_0} d\tau \int_G k(t + t_0 - \tau, x, z)u(\tau, z)^p d\mu(z).$$

(4.15)

Combining (4.14) and (4.15),

$$P_t u(t_0, x) - u(t_0, x) = u(t + t_0, x) - u(t_0, x)$$

$$- \int_{t_0}^{t + t_0} d\tau \int_G k(t + t_0 - \tau, x, z)u(\tau, z)^p d\mu(z)$$

(4.16)

for all $x \in G$ and $t > 0$. Since $u(t, x)$ is continuous and bounded, it follows by $(K_5)$ that

$$\lim_{t \to 0} \frac{1}{I} \int_{t_0}^{t + t_0} d\tau \int_G k(t + t_0 - \tau, x, z)u(\tau, z)^p d\mu(z) = u(t_0, x)^p$$

for each $x \in G$. Therefore, from (4.16) we deduce that

$$\lim_{t \to 0} \frac{1}{I} (P_t u(t_0, x) - u(t_0, x)) = \frac{\partial u}{\partial t}(t_0, x) - u(t_0, x)^p$$
for each \( x \in G \). The limit here is pointwise, but using the uniform integrability of \( u(t, x) \) and \( \partial u / \partial t(t, x) \) and the dominated convergence theorem, the limit also exists in the \( L^2 \)-norm, so the result follows by the definition (1.7) of the infinitesimal generator \( \Delta \). 

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