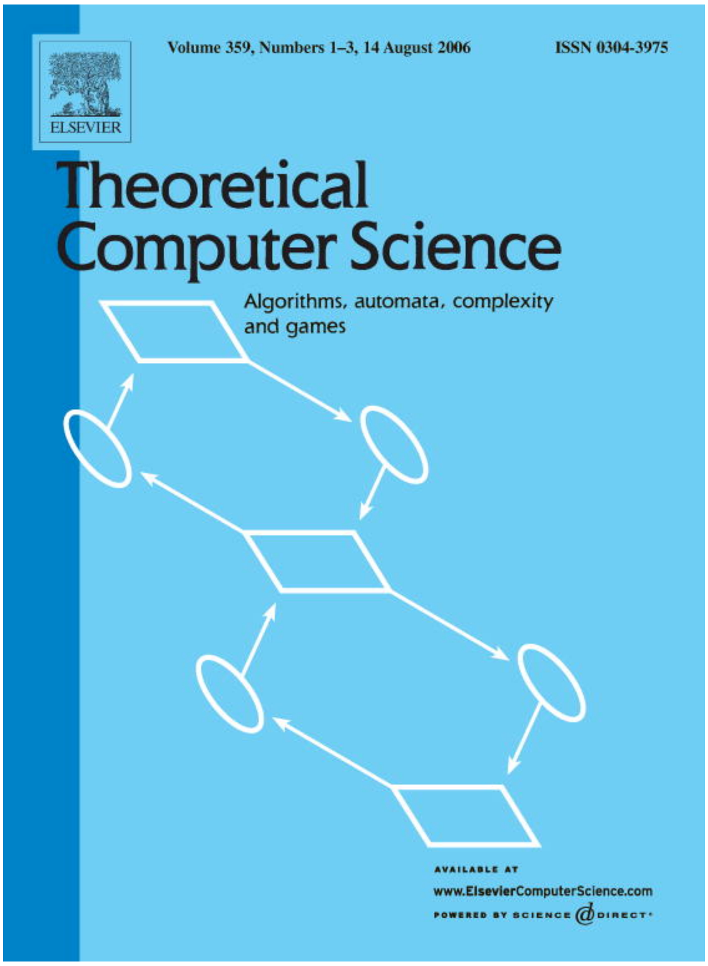
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Fundamental study

Maximal pattern complexity of two-dimensional words

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Abstract

The maximal pattern complexity of one-dimensional words has been studied in several papers [T. Kamae, L. Zamboni, Sequence entropy and the maximal pattern complexity of infinite words, Ergodic Theory Dynam. Systems 22(4) (2002) 1191–1199; T. Kamae, L. Zamboni, Maximal pattern complexity for discrete systems, Ergodic Theory Dynam. Systems 22(4) (2002) 1201–1214; T. Kamae, H. Rao, Pattern Complexity over ℓ letters, E. Comb. J., to appear; T. Kamae, Y.M. Xue, Two dimensional word with $2k$ maximal pattern complexity, Osaka J. Math. 41(2) (2004) 257–265]. We study the maximal pattern complexity $p_\alpha^*(k)$ of two-dimensional words α. A two-dimensional version of the notion of *strong recurrence* is introduced. It is shown that if α is strongly recurrent, then either α is doubly periodic or $p_\alpha^*(k) \geqslant 2k$ $(k = 1, 2, \ldots)$. Accordingly, we define a *two-dimensional pattern Sturmian word* as a strongly recurrent word α with $p_\alpha^*(k) = 2k$. Examples of pattern Sturmian words are given.

Keywords: Two-dimensional words; Maximal pattern complexity; Strong recurrence

1. Introduction

The study of complexity of words has a very long history. Especially the words with low complexity have raised special interest. Recently, the study of complexity of words has been extended in two different directions. One direction is to study the complexity of higher-dimensional words (cf. [1–5,12,7,15,16]). Another is to consider a new complexity, the so-called *maximal pattern complexity* [10,11,8]. However, in this paper we will combine the above two efforts. We study the maximal pattern complexity of two-dimensional words.

1.1. Maximal pattern complexity

Let A be a finite alphabet. An element $\alpha = \alpha_0\alpha_1\alpha_2 \cdots \in A^{\mathbb{N}}$, where $\mathbb{N} := \{0, 1, 2, \ldots\}$, is called a *one-sided word* over A if every letter of A appears in α.

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Let k be a positive integer. By a *k-window* τ, we mean a sequence of integers of length k with

$$0 = \tau(0) < \tau(1) < \tau(2) < \cdots < \tau(k-1).$$

The k-window $\tau = \{0, 1, \ldots, k-1\}$ is called the *k-block window*. For a k-window $\tau : 0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ and a word α, the word

$$\alpha[n+\tau] := \alpha_{n+\tau(0)}\alpha_{n+\tau(1)}\cdots\alpha_{n+\tau(k-1)}$$

is the pattern of α through the window τ at position n. We denote by $F_\alpha(\tau)$ the set of all patterns of α through the window τ, i.e.,

$$F_\alpha(\tau) := \{\alpha[n+\tau];\, n = 0, 1, 2, \ldots\}.$$

In particular, we denote $F_\alpha(k) := F_\alpha(\tau)$ for the k-block window τ.

The *maximal pattern complexity function* p^*_α for a word α is introduced by the first author together with Zamboni [10] as

$$p^*_\alpha(k) := \sup_\tau \sharp F_\alpha(\tau) \quad (k = 1, 2, 3, \ldots),$$

where the supremum is taken over all k-windows τ, while the classical complexity function p_α is defined as $p_\alpha(k) = \sharp F_\alpha(k)$.

A classical result by Morse and Hedlund says that

Theorem A (*Morse and Hedlund [13]*). *For a word* α, *the following statements are equivalent*:
(i) α *is eventually periodic*,
(ii) $p_\alpha(k)$ *is bounded in* k,
(iii) $p_\alpha(k) < k+1$ *for some* $k = 1, 2, \ldots$.

The following parallel result with respect to the maximal pattern complexity function is proved in [10].

Theorem B (*Kamae and Zamboni [10]*). *For a word* α, *the following statements are equivalent*:
(i) α *is eventually periodic*,
(ii) $p^*_\alpha(k)$ *is bounded in* k,
(iii) $p^*_\alpha(k) < 2k$ *for some* $k = 1, 2, \ldots$.

A word α with block complexity $p_\alpha(k) = k+1$ $(k = 1, 2, 3, \ldots)$ is known as a *Sturmian word* and is studied extensively (see for example Berthé [2] and the references therein). Naturally, a word α with maximal pattern complexity $p^*_\alpha(k) = 2k$ $(k = 1, 2, 3, \ldots)$ is called a *pattern Sturmian word*. It is interesting that the classical Sturmian words are also pattern Sturmian words, and the class of pattern Sturmian words is larger than the class of Sturmian words [10].

