Iterated Function Systems With Overlaps And Self-Similar Measures

by

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Abstract

In this paper we consider the iterated function systems (IFS) of similitudes which satisfy a separation condition weaker than the open set condition in that it allows overlaps in the iteration. Such systems include the well-known Bernoulli convolutions associated with the PV numbers, and the contractive similitudes associated with integral matrices. The latter appears frequently in wavelet analysis and the theory of tilings. We will study one of the basic questions: the absolute continuity and singularity of the self-similar measures generated by such systems. We give various conditions to determine the dichotomy.

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1 Introduction

We will call a family of contractive maps $\{S_j\}_{j=1}^N$ on $\mathbb{R}^d$ an *iterated function system* (IFS). An IFS will generate an invariant compact subset $K = \bigcup_{j=1}^N S_j K$, and if further, the system is associated with a set of probability weights $\{w_j\}_{j=1}^N$, then it will generate an invariant measure

$$\mu = \sum_{i=1}^{N} w_j \mu \circ S_j^{-1}. \tag{1.1}$$

In order to obtain sharp results on the invariant set K or the invariant measure μ, it is often assumed that the maps are similitudes (or the extension to self-conformal maps). The corresponding K and μ are called *self-similar set* and *self-similar measure* respectively. For the iteration, it is often assumed that the IFS satisfies the *open set condition* (OSC) [Hut]. One of the advantages of the OSC is that the "generic" points of the set K can be uniquely represented in a symbolic space and the dynamics of the IFS can be identified with the shift operation in the symbolic space. Without the OSC, the iteration has overlaps and such representation will break down and it is more difficult to handle the situation.

The simplest case of an IFS with overlaps is provided by the maps

$$S_1 x = \rho x, \; S_2 x = \rho x + 1, \quad x \in \mathbb{R} \qquad \text{with} \qquad \frac{1}{2} < \rho < 1. \tag{1.2}$$

The invariant measure μ associated with the weights $\frac{1}{2}$ has been studied for a long time in the context of Bernoulli convolutions [S]. Recently Solomyak ([So], [PS1]) proved that for almost all such ρ, the measure μ is absolutely continuous. This solves a conjecture of Erdös. The "transversality" argument used in [PS1] has also been used to consider a variety of IFS with overlaps (Peres and Solomyak [PS2], Peres and Schlag [PSc]). Other considerations on IFS with overlaps related to digit expansion can be found in Keane *et al.* [KSS], Pollicott and Simon [PoS], Kenyon [Ke], Rao and Wen [RW].

In [LN1] Lau and Ngai introduced a *weak separation condition* (WSC) on the IFS of similitudes that has overlaps. This condition is weaker than the open set condition and includes many of the important overlapping cases (see the examples in Section 2). In particular it includes the IFS in (1.2) where ρ^{-1} is a PV number. The multifractal structure of μ related to these numbers has been studies in detail in [LN1, 2, 3], [Hu], [HuL].

In this paper we continue the study of the WSC (see Definition 2.1). We will restrict our attention to similitudes $\{S_j\}_{j=1}^N$ on $\mathbb{R}^d$ with the same contraction ratio, i.e.,

$$S_j x = A_j x + b_j = \rho R_j x + b_j \tag{1.3}$$

where $0 < \rho < 1$ and R_j is an orthogonal transformation. It is known that the self-similar measure μ is either absolutely continuous or continuously singular, but it is rather difficult to determine the dichotomy. Our main purpose in this paper is to study this problem. We prove

Theorem 1.1 *Let $\{S_j\}_{j=1}^N$ be an IFS as in (1.3) and assume it satisfies the WSC. Suppose $w_j > \rho^d$ for at least one j. Then the self-similar measure is singular.*

For the proof of the theorem we observe that the WSC implies that any ball $B_{\rho^n}(x)$ contains a (uniformly) bounded number of $S_\sigma(x_0), |\sigma| = n$. This property yields the key Proposition 2.4 which allows us to retain some control on counting the overlaps. We can then use it to handle the product measure on the symbolic space and the self-similar μ as its projection.