The pattern complexity of a word α with more than two letters has been investigate in [8]. Let 1_S stand for the indicator function of the set S. A word α over A is called *periodic by projection* if there exists S with $\emptyset \neq S \subsetneq A$ such that the word

$$1_S \circ \alpha := 1_S(\alpha_0)1_S(\alpha_1)1_S(\alpha_2)\cdots \in \{0, 1\}^{\mathbb{N}}$$

is eventually periodic. Note that if $\ell = 2$, then α is periodic by projection if and only if α is eventually periodic. It is shown that

Theorem C (*Kamae and Rao [8]*). *Let* α *be a word over* ℓ *letters with* $\ell \geqslant 2$. *If* $p^*_\alpha(k) < \ell k$ *holds for some* $k = 1, 2, \ldots$, *then* α *is periodic by projection*.

Accordingly, a word over ℓ letters is said to be a *pattern Sturmian word* if $p^*_\alpha(k) = \ell k$ and it is not periodic by projection.

Example 1.1. Let θ be an irrational number and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\ell \geqslant 2$ and $\mathcal{P} = \{I_0, I_1, \ldots, I_{\ell-1}\}$ be a partition of $\mathbb{T}$ into ℓ intervals with non-empty interiors. Define $\alpha \in \{0, 1, \cdots, \ell - 1\}^{\mathbb{N}}$ by

$$\alpha(n) = i \quad \text{if } n\theta \in I_i.$$

Then α is a pattern Sturmian word [8].

Other classes of pattern Sturmian words over ℓ letters are given by [8]. In the case $\ell = 2$, *Toeplitz words* are also considered in [10]. It would be interesting to know some new examples of pattern Sturmian words.

1.2. Complexity for two-dimensional words

For a two-dimensional word α over A, we consider a mapping from $\mathbb{Z}^2$ to A instead of from $\mathbb{N}^2$ to A since the former is more natural and simpler than the latter.

Recently, the question of generalizing the Sturmian words to higher dimensions has raised. Many new difficulties appear already in dimension 2, beginning with the choice of the right definition for the complexity function. A natural extension of the classical complexity is the so called *rectangle complexity* $p(m, n)$ which counts the number of distinct rectangular blocks of size $m \times n$ that occur in a two-dimensional word. With these conventions, Nivat [14] proposed the following conjecture:

Conjecture. *Let $\alpha \in A^{\mathbb{Z}^2}$ be a two-dimensional word. If $p_\alpha(m, n) \leqslant mn$ holds for some m, n, then α is periodic, i.e., there is a vector ξ such that $\alpha(x) = \alpha(x + \xi)$ holds for any $x \in \mathbb{Z}^2$.*

This conjecture is still open; some results related to this conjecture can be found in Sander and Tijdeman [15], and Epifanio et al. [7]. Cassaigne [6] characterized all the two-dimensional words with rectangle complexity $mn + 1$; the structure of these words is rather simple and it seems that they are not good candidates for two-dimensional Sturmian words.

A class of two-dimensional words (over a three-letter alphabet) obtained by coding discrete planes has been studied by Arnoux et al. and Vuillon [16,5,1]. It is shown that these words can also obtained by two-rotations on the unit circle [5], the rectangle complexity is $mn + m + n$ [16,5]. Recurrence properties and balanced property of these words are studied. They are proposed to be a possible candidates for two-dimensional Sturmian words.

Balance property of higher-dimensional words has been studied by Berthé and Tijdeman [3].

1.3. Pattern complexity of two-sided words

One advantage of the notion of the maximal pattern complexity is that it can be generalized easily to higher dimensions. For comparison, let us first have a look at one-dimensional two-sided words, where a *two-sided word*

$$\alpha = \cdots \alpha(-2)\alpha(-1)\alpha(0)\alpha(1)\alpha(2) \cdots \in A^{\mathbb{Z}}$$

over A is a mapping from $\mathbb{Z}$ to A.

A one-sided one-dimensional words is recurrent if every factor occurs infinitely often. We extend this definition to the two-sided one-dimensional words α, that is, α is recurrent if every factor occurs infinitely often. It is easy to show that a two-sided word α is recurrent if and only if every factor occurs at least two times. Hence α is recurrent if for any positive integer N, there exists an integer $L \neq 0$ such that $\alpha(i) = \alpha(L + i)$ for any $i \in \mathbb{Z}$ with $|i| \leqslant N$. It may happen that a factor occurs infinitely often in α, but only in one direction.

It is not hard to generalize Theorem A to the case of two-sided words.

Proposition 1.2. *For a two-sided one-dimensional word α, the following conditions are equivalent*:

(i) *α is periodic*.
(ii) *$p^*_\alpha(k)$ is bounded in k*.
(iii) *α is recurrent and $p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \ldots$.*

Remark 1.3. The condition of recurrence cannot be dropped, for it is easy to check that word $\alpha = (\ldots 0001000 \ldots)$, with 1 in the origin and 0 in the other places, is not periodic but $p^*_\alpha(k) = k + 1$ for any $k = 1, 2, 3, \ldots$.

1.4. Pattern complexity of two-dimensional words, strong recurrence

The main purpose of this paper is to generalize Proposition 1.2 to two-dimensional words. First, let us give some notations and definitions.

Definitions. Let $\alpha \in A^{\mathbb{Z}^2}$ be a two-dimensional word. By a *k-window*, we mean a subset τ of $\mathbb{Z}^2$ with $\sharp\tau = k$ and $O \in \tau$, where $O = (0, 0)$ is the origin. Let $\xi \in \mathbb{Z}^2$ and τ be a k-window. We denote

$$\alpha[\xi + \tau] := (\alpha(\xi + x))_{x\in\tau} \in A^\tau,$$

which is called a *τ-factor* of α. Sometimes we also call it a pattern of τ in α. Let $F_\alpha(\tau)$ be the set of τ-factors of α. We define the *maximal pattern complexity* by

$$p^*_\alpha(k) := \sup_{\tau;\sharp\tau=k} \sharp F_\alpha(\tau) \quad (k = 1, 2, 3, \ldots).$$

For any positive integer N, we denote $\Lambda_N := ([-N, N] \times [-N, N]) \cap \mathbb{Z}^2$, which is a $(2N + 1) \times (2N + 1)$ square.

For $u = (u_1, u_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, let $\|u\| := \sqrt{u_1^2 + u_2^2}$ be the Euclidean norm and $\mathrm{Arg}(u) := \mathrm{Arg}(u_1 + u_2\sqrt{-1})$ be the argument of the complex number $u_1 + u_2\sqrt{-1}$. We also call $\mathrm{Arg}(u)$ the *angle* of u.

A non-zero vector $\xi \in \mathbb{Z}^2$ is called a *period* of a word α if $\alpha(x) = \alpha(x + \xi)$ for any $x \in \mathbb{Z}^2$. A word α is called *two-periodic* or doubly periodic if there exist two linearly independent vectors ξ and η which are periods of α.

A word α is called *two-recurrent* if for any positive integer N, there exist two linearly independent vectors ξ and η in $\mathbb{Z}^2$ such that $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N] = \alpha[\eta + \Lambda_N]$.

Definition 1.4. A word α is called strongly recurrent if there exists $\delta > 0$ such that for any positive integer N, there exist two nonzero vectors ξ and η in $\mathbb{Z}^2$ such that
(i) $\delta < |\mathrm{Arg}(\xi) - \mathrm{Arg}(\eta)| < \pi - \delta$,
(ii) $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N] = \alpha[\eta + \Lambda_N]$.

Now we can state our main result.

Theorem 1.5. *For a two-dimensional word α, the following statements are equivalent to each other.*
(i) *α is two-periodic.*
(ii) *$p^*_\alpha(k)$ is bounded in k.*
(iii) *α is strongly recurrent and $p^*_\alpha(k) < 2k$ holds for some $k = 1, 2, 3, \ldots$.*

According to Theorem 1.5, we define

Definition 1.6. A two-dimensional words $\alpha \in \{0, 1\}^{\mathbb{Z}}$ is called a *pattern Strumian word* if it is strong recurrent and $p^*_\alpha(k) = 2k$ for $k = 1, 2, 3, \ldots$.

Examples of such words are given in Section 6. The following example shows that the strongly recurrent cannot be replaced by the two-recurrent.

Example 1.7. Let $a \neq 0$ be a real number. Let $\alpha \in \{0, 1\}^{\mathbb{Z}^2}$ be the word over the two-letter alphabet $\{0, 1\}$ such that $\alpha(m, n) = 1$ if $n \leqslant am$, and $\alpha(m, n) = 0$ otherwise.

First, the pattern complexity $p^*_\alpha(k) = k + 1$. For when we move a k-window τ in the plane, the pattern in τ is completely determined by the number of 0 in it, which can only take values $0, 1, \ldots, k$. Second, it is seen that α is not two-recurrent when a is rational, and we will see α is two-recurrent but not strongly recurrent when a is irrational by Lemma 1.8. Such two-dimensional words have been studied in [6] and it is shown there that their rectangle complexities are $mn + 1$.

Recurrence direction. Motivated by these examples, we define recurrence directions of a word α. A angle θ is said to be a *recurrence direction* of a two-dimensional word α, if for any $\delta > 0$ and any $N > 0$, there exists a vector $\xi = \xi(\delta, N)$ such that $\alpha[\Lambda_N] = \alpha[\Lambda_N + \xi]$ and $|(\mathrm{Arg}(\xi) - \theta)(\mathrm{mod}\, \pi)| < \delta$. Then

Lemma 1.8. *A two-dimensional word is strongly recurrent if and only if it has at least two recurrence directions.*

Proof. If a word has two recurrence directions, then clearly it is strongly recurrent.

Now suppose a word α is strongly recurrent. Let δ be the constant in Definition 1.4. Then for any $N > 0$, there exists two non-zero vectors ξ_N and η_N satisfying the condition of Definition 1.4. Let θ be a convergence point of $\{\mathrm{Arg}(\xi_N)\}_{N \geqslant 1}$, that is, θ is a recurrence direction of α. Then for N large, we have that

$$|\mathrm{Arg}(\eta_N) - \theta(\mathrm{mod}\, \pi)| > \delta/2.$$

Let θ' be a convergence point of $\{\mathrm{Arg}(\eta_N)\}_{N \geqslant 1}$, then θ' is another recurrence direction and $\theta' \neq \theta$. The lemma is proved. □

For the word in Example 1.7, the only recurrence direction is $\mathrm{Arg}(1, a)$ and hence it is not strongly recurrent.

1.5. Outline of the paper

The equivalence of (i) and (ii) in Theorem 1.5 is an easy matter and it is proved in Section 2 as Proposition 2.1. Item (i)⇒(iii) is trivial, hence to prove Theorem 1.5, it remains to show that (iii) implies (i). For this purpose, we can reduce to the case that $A = \{0, 1\}$ according to the following lemma.

Lemma 1.9. *Let α be a two-dimensional word over A.*
(i) *If α is strongly two-recurrent, then the word $1_{\{a\}} \circ \alpha$ is strongly two-recurrent for any $a \in A$.*
(ii) *If the words $1_{\{a\}} \circ \alpha$, $a \in A$, are two-periodic, then α is two-periodic.*

The proof is straightforward and we omit it. Therefore, we need only consider a word α over $\{0, 1\}$ and satisfying

Assumption I. There is a positive integer k such that $p^*_\alpha(k) < 2k$.

In Section 3 we show that our main theorem is valid for a class of very special words, the monotone words. To show this, Lemma 3.2, a lemma of Abel type plays a crucial role.

Then, in Section 4, we show that a two-recurrent word with low pattern complexity has a monotone structure.

Based on this monotone structure, we decompose α into several lattice subwords in Section 5. These lattice subwords are essentially monotone words and hence are two-periodic, which yields the two-periodicity of α.

Some examples of words with $p^*_\alpha(k) = 2k$ are given in Section 6.

2. Preliminaries

In this section we prove the equivalence of (i) and (ii) in Theorem 1.5. Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be the canonical basis of $\mathbb{Z}^2$.

Proposition 2.1. *A two-dimensional word α is two-periodic if and only if $p^*_\alpha(k)$ is bounded in k.*

Proof. Let ξ and η be linearly independent periods of α. Let

$$\Omega := \{s\xi + t\eta \in \mathbb{Z}^2;\ 0 \leqslant s < 1 \text{ and } 0 \leqslant t < 1\} = \{x_0, \ldots, x_{q-1}\}.$$

For any $x \in \mathbb{Z}^2$ and any window τ, the periodicity of α yields

$$\alpha[x + \tau] = \alpha[x_i + \tau],$$

where $x_i \in \Omega$ is the element such that $x = x_i + m\xi + n\eta$ for some integers m, n. Hence $p^*_\alpha(k) \leqslant \sharp\Omega$ holds for any $k = 1, 2, 3, \ldots$.

Now suppose that $\sup_k p^*_\alpha(k) = M < \infty$. Recall that $\Lambda_k = \{(m, n) \in \mathbb{Z}^2; -k \leqslant m, n \leqslant k\}$ is a square window. Then at least one pattern appears twice among $\alpha[\Lambda_k], \alpha[\mathbf{e}_1 + \Lambda_k], \ldots, \alpha[M\mathbf{e}_1 + \Lambda_k]$. Hence $\alpha[i_k\mathbf{e}_1 + \Lambda_k] = \alpha[j_k\mathbf{e}_1 + \Lambda_k]$ for some $0 \leqslant i_k < j_k \leqslant M$. It follows that $\alpha[\Lambda_{k-M}] = \alpha[(j_k - i_k) + \Lambda_{k-M}]$. Now there exists a subsequence of k such that $j_k - i_k = i$ is a constant. Since $\lim_{k\to\infty} \Lambda_k = \mathbb{Z}^2$, it is seen that $i\mathbf{e}_1$ is a period of α.

Applying the same argument to the $\mathbf{e}_2$ direction, we obtain another period of α of the form $j\mathbf{e}_2$. Hence α is two-periodic. $\square$

Proof of Proposition 1.2. Let $\alpha = \cdots \alpha(-2)\alpha(-1)\alpha(0)\alpha(1)\alpha(2) \cdots \in A^{\mathbb{Z}}$.

The proof of (i) $\Leftrightarrow$ (ii) is an analogue of Proposition 2.1 but simpler. (i) $\Rightarrow$ (iii) is trivial. In the following, we prove (iii) $\Rightarrow$ (i).