By using the Lebesgue density theorem and Theorem 1.1 we have the following interesting result

Theorem 1.2 *Let $\{S_j\}_{j=1}^N$ be as above. If the self-similar measure μ is absolutely continuous, then the density function $f = D\mu \in L^1(\mathbb{R}^d)$ is actually in $L^\infty(\mathbb{R}^d)$.*

The second theorem allows us to determine the absolute continuity by checking the existence of the L^2-density, which is simpler for self-similar measures. By assuming a slightly stronger condition on the IFS, which we call WSC* (see §4), we can make use of the self-similar identity (1.1) and the correlation of μ to construct a nonnegative, irreducible transition matrix T_I (I stands for the identity map on $\mathbb{R}^d$) and prove

Theorem 1.3 *Suppose $\{S_j\}_{j=1}^N$ is an IFS as in (1.3) and satisfies the WSC*. Then μ is absolutely continuous if and only if*

$$\lambda_{\max} = \rho^d \tag{1.4}$$

where $\lambda_{\max}$ is the maximal eigenvalue of T_I.

We will see that $\lambda_{\max} \geq \rho^d$ always holds (Proposition 4.7). When $\lambda_{\max} > \rho^d$, we can use $\lambda_{\max}$ to determine the L^2-dimension of μ: $\dim_2(\mu) = |\log \lambda_{\max}/\log \rho|$ (Theorem 4.5). The construction of the T_I is quite direct and can be implemented for the concrete cases.

We organize the paper as follows. We give the definition of the WSC in Section 2 together with some examples and properties. Theorems 1.1 and 1.2 are proved in Section 3. In Sections 4 we will define the transition matrix T_I and prove Theorem 1.3. We also discuss some examples and the construction of the matrix T_I. To conclude the paper, we consider in Section 5 an extension of the classical Bernoulli convolution associated with the PV numbers and prove the singularity in such case. This extends a result of Erdös [S].

We remark that for the case $S_j(x) = \frac{1}{2}(x+j),\ j = 0, \cdots, N$ on $\mathbb{R}$, if the μ in Theorem 1.3 is absolutely continuous and if we let $f = D\mu$ be the Radon Nikodym derivative of μ, then the eigenvalue of T_I with the second largest magnitude can be used to determine the regularity of f. Namely if we let

$$\tilde{\lambda} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T_I,\ \lambda \neq \lambda_{\max}\},$$

then the L^2-Lipschitz exponent of f is given by $\frac{1}{2}\left(\left|\frac{\log\tilde{\lambda}}{\log 2}\right|-1\right)$. The result is a special case in [LMW]. The proof is more complicated than that of Theorem 1.3 because the corresponding eigenvalue and eigenvector may not be positive. We conjecture that the result is still true in the present setup of WSC.

2 The weak separation condition

Throughout the paper we assume that $S_j : \mathbb{R}^d \to \mathbb{R}^d,\ 1 \le j \le N$, are contractive similitudes defined by

$$S_j x = A_j(x + d_j) = A_j x + b_j,$$

where $b_j = A_j d_j \in \mathbb{R}^d$, $A_j = \rho R_j$ with $0 < \rho < 1$ and R_j orthonormal matrices. We will use Σ_n to denote the set of multi-indices $\sigma = (j_1, \cdots, j_n)$ and use $|\sigma| = n$ to denote the length of σ. Let $S_\sigma = S_{j_1} \circ \cdots \circ S_{j_n}$. Then

$$\begin{aligned} S_\sigma(x) &= A_{j_1\cdots j_n} x + A_{j_1\cdots j_{n-1}} b_{j_n} + A_{j_1\cdots j_{n-2}} b_{j_{n-1}} + \cdots + b_{j_1} \\ &= A_{j_1\cdots j_n} x + \sum_{i=1}^{n} A_{j_1\cdots j_{i-1}} b_{j_i}. \end{aligned}$$

In particular if $A_j = A$ for all j then $S_\sigma(x) = A^n x + \sum_{i=1}^n A^{i-1} b_{j_i}$.