Without loss of generality, we may assume that α is recurrent in the positive direction, i.e., for any $N > 1$, there is a positive integer L such that

$$\alpha_{-N}\alpha_{-N+1} \ldots \alpha_0 \ldots \alpha_N = \alpha_{-N+L}\alpha_{-N+1+L} \ldots \alpha_L \ldots \alpha_{N+L}.$$

For otherwise we can consider the mirror reflection of α instead of α. Notice that the one-sided word $\alpha^+ := \alpha(0)\alpha(1)\alpha(2) \cdots$ satisfies

$$p^*_{\alpha^+}(k) \leqslant p^*_\alpha(k) < 2k$$

for some $k = 1, 2, 3, \ldots$. Hence α^+ is eventually periodic by Theorem B.

Let $p > 0$ be a period of α^+. We prove that p is a period of α. Suppose that this is not the case. Then, there exists i such that $\alpha(i) \neq \alpha(i + p)$. Hence

$$i_0 := \sup\{i \in \mathbb{Z}; \quad \alpha(i) \neq \alpha(i + p)\}$$

is a finite number. Let $N := |i_0| + p$, then $\tau = \{-N, \ldots, 0, \ldots, N\}$ is a window containing i_0 and $i_0 + p$. Since α is recurrent in positive direction, we have $\alpha[\tau] = \alpha[i + \tau]$ for some $i \geqslant 1$. Now $\alpha(i_0) \neq \alpha(i_0 + p)$ implies $\alpha(i_0 + i) \neq (i_0 + i + p)$, which contradicts the maximality of i_0. Hence p must be a period of α. $\square$

3. Monotone word

In this section, we show that our main theorem is valid for a class of very special words, the monotone words. A two-dimensional word α over $\{0, 1\}$ is said to be a *monotone word* if

$$\alpha(x + \mathbf{e}_1) \geqslant \alpha(x), \quad \alpha(x + \mathbf{e}_2) \geqslant \alpha(x) \tag{3.1}$$

holds for all $x \in \mathbb{Z}^2$. We will show that a monotone word satisfying Assumption I has a structure very similar to the words in Example 1.7.

For $x \in \mathbb{Z}^2$, we denote by $C^-_x := x - \mathbb{N}\mathbf{e}_1 - \mathbb{N}\mathbf{e}_2$ the negative cone with vertex x, and $C^+_x := x + \mathbb{N}\mathbf{e}_1 + \mathbb{N}\mathbf{e}_2$ the positive cone. The following lemma is trivial.

Lemma 3.1. *Let α be a monotone word. If $\alpha(x) = 1$, then α is constantly 1 in the cone C^+_x; if $\alpha(x) = 0$, then α is constantly 0 in the cone C^-_x.*

Let θ be a rational angle, i.e., $\tan\theta$ is a rational number. We denote by l_θ the ray which starts from origin O and forms an angle θ with the $\mathbf{e}_1$ direction. Let L_θ be the set of integer points on the ray, which we may write as $\{O = P_0, P_1, P_2, \ldots\}$ such that $\|P_n - P_0\| = n\|P_1 - P_0\|$. By, restricting α on the set L_θ, we obtain a one-dimensional one-sided word φ_θ by setting $\varphi_\theta(n) = \alpha(P_n)$, $n = 0, 1, 2, \ldots$.

If α satisfies $p^*_\alpha(k) < 2k$ for some k, then $p^*_{\varphi_\theta}(k) \leqslant p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \ldots$, and hence φ_θ is eventually periodic by Theorem B.

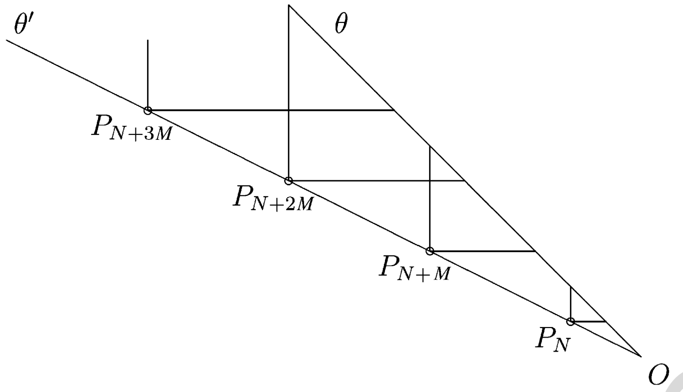

Fig. 1.

Type of φ_θ. We say φ_θ is of *type*-0 if it contains only finitely many 1; is of *type*-1 if it contains only finitely many 0. We say φ_θ is *mixed* if it is neither of type-0 nor of type-1.

From now on, we assume that α is a non-constant monotone word. Then by Lemma 3.1, it is clear that φ_θ is of type-1 if $0 < \theta < \pi/2$; φ_θ is of type-0 if $\pi < \theta < 3\pi/2$. When θ belongs to $[\pi/2, \pi] \cup [3\pi/2, 2\pi]$, the type of φ_θ is not clear. But we have the following lemma which is very similar to Abel Theorem on convergence of power series.

Lemma 3.2. *Let α be a non-constant monotone word satisfying $p^*_\alpha(k) < 2k$ for some k. If $\varphi_{\theta'}$ is of type-1 or mixed for some $\theta' \in [\pi/2, \pi]$, then φ_θ is of type-1 for all $\theta \in [\pi/2, \theta')$.*

Proof. Take $\theta \in [\pi/2, \theta')$. Write $L_\theta = (P_0, P_1, P_2, \ldots)$. By the assumption $p^*_\alpha(k) < 2k$ for some k, $\varphi_{\theta'}$ is eventually periodic. Since $\varphi_{\theta'}$ is not of type-0, it must contain infinitely many 1. So there exist positive integers N, M such that $\varphi_{\theta'}(P_{N+iM}) = 1$ for any $i = 0, 1, 2, \ldots$. Hence α is constant and takes the value 1 in the cones $C^+_{P_{N+iM}}$. Notice that when $\theta \in (0, \theta')$, the ray l_θ is contained in the union of those cones except a finite part of the ray near O (see Fig. 1). Thus, φ_θ is of type-1. □

Similarly, under the assumption of Lemma 3.2, we have that:
If $\varphi_{\theta'}$ is of type-1 or mixed for some $\theta' \in [3\pi/2, 2\pi]$, then φ_θ is of type-1 for all $\theta \in (\theta', 2\pi]$.
If $\varphi_{\theta'}$ is of type-0 or mixed for some $\theta' \in [\pi/2, \pi]$, then φ_θ is of type-0 for all $\theta \in (\theta', \pi]$.
If $\varphi_{\theta'}$ is of type-0 or mixed for some $\theta' \in [3\pi/2, 2\pi]$, then φ_θ is of type-0 for all $\theta \in [3\pi/2, \theta')$.

Accordingly, there exist two angles $\theta_1 \in [\pi/2, \pi]$ and $\theta_2 \in [3\pi/2, 2\pi]$ such that φ_θ is of type-1 when $\theta_2 - 2\pi < \theta < \theta_1$ and φ_θ is of type-0 when $\theta_1 < \theta < \theta_2$. We call θ_1, θ_2 the *critical angles* of the word α. As Example 1.7 shows, the angles θ_1, θ_2 may be irrational. Using θ_1 and θ_2, we have

Lemma 3.3. *Let α be a non-constant monotone word satisfying $p^*_\alpha(k) < 2k$ for some k. Let θ_1, θ_2 be the critical angles of α. Then for any $\varepsilon > 0$ there exists $C > 0$ such that for any $\|x\| \geqslant C$,*

$$\alpha(x) = \begin{cases} 0 & \text{if } \theta_1 + \varepsilon < \mathrm{Arg}(x) < \theta_2 - \varepsilon, \\ 1 & \text{if } \theta_2 + \varepsilon < \mathrm{Arg}(x) < \theta_1 + 2\pi - \varepsilon. \end{cases} \tag{3.2}$$

Proof. Choose two angles θ_3 and θ_4 with $\theta_1 - \varepsilon < \theta_3 < \theta_1$ and $\theta_2 < \theta_4 < \theta_2 + \varepsilon$. Then both of φ_{θ_3} and φ_{θ_4} are of type-1.

Let us denote the set L_{θ_3} by $\{P_0, P_1, P_2, \ldots\}$ and denote the set L_{θ_4} by $\{Q_0, Q_1, Q_2, \ldots\}$. There exists a $N > 0$ such that $\alpha(P_n) = \alpha(Q_n) = 1$ for $n \geqslant N$. It is easy to show that the area $\{x; \theta_2 + \varepsilon < \mathrm{Arg}(x) < \theta_1 + 2\pi - \varepsilon\}$ is covered by the cones $C^+_{P_n}$ and $C^+_{Q_n}$ $(n \geqslant N)$ except a finite part near O (see Fig. 2). Therefore, we have a half part of (3.2). The other half part can be proved similarly. □

Lemma 3.4. *Let α be a non-constant monotone word satisfying $p^*_\alpha(k) < 2k$ for some k. Let θ_1, θ_2 be the critical angles of α. Then $\theta_2 = \theta_1 + \pi$.*

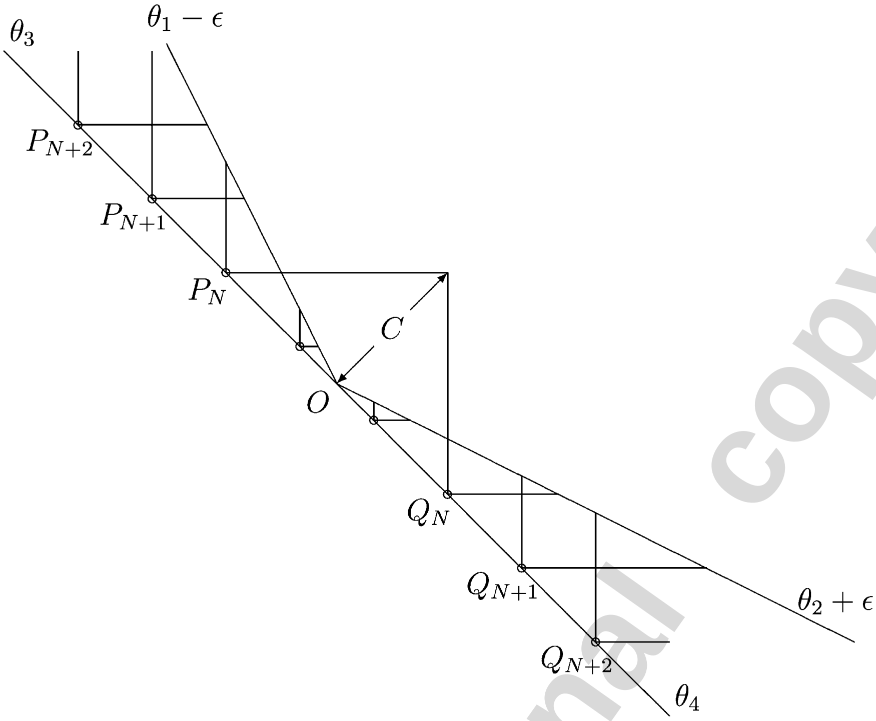

Fig. 2.

Proof. Suppose that $\theta_2 \neq \theta_1 + \pi$. Then, either $\theta_2 < \theta_1 + \pi$ or $\theta_2 > \theta_1 + \pi$. Without loss of generality, let us assume $\theta_2 > \theta_1 + \pi$. Take $\varepsilon > 0$ small. By Lemma 3.3, there exists $C > 0$ such that for any $\|x\| \geqslant C$, it holds

$$\alpha(x) = \begin{cases} 0 & \text{if } \theta_1 + \varepsilon < \mathrm{Arg}(x) < \theta_2 - \varepsilon \text{ and } \|x\| \geqslant C, \\ 1 & \text{if } \theta_2 + \varepsilon - 2\pi < \mathrm{Arg}(x) < \theta_1 - \varepsilon \text{ and } \|x\| \geqslant C. \end{cases}$$

Let us call these two regions Region 0 and Region 1, respectively.

Take $N > 0$ and take a line ℓ which passes through the point $N\mathbf{e}_1$ and has slope -1. Clearly ℓ intersects both Region 0 and Region 1. We choose N large enough so that ℓ does not intersect the circle with center O and radius C (see Fig. 3). Hence the configuration of α on the line ℓ is (from left to right)

$$\cdots 0\,0\,0 \; * \cdots * \, 1 \cdots 1 \, * \cdots * \, 0\,0\,0 \cdots,$$

where $*$ represents an undetermined letter 0 or 1 (Fig. 3). The number of consecutive 1 in the middle can be arbitrarily large if we choose N large. Furthermore, the ratio of the number of $*$ against the number of consecutive 1 in the middle can be arbitrarily small if $\varepsilon > 0$ is sufficiently small.

Take $k > 0$ such that k is larger than the size of the $*$-part, let τ be the k-window

$$\tau = \{ik(\mathbf{e}_1 - \mathbf{e}_2) \in \mathbb{Z}^2; \quad i = 0, 1, \ldots, k-1\}.$$

By moving this window along ℓ, we obtain at least the following patterns:

$$\{\underbrace{0\cdots 0}_{i}\underbrace{1\cdots 1}_{k-i};\ i = 1, 2, \ldots, k\} \cup \{\underbrace{1\cdots 1}_{i}\underbrace{0\cdots 0}_{k-i};\ i = 1, 2, \cdots, k\} \subset F_\alpha(\tau).$$

Hence, $\sharp F_\alpha(\tau) \geqslant 2k$, which contradicts the assumption $p_\alpha^*(k) < 2k$ for some k. The lemma is proved. □

Theorem 3.5. *If α is a non-constant monotone word satisfying $p_\alpha^*(k) < 2k$ for some k, then α has at most one recurrence direction.*

Proof. Let us denote by θ^* the common value of θ_1 and $\theta_2 - \pi$ in Lemma 3.4. We prove that θ^* is the only recurrence direction of α.