Let $\{w_j\}_{j=1}^N$ be a set of probability weights associated with the IFS $\{S_j\}_{j=1}^N$ and let μ be the self-similar measure defined by $\mu = \sum_{j=1}^N w_j \mu \circ S_j^{-1}$ as in (1.1). For a fixed x_0, we define the discrete measure μ_n by

$$\mu_n\{x\} = \sum \{w_\sigma : \ S_\sigma(x_0) = x,\ |\sigma| = n\}, \quad x \in \mathbb{R}^d$$

where $w_\sigma = w_{j_1} \cdots w_{j_n}$. It is well known that $\{\mu_n\}$ converges to μ weakly. Moreover if we let $c = \max_j |b_j|/(1-\rho)$, then supp μ is contained in a ball of radius c. For $x = \sum_{i=1}^\infty A_{j_1\cdots j_{i-1}} b_{j_i}$, it is easy to show that

$$\mu(B_{\rho^n}(x)) \le \mu_n(B_{c\rho^n}(x)) \le \mu(B_{2c\rho^n}(x))$$

where $B_r(x)$ denotes the open ball of radius r centered at x [LN1]. If the IFS satisfies the open set condition, then the term $\mu_n(B_{r\rho^n}(x))$ is quite easy to handle. In the following we define a condition on the IFS which extends the open set condition and allows us to handle the term.

Definition 2.1 *We say that $\{S_j\}_{j=1}^N$ satisfies the weak separation condition (WSC) if there exists $x_0 \in \mathbb{R}^d$ and a constant $a > 0$ such that for $|\sigma|, |\sigma'| = n$, then either*

$$S_\sigma(x_0) = S_{\sigma'}(x_0) \quad \text{or} \quad |S_\sigma(x_0) - S_{\sigma'}(x_0)| \ge a\rho^n. \tag{2.1}$$

By a suitable translation we can assume $x_0 = 0$. Also note that the IFS is invariant on the subspace in $\mathbb{R}^d$ spanned by $S_\sigma, |\sigma| = n,\ n \in \mathbb{N}$. Hence by restricting the IFS to the subspace, we can assume without loss of generality that for some n large enough,

$\{S_\sigma(x_0) : |\sigma| = n\}$ spans $\mathbb{R}^d$. The definition says that after iterating x_0 by $\{S_j\}_{j=1}^N$ for n times, all the points $S_\sigma(x_0), |\sigma| = n$ are either identical or separated by a distance $a\rho^n$. This definition was introduced in [LN1] under the more general setting that the similitudes can have different contraction ratios. Since for all the practical examples considered here, the maps in the IFS have the same contraction ratio, we hence impose it in the definition for simplicity. The following proposition is immediate from the definition.

Proposition 2.2 *Suppose the IFS $\{S_j\}_{j=1}^N$ satisfies the WSC. Then any ball $B_{c\rho^n}(x)$ contains at most ℓ distinct $S_\sigma(x_0),\ \sigma \in \Sigma_n$.*

It is also easy to see that (2.1) is equivalent to:

$$\text{either} \;\; S_\sigma^{-1} S_{\sigma'}(x_0) = x_0 \;\; \text{or} \;\; |S_\sigma^{-1} S_{\sigma'}(x_0) - x_0| \geq a, \quad \forall\, |\sigma| = |\sigma'|. \tag{2.2}$$

In [BG] Bandt and Graf showed that $\{S_j\}_{j=1}^N$ satisfies the open set condition if and only if there exists $x_0 \in \mathbb{R}^d$ and $a > 0$ such that $|S_\sigma^{-1} S_{\sigma'}(x_0) - x_0| \geq a$ for all incomparable σ and σ'. It follows that

Proposition 2.3 *If $\{S_j\}_{j=1}^N$ satisfies the open set condition, then it satisfies the WSC.*

Our main interest is on the IFS's that do not satisfy the open set condition. In the following we exhibit a list of such examples with the WSC.