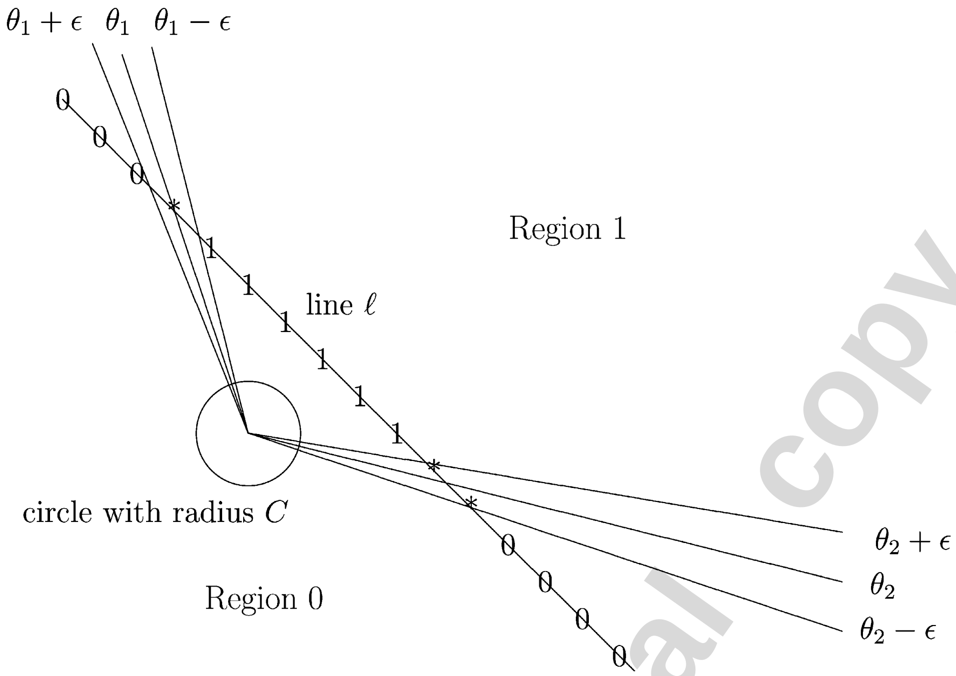

Fig. 3.

Suppose θ is a recurrence direction which is different from θ^*. Precisely, $\theta \neq \theta^*$ and $\theta \neq \theta^* + \pi$. Without loss of generality let us assume that $\theta^* < \theta < \theta^* + \pi$. Let $\delta = \min\{\theta - \theta^*, \theta^* + \pi - \theta\}/4$. We choose N_0 large enough so that $\alpha[\Lambda_{N_0}]$ contains at least one 1. For any $N > N_0$, there is a vector λ_N such that

$$\alpha[\Lambda_N] = \alpha[\Lambda_N + \lambda_N] \quad \text{and} \quad |(\mathrm{Arg}(\lambda_N) - \mathrm{Arg}(\theta))(\mathrm{mod}\,\pi)| < \delta.$$

Particularly, $\theta^* + 3\delta < \mathrm{Arg}(\lambda_N) < \theta^* + \pi - 3\delta$.

Case 1: Suppose the sequence $\{\lambda_N\}_{N=1}^{\infty}$ is bounded. Then there is a vector λ which appears infinitely many times in the sequence. Since $\Lambda_N \to \mathbb{Z}^2$ as $N \to \infty$, we have $\alpha[\mathbb{Z}^2] = \alpha[\lambda + \mathbb{Z}^2]$ and hence λ is a period of α.

Take any point x with $\alpha(x) = 1$. Let us denote by $l_{\mathrm{Arg}(\lambda),x}$ the ray starting from x and is parallel to λ. Since $\theta^* < \mathrm{Arg}(\lambda) < \theta^* + \pi$, sooner or later the ray $l_{\mathrm{Arg}(\lambda),x}$ will fall into the Region 0. Hence the configuration of α on $l_{\mathrm{Arg}\lambda,x}$ is eventually 0. On the other hand, the configuration of α on $l_{\mathrm{Arg}(\lambda),x}$ contains infinitely many 1 by periodicity. We obtain a desired contradiction in this case.

Case 2: Suppose that $\{\lambda_N\}_{N=1}^{\infty}$ is unbounded. It it easy to see that when $\|\lambda_N\|$ is very large, the square $\Lambda_{N_0} + \lambda_N$ is contained in the cone $\{x;\, \theta + 2\delta < \mathrm{Arg}(x) < \theta - 2\delta\}$. Let C be the constant in Lemma 3.3 when one takes $\varepsilon = 2\delta$. Then if $\|\lambda_N\|$ is larger than $C + 3N_0$, $\alpha[\Lambda_{N_0} + \lambda_N]$ is purely 0. But $\alpha[\Lambda_{N_0} + \lambda_N] = \alpha[\Lambda_{N_0}]$ implies $\alpha[\Lambda_{N_0} + \lambda_N]$ contains at least one 1. Again we get a contradiction. □

4. Critical window and monotonity lemma

The main purpose of this section is to prove the following theorem.

Theorem 4.1. *If a word α over $\{0, 1\}$ is two-recurrent and satisfies $p^*_\alpha(k) < 2k$ for some k, then there exist linearly independent vectors ξ and η in $\mathbb{Z}^2$ such that*

$$\alpha(x + \xi) \geqslant \alpha(x), \quad \alpha(x + \eta) \geqslant \alpha(x) \tag{4.1}$$

for all $x \in \mathbb{Z}^2$.

Critical window. Let τ be a k-window. We call τ' an *immediate extension* of τ if τ' is a $k+1$-window containing τ as a subset.

Lemma 4.2. *If α is a word satisfying Assumption I, then there exists a window τ such that*

$$\sharp F_\alpha(\tau') \leqslant \sharp F_\alpha(\tau) + 1 \tag{4.2}$$

holds for any immediate extension τ'.

We call such a window τ a *critical window*. This idea comes from [11].

Proof. If $p^*_\alpha(k) < 2k$ for $k = 1$, then the word α consists of just one letter and the lemma is true. So we assume that $p^*_\alpha(1) \geqslant 2$. Let k be the smallest integer such that $p^*_\alpha(k+1) < 2(k+1)$. Then, $p^*_\alpha(k) \geqslant 2k$. Let τ be a k-window which attains $p^*_\alpha(k)$. That is, $\sharp F_\alpha(\tau) = p^*_\alpha(k)$. Then, for any immediate extension $\tau'$of τ, we have

$$\sharp F_\alpha(\tau') \leqslant p^*_\alpha(k+1) \leqslant p^*_\alpha(k) + 1 = \sharp F_\alpha(\tau) + 1,$$

which proves (4.2). □

For a window τ, we say a square Λ_N is a *sample square* if

$$\{\alpha[x+\tau]; \quad x \in \Lambda_N\} = F_\alpha(\tau).$$

It is amount to say that all the patterns of τ appear in a neighborhood of Λ_N.