Example 2.1. Let $\{S_j\}_{j=1}^N$ be defined on $\mathbb{R}$ with $S_j x = \frac{1}{k}x + b_j$ where $k \geq 2$ is an integer, and $b_j = c r_j$ with $c \in \mathbb{R}$ and r_j rationals. We take $x_0 = 0$. Then for $\sigma = (j_1, \cdots, j_n)$,

$$S_\sigma(0) = c \sum_{i=1}^n \frac{r_{j_i}}{k^{i-1}} \;\; = \;\; \frac{c}{q} \sum_{i=1}^n \frac{t_{j_i}}{k^{i-1}}$$

where $t_j = q r_j,\ 1 \leq j \leq N$, are integers. It is easy to see that the WSC is satisfied by taking $a = c/q$ in the definition.

Note that the case with contraction ratio $\rho = 1/2$ has been studied in great detail in wavelet theory in connection with the *dilation equation*

$$f(x) = \sum_{j=1}^N c_j f(2x - (j-1)).$$

The function f can be considered as the density function of the corresponding absolutely continuous self-similar measure μ in (1.1), with $c_j = 2w_j$. In wavelet theory the c_j may be negative but $\sum c_j$ must be 2.

Kenyon [Ke] and Rao and Wen [RW] have studied the IFS for the contraction ratio $\rho = 1/3$, with $N = 3$. They are interested in the dimension of the invariant set K, and the tiling property related to the translation numbers r_j. Hu and Lau [HuL] and Fan *et al.* [FLN] had also given a detailed analysis of the multifractal structure of the self-similar measure generated by the IFS with such ρ.

Example 2.2. Let $\{S_j\}_{j=1}^N$ be defined on $\mathbb{R}$ with $S_jx = \rho x + b_j$ where $\beta = \rho^{-1}$ is a PV number, and $b_j = cr_j$ with $c \in \mathbb{R}$ and r_j rationals. (Recall that $\beta > 1$ is a PV number if it is an algebraic integer such that all its algebraic conjugates have modulus less than 1 [S], e.g., the golden ratio $(\sqrt{5}+1)/2$.)

Similar to the above we can write $S_\sigma(0) = \frac{c}{q}\sum_{i=1}^n t_{j_i}\rho^{i-1},\ t_{j_i} \in \mathbb{Z}$. The WSC follows from a lemma of Garsia [G, Lemma 1.51]. The self-similar measure corresponding to $S_1x = \rho x$ and $S_2x = \rho x + (1-\rho)$ with weights $\frac{1}{2}$ each is the classical Bernoulli convolution [S]. The entropy dimension, L^p-spectrum, local dimension spectrum and the multifractal formalism of the Bernoulli convolution associated with the PV numbers, in particular with the golden ratio, have been studied in great detail by many authors (e.g. [AY], [AZ], [Hu], [La], [L], [LN1,2], [LP]).

In the above two examples the contraction ratios ρ are algebraic integers. It is not difficult to construct an example with a more arbitrary ρ.

Example 2.3. Let $0 < \rho < \frac{1}{3}$, $S_1x = \rho x,\ S_2x = \rho x + \rho$, and $S_3x = \rho x + 1, x \in \mathbb{R}$. Then $S_1 \circ S_3(x) = S_2 \circ S_1(x) = \rho^2 x + \rho$. This implies that the OSC is not satisfied according to the result of Bandt and Graf stated after (2.2). However the IFS satisfies the WSC: it is easy to show that

$$S_\sigma(0) - S_{\sigma'}(0) = \sum_{i=0}^{n-1} \rho^i \epsilon_i, \qquad \epsilon_i = 0,\ \pm\rho,\ \pm 1,\ \pm(1-\rho)$$

can be written into the form $\sum_{i=0}^n \rho^i \eta_i$ where $\eta_i = 0,\ \pm 1,\ \pm 2$. If $\sum_{i=0}^n \rho^i\eta_i \neq 0$ and k is the smallest index such that $\eta_k \neq 0$, then since $0 < \rho < \frac{1}{3}$,

$$|S_\sigma(0) - S_{\sigma'}(0)| \geq |\rho^k - 2\sum_{i=k+1}^{\infty} \rho^i| \geq \big(1 - \frac{2\rho}{1-\rho}\big)\rho^k \geq a\rho^n > 0.$$

where $a = (1-3\rho)/(1-\rho)$.