Lemma 4.3 (*Monotone Lemma*). *Let τ be a critical window for a word α, and Λ_N a sample square for τ. If a non-zero vector $\xi \in \mathbb{Z}^2$ satisfies $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N]$, then we have the following dichotomy*:

$$\textit{either } \alpha(x+\xi) \geqslant \alpha(x) \quad \textit{for all } x \in \mathbb{Z}^2 \tag{4.3}$$

$$\textit{or } \alpha(x-\xi) \geqslant \alpha(x) \quad \textit{for all } x \in \mathbb{Z}^2. \tag{4.4}$$

Proof. First, let us consider the case $\xi \in \tau$. Take any $\omega \in F_\alpha(\tau)$, then there exists $x \in \Lambda_N$ such that $\omega = \alpha[x+\tau]$. The assumption $\alpha[\xi + \Lambda_N] = \alpha[\Lambda_N]$ yields $\omega(O) = \alpha(x) = \alpha(x+\xi) = \omega(\xi)$. Since ω is an arbitrary pattern of τ, we conclude that $\alpha(x) = \alpha(x+\xi)$ holds for any $x \in \mathbb{Z}^2$. The lemma is proved in this case.

Next, we consider the case $\xi \notin \tau$. Let $\tau' = \tau \cup \{\xi\}$. For any $\omega \in F_\alpha(\tau)$, we define a candidate pattern ω' of τ' by setting $\omega'(z) = \omega(z)$ for any $z \in \tau$ and $\omega'(\xi) = \omega(O)$. Indeed ω' is a pattern of τ' in α. For let $x \in \Lambda_N$ be a point such that $\omega = \alpha[x+\tau]$, then $\alpha(x) = \alpha(x+\xi)$ yields $\omega' = \alpha[x+\tau'] \in F_\alpha(\tau')$.

Thus we have proved that $\{\omega'; \omega \in F_\alpha(\tau)\} \subset F_\alpha(\tau')$. Since $\sharp F_\alpha(\tau') \leqslant \sharp F_\alpha(\tau) + 1$ and

$$\sharp F_\alpha(\tau') \geqslant \sharp F_\alpha(\tau) + \sharp\{(\alpha(x), \alpha(x+\xi));\ x \in \mathbb{Z}^2 \text{ and } \alpha(x) \neq \alpha(x+\xi)\},$$

we have

$$\sharp\{(\alpha(x), \alpha(x+\xi));\ x \in \mathbb{Z}^2 \text{ and } \alpha(x) \neq \alpha(x+\xi)\} \leqslant 1.$$

This implies that either

$$\{(\alpha(x), \alpha(x+\xi));\ x \in \mathbb{Z}^2 \text{ and } \alpha(x) \neq \alpha(x+\xi)\} = \{(0,1)\}$$

or

$$\{(\alpha(x), \alpha(x+\xi));\ x \in \mathbb{Z} \text{ and } \alpha(x) \neq \alpha(x+\xi)\} = \{(1,0)\}.$$

The two formulas imply (4.3) and (4.4), respectively. □

Proof of Theorem 4.1. Since $p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \ldots$, there exists a critical window τ with a sample square Λ_N. Since α is two-recurrent, there exist linearly independent integer vectors ξ and η such that $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N] = \alpha[\eta + \Lambda_N]$. If (4.3) holds for ξ, we use ξ; otherwise we replace ξ by $-\xi$. Do the same for η. □

5. Lattice decomposition

Let α be a strongly recurrent word satisfying $p^*_\alpha(k) < 2k$ for some k. Then there exist ξ and η such that (4.1) hold.

Let $\Omega := \{s\xi + t\eta \in \mathbb{Z}^2;\ 0 \leqslant s < 1 \text{ and } 0 \leqslant t < 1\}$ be the set of integers in the polygon with vertices O, ξ, η and $\xi + \eta$. Let us write $\Omega = \{x_0, x_2, \ldots, x_{q-1}\}$. Take a $x_i \in \Omega$, we define a *lattice subword* of α by restricting α on the lattice $x_i + \xi\mathbb{Z} + \eta\mathbb{Z}$. We denote this subword by $\alpha^{(i)}$. Clearly α is the union the subwords $\alpha^{(i)}$, $i = 0, \ldots, q-1$.

According to $\alpha^{(i)}$, we define a word β_i as

$$\beta_i(m, n) = \alpha(x_i + m\xi + n\eta).$$

Then it is easy to see that β_i are monotone words satisfying Assumption I. We are going to show that β_i is strongly recurrent, and hence is a constant word by Theorem 3.5. The next lemma says that if θ is a recurrence direction of α, then it is also a recurrence direction of $\alpha^{(i)}$.

Lemma 5.1. *Suppose θ is a recurrence direction of α. Then for any $N > 0, \delta > 0$, there is a vector $\lambda \in \xi\mathbb{Z} + \eta\mathbb{Z}$ such that*

$$\alpha[\Lambda_N + \lambda] = \alpha[\Lambda_N] \quad \textit{and} \quad |\mathrm{Arg}(\lambda) - \theta(\mathrm{mod}\,\pi)| < \delta.$$

Proof. Since θ is a recurrence direction, there exists a vector λ_0 such that

$$\alpha[\Lambda_N + \lambda_0] = \alpha[\Lambda_N], \quad |\mathrm{Arg}(\lambda_0) - \theta(\mathrm{mod}\,\pi)| < \delta/q.$$

We choose N_1 large so that $\Lambda_N + \lambda_0$ is contained in the square Λ_{N_1}. Then we choose λ_1 such that

$$\alpha[\Lambda_{N_1} + \lambda_1] = \alpha[\Lambda_{N_1}], \quad |\mathrm{Arg}(\lambda_1) - \theta(\mathrm{mod}\,\pi)| < \delta/q.$$

Recurrently, if N_k and λ_k are chosen, we can choose N_{k+1} large so that $\Lambda_{N_k} + \lambda_k \subset \Lambda_{N_{k+1}}$, and then choose λ_{k+1} to be a vector satisfying

$$\alpha[\Lambda_{N_{k+1}} + \lambda_{k+1}] = \alpha[\Lambda_{N_{k+1}}], \quad |\mathrm{Arg}(\lambda_{k+1}) - \theta(\mathrm{mod}\,\pi)| < \delta/q.$$

Set $\lambda'_k = \lambda_0 + \lambda_1 + \cdots \lambda_k$, then among $\lambda'_0, \lambda'_1, \ldots, \lambda'_q$, at least two of them are in the same residue class modulo the lattice $\xi\mathbb{Z} + \eta\mathbb{Z}$. Say, λ'_i and λ'_j $(i < j)$ are in the same residue class, then $\lambda := \lambda'_j - \lambda'_i = \lambda_{i+1} + \cdots \lambda_j$ belongs to the lattice $\xi\mathbb{Z} + \eta\mathbb{Z}$. It is easy to check that

$$\alpha[\Lambda_N + \lambda] = \alpha[\Lambda_N] \text{ and } |\mathrm{Arg}(\lambda) - \theta(\mathrm{mod}\,\pi)| < (j - i)\delta/q \leqslant \delta. \qquad \square$$

Let A be the linear transformation determined by $A\mathbf{e}_1 = \xi$, $A\mathbf{e}_2 = \eta$.

Lemma 5.2. *If a $\mathrm{Arg}(u)$ is a recurrence direction of α, then $\mathrm{Arg}(A^{-1}u)$ is a recurrence direction of β_i for any $0 \leqslant i \leqslant q-1$.*

Proof. The lemma is obvious, but still we give a proof. Take any $i \in \{0, 1, \ldots, q-1\}$. We define a new word α_i, the translate of α by x_i, to be

$$\alpha_i(x) = \alpha(x + x_i).$$

If λ is a vector such that $\alpha[\Lambda_N + \lambda] = \alpha[\Lambda_N]$, then we have that $\alpha_i[\Lambda_{N-2\|x_i\|} + \lambda] = \alpha_i[\Lambda_{N-2\|x_i\|}]$. Hence $\mathrm{Arg}(u)$ is also a recurrence direction of α_i.

We decompose the word α_i into $\alpha_i^{(j)}$, $j = 0, \ldots, q-1$ as before. Now applying a linear transformation A^{-1} to $\alpha_i^{(0)}$, we obtain the word β_j. For any square Λ_N, there is a N' such that $A^{-1}(\Lambda_N)$ is contained in $\Lambda_{N'}$. Let λ' be a vector such that

$$\lambda' \in \xi\mathbb{Z} + \eta\mathbb{Z}, \quad \alpha_i[\Lambda_{N'} + \lambda'] = \alpha_i[\Lambda_{N'}], \quad |\mathrm{Arg}(\lambda') - \theta(\mathrm{mod}\,\pi)| < \delta.$$

Such a vector exists by Lemma 5.1. Then $\lambda = A^{-1}\lambda'$ satisfies $\beta_i[\Lambda_N + \lambda] = \beta_i[\Lambda_N]$. It is easy to see that $\mathrm{Arg}(\lambda) - \mathrm{Arg}(A^{-1}u)$ is small if $\mathrm{Arg}(\lambda') - \mathrm{Arg}(u)$ is small. Hence $\mathrm{Arg}(A^{-1}u)$ is a recurrence direction of β_i. $\square$

Proof of Theorem 1.5. The equivalence of (i) and (ii) in Theorem 1.5 is proved in Proposition 2.1. Item (i)$\Rightarrow$ (iii) is trivial, hence to prove Theorem 1.5, it remains to show that (iii) implies (i).

By the assumption $p^*_\alpha(k) < 2k$ for some k, there exist linearly independent vectors ξ and η in $\mathbb{Z}^2$ such that

$$\alpha(x+\xi) \geqslant \alpha(x), \quad \alpha(x+\eta) \geqslant \alpha(x)$$

for all $x \in \mathbb{Z}^2$. Then we decompose α into q lattice subwords $\alpha^{(i)}$, $q = 0, \ldots, q-1$. Consequently, we obtain monotone words β_i.

Obviously the words β_i still satisfies the assumption $p^*_\alpha(k) < 2k$ for some k. Besides, since the word α has at least two recurrence directions, each word β_i $(1 \leqslant i \leqslant q)$ has at least two recurrence directions (Lemma 5.2). Due to Theorem 3.5, the word β_i is constant. Consequently the lattice subwords $\alpha^{(i)}$ must be constant word, which implies that α is two-periodic. Actually, this proves that ξ and η are two linear independent periods of α. □

6. Examples of two-dimensional pattern Sturmian word

Example 6.1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the torus. Let $0 < \theta < 1$ be a real number. Let r, s be two real numbers. Let $\mathcal{P} = \{I_0, I_1\}$ be a partition of $\mathbb{T}$, where $I_0 = [0, \theta)$, $I_1 = [\theta, 1)$. Define a word α by

$$\alpha(m,n) = i \quad \text{if } mr + ns \in I_i.$$

Here, we show that α is a two-dimensional pattern Sturmian word provided at least one of r, s is irrational.

Let us use the notation $S - x := \{s - x;\ s \in S\}$. Let

$$\tau = \{(m_0, n_0) = (0,0), (m_1, n_1), \ldots, (m_{k-1}, n_{k-1})\}$$

be a k-window. Then a word $a_0 a_1 \cdots a_{k-1}$ is a pattern of τ in α if and only if

$$I_{a_0} \cap (I_{a_1} - m_1 r - n_1 s) \cap \cdots \cap (I_{a_{k-1}} - m_{k-1} r - n_{k-1} s) \neq \emptyset.$$

Therefore,

$$\sharp F_\alpha(\tau) \leqslant \sharp(\mathcal{P} \vee (\mathcal{P} - m_1 r - n_1 s) \vee \cdots \vee (\mathcal{P} - m_{k-1} r - n_{k-1} s)), \tag{6.1}$$

where "$\vee$" is the common refinement of partitions. Since the right side of (6.1) is no greater than the number of the end points of the intervals

$$I_i - m_j r - n_j s \quad (i = 0, 1;\ j = 0, 1, \ldots, k-1),$$

we have $p^*_\alpha(k) \leqslant 2k$ $(k = 1, 2, \ldots)$. Suppose r is irrational, then the factor $\alpha(m, 0), m \in \mathbb{Z}$ is an one-dimensional pattern Sturmian word, and hence $p^*_\alpha(k) = 2k$ $(k = 1, 2, \ldots)$.

We remain to show that α is strongly recurrent. In the following, we will show that α is *uniformly recurrent*. A two-dimensional word α is said to be uniformly recurrent, if for any $N > 0$, there exist a constant $C > 0$ such that the pattern $\alpha[\Lambda_N]$ occurs in every ball with radius C.

Recall the left end points of I_0 and I_1 are θ and 1, respectively. For $x \in [0, 1)$, define $\Delta(x) = \theta - x$ if $x \in I_0$, $\Delta(x) = 1 - x$ if $x \in I_1$. Set

$$\varepsilon = \min\{\Delta(mr + ns);\ (m,n) \in \Lambda_N\}.$$

Let $(p, q) \in \mathbb{Z}^2$ such that $pr + qs \pmod 1 < \varepsilon$. It is not difficult to show that there exists a constant C such that every ball with radius C contains at least one such (p, q).

Finally we proof that $\alpha[\Lambda_N] = \alpha[\Lambda_N + (p, q)]$. Take any $(m, n) \in \Lambda_N$. Suppose $mr + ns \pmod 1 \in I_j$, $j = 0$ or 1. Then

$$mr + ns \pmod 1 \leqslant (p+m)r + (q+n)s \pmod 1 \leqslant mr + ns + \varepsilon \pmod 1.$$

Hence $(p+m)r + (q+n)s$ is still in the interval I_j. This completes the proof.

Example 6.2. For an integer n, we define $v_2(n)$ to be the largest integer such that $2^p|n$. Define α as

$$\alpha(m,n) = \begin{cases} 1 & \text{if } v_2(m) = v_2(n), \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in [9] that $p^*_\alpha(k) = 2k$.

This word is also uniformly recurrent and thus strongly recurrent. Fix a square Λ_N. Let κ be an integer such that $2^\kappa > N$. Let $(p,q) \in \mathbb{Z}^2$ such that both p and q are multiples of 2^κ. It is easy to see that $\alpha[\Lambda_N] = \alpha[\Lambda_N + (p,q)]$.

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