Example 2.4. In $\mathbb{R}^d$, we let $S_jx = A(x+d_j)$ where $B = A^{-1}$ is an integral expanding similarity matrix, with $d_j \in \mathbb{Z}^d$ and $d_1 = 0$. Then under a suitable norm on $\mathbb{R}^d$, A is a contraction [LWa]. It is easy to show that $\{S_j\}_{j=1}^N$ satisfies the WSC. Indeed if we let $x_0 = 0$ and if $S_\sigma^{-1}S_{\sigma'}(0) \neq 0$, then being an element in the integer lattice, $|S_\sigma^{-1}S_{\sigma'}(0)| \geq 1$ and (2.2) applies.

This class of IFS has been studied in detail in connection with the theory of tiles under the more general setting using self-affine maps (see e.g. [LWa] and the references there). In the case of tiles, it is necessary to take $N = |\det B|$.

Example 2.5. Let A be as above, let Γ be a finite group of integral matrices γ with $\det\gamma = \pm 1$ and assume Γ satisfies $\Gamma A = A\Gamma$. Let $S_jx = A_j(x+d_j)$ where $A_j = \gamma_j A,\ \gamma_j \in \Gamma,\ d_j \in \mathbb{Z}^d$. Then the above argument also implies that $\{S_j\}_{j=1}^N$ satisfies the WSC. This class of IFS was used by Bandt [B] and Xu [X] to study tiles that involve rotations and reflections. For example let

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \Gamma = \Big\{ \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 \\ \epsilon_2 & 0 \end{bmatrix} :\ \epsilon_i = \pm 1 \Big\},$$

$S_1 x = Ax,\ S_2 x = \gamma A x + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ with $\gamma = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then the corresponding invariant set is the *Lévy dragon* [B].

To conclude this section we prove some useful consequences of the WSC that will be needed in the following sections. In order to avoid confusion in the counting, we will identify S_σ and $S_{\sigma'}$, $|\sigma| = |\sigma'| = n$, if they are equal, and denote the set of such distinct S_σ by $\mathcal{A}_n$. For $\sigma \in \Sigma_n$, we will use $[\sigma]$ to denote the equivalence class $\{\sigma' \in \Sigma_n : S_{\sigma'} = S_\sigma\}$.

Proposition 2.4 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. Then for any bounded subset $D \subset \mathbb{R}^d$, there exists $\gamma > 0$ such that for any $x \in \mathbb{R}^d,\ n \in \mathbb{N}$,*

$$\#\{S \in \mathcal{A}_n :\ x \in S(D)\} < \gamma.$$

Proof. Suppose the proposition is false, then there exists a bounded D such that for any $M > 0$,

$$\#\{S \in \mathcal{A}_n :\ x \in S(D)\} > M$$

for some n and x. By enlarging the set D we can assume the x_0 in the definition of WSC is in D and $\bigcup_{j=1}^N S_j(D) \subset D$. It follows that $\#\{S \in \mathcal{A}_n :\ S(D) \subseteq B_{\rho^n|D|}(x)\} > M$ and hence

$$\#\{S \in \mathcal{A}_n :\ S(x_0) \in B_{\rho^n|D|}(x)\} > M.$$

Since $B_{\rho^n|D|}(x)$ contains at most l distinct $S_\sigma(x_0)$, $|\sigma| = n$ (Proposition 2.2), there exists s such that

$$\#\{S \in \mathcal{A}_n :\ S(x_0) = s\} > \frac{M}{l}.$$

Let $E_s = \{S \in \mathcal{A}_n : S(x_0) = s\}$. For $S_\sigma, S_{\sigma'} \in E_s$, we have

$$\begin{aligned} S_\sigma(x) - S_{\sigma'}(x) &= (A_\sigma(x - x_0) + S_\sigma(x_0)) - (A_{\sigma'}(x - x_0) + S_{\sigma'}(x_0)) \\ &= (A_\sigma - A_{\sigma'})(x - x_0) = \rho^n (R_\sigma - R_{\sigma'})(x - x_0). \end{aligned}$$

Since M can be arbitrarily large and there are at least M distinct S_σ in $\mathcal{A}_n$, we can choose distinct $R_\sigma, R_{\sigma'}$ so that $\|R_\sigma - R_{\sigma'}\|$ is arbitrarily small (depending on M).

We can assume without loss of generality that there exists n_0 such that $\{S_\tau(x_0) : |\tau| = n_0\}$ contains 0 and spans $\mathbb{R}^d$ (see the remark after the definition of the WSC). We claim that for any $\sigma, \sigma' \in \Sigma_n$ with $S_{\sigma'} \neq S_\sigma$, we have $S_{(\sigma,\tau)}(x_0) \neq S_{(\sigma',\tau)}(x_0)$ for at least one of the τ. For otherwise, we let $T = S_{\sigma'}^{-1} S_\sigma$, then $S_\tau(x_0), |\tau| = n_0$ are fixed points of T. Recall $\{S_\tau(x_0) : |\tau| = n_0\}$ spans $\mathbb{R}^d$ and T is an isometry of $\mathbb{R}^d$. This forces T to be the identity map, which is a contradiction.

Now for any $a > 0$, we choose σ, σ' with $S_\sigma, s'_\sigma \in E_s$ such that

$$\|R_\sigma - R_{\sigma'}\| \leq a\rho^{n_0} \min\{\|S_\tau(x_0) - x_0\|^{-1} : |\tau| = n_0\}.$$

Then for τ satisfying $S_{(\sigma,\tau)}(x_0) \neq S_{(\sigma',\tau)}(x_0)$ we have

$$\begin{aligned} 0 &< \|S_{(\sigma,\tau)}(x_0) - S_{(\sigma',\tau)}(x_0)\| = \|S_\sigma S_\tau(x_0) - S_{\sigma'} S_\tau(x_0)\| \\ &= \rho^n \|(R_\sigma - R_{\sigma'})(S_\tau(x_0) - x_0)\| \leq a\rho^{(n+n_0)}. \end{aligned}$$

This contradicts the WSC. □

Note that if the IFS satisfies the OSC with an associated open set U and if we take $D = U$, then the γ in the above proposition is 1. In later applications we actually assume that

$$D \text{ is closed, } D^\circ \neq \emptyset, \ \ x_0 \in D^\circ \ \text{ and } \ \bigcup_{j=1}^N S_j(D) \subset D.$$

We will call such D a *basic region*. It follows that the invariant set K is contained in D.

Corollary 2.5 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. Let D be a basic region and let γ be as in Proposition 2.4, then $\#\mathcal{A}_n \leq \gamma\rho^{-dn}$.*

Proof. It follows from the assumption on D that $\bigcup_{S\in\mathcal{A}_n} S(D) \subset D$ and $m(D) > 0$ where m is the Lebesgue measure. By the above proposition, we see that each $x \in D$ is covered by at most γ of $S(D)$ with $S \in \mathcal{A}_n$. It follows that

$$\sum_{S\in\mathcal{A}_n} m(S(D)) \leq \gamma m(D).$$

The self-similarity implies that $m(S(D)) = \rho^{dn} m(D)$. Therefore $\#\mathcal{A}_n \leq \gamma\rho^{-dn}$ as claimed. □

Corollary 2.6 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. Let D be a basic region. Then for any $c > 0$, there exists $c_1 > 0$ such that for any n and x,*

$$\#\{S \in \mathcal{A}_n : \ S(D) \cap B_{c\rho^n}(x) \neq \emptyset\} < c_1.$$

Proof. Let $r_n = c\rho^n + |D|\rho^n$. Then $S(D) \cap B_{c\rho^n}(x) \neq \emptyset$ implies $S(D) \subset B_{r_n}(x)$. Also by Proposition 2.4, we know that every point is covered by at most γ of the $S(D), \ S \in \mathcal{A}_n$. It follows that

$$\begin{aligned}\gamma \cdot m(B_{r_n}(x)) \geq & \sum_{S\in\mathcal{A}_n} \{m(S(D)) : \ S(D) \cap B_{c\rho^n}(x) \neq \emptyset\} \\ & = \rho^{dn} m(D) \cdot \#\{S \in \mathcal{A}_n : \ S(D) \cap B_{c\rho^n}(x) \neq \emptyset\}.\end{aligned}$$

Hence

$$\#\{S \in \mathcal{A}_n : \ S(D) \cap B_{c\rho^n}(x) \neq \emptyset\} \leq \gamma(m(D)\rho^{dn})^{-1} m(B_{r_n}(x))$$

and we can choose a bound c_1 for the last expression. □

3 A sufficient condition for singularity

Let $\{S_j\}_{j=1}^N$ be an IFS with associated weights $\{w_j\}_{j=1}^N$ and let μ be the self-similar measure as defined in the last section. Our first theorem in this section gives a simple criterion for such measure to be singular. First we will introduce some notations in the symbolic space. Let

$$\Sigma = \{\overline{\sigma} = (j_1, j_2, \cdots) : \ j_i \in \{1, \cdots, N\}\},$$

and let Σ_n be the set of σ with length n. Let π be the projection of Σ to $\mathbb{R}^d$ defined by

$$\pi(\overline{\sigma}) = \bigcap_{n=1}^{\infty} S_{j_1\cdots j_n}(K), \quad \overline{\sigma} = (j_1, j_2, \cdots),$$

where K is the invariant set of the IFS. For C_σ a cylinder set with base $\sigma \in \Sigma_n$, it is clear that $\pi(C_\sigma) = S_\sigma(K)$. We let P be the probability measure on Σ and P_n the probability on Σ_n. For $\Lambda \subset \Sigma_n$, we will use the abbreviated notation $P(\Lambda)$ to denote $P_n(\Lambda)$ i.e., $P(\Lambda) = P_n(\Lambda) = P(C_\Lambda)$, C_Λ is the cylinder set with base Λ. Note that $\mu = P \circ \pi^{-1}$, and for $\sigma \in \Sigma_n$,

$$\mu(\pi(C_\sigma)) = P(C_{[\sigma]}) = \sum_{\sigma' \in [\sigma]} w_{\sigma'}.$$

Lemma 3.1 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. For any $\Lambda \subseteq \Sigma_n$, we let*

$$\tilde{\Lambda} = \{\sigma \in \Lambda : \sum_{\sigma' \in [\sigma] \cap \Lambda} w_{\sigma'} > \frac{\rho^{dn}}{4\gamma}\}.$$

Then $P(\Lambda) > \frac{1}{2}$ implies that $P(\tilde{\Lambda}) > \frac{1}{4}$.

Proof. By Corollary 2.5 we have $\#\mathcal{A}_n \leq \gamma\rho^{-dn}$. This implies that

$$P(\Lambda \setminus \tilde{\Lambda}) = \sum \{w_\sigma : \sigma \in \Lambda \setminus \tilde{\Lambda}\} = \sum_{[\sigma]} \sum_{\sigma' \in [\sigma]} \{w_{\sigma'} : \ \sigma' \in \Lambda \setminus \tilde{\Lambda}\} \leq \#\mathcal{A}_n \cdot \frac{\rho^{dn}}{4\gamma} \leq \frac{1}{4},$$

and $P(\tilde{\Lambda}) = P(\Lambda) - P(\Lambda \setminus \tilde{\Lambda}) > 1/2 - 1/4 = 1/4$. □

Theorem 3.2 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC and suppose $w_j > \rho^d$ for at least one $1 \leq j \leq N$. Then μ is singular.*

Proof. Our aim is to choose for any $\epsilon > 0$, a subset $E \subseteq \mathbb{R}^d$ such that $\mu(E) \geq \frac{1}{2}$ and $m(E) < \epsilon$. Without loss of generality we assume that $w_1 > \rho^d$. Let D be a basic region as in the last section and let $p \in \mathbb{N}$ be such that

$$4\gamma m(D) \left(\frac{\rho^d}{w_1}\right)^p < \epsilon.$$

Let

$$\begin{aligned}
\Lambda_1 = &\ \Sigma_1 = \{1, \cdots, N\}, \\
\tilde{\Lambda}_1 = &\ \{\sigma \in \Lambda_1 : \ w_\sigma > \frac{\rho^d}{4\gamma}\}, \\
\Lambda_1^* = &\ \{(\sigma, 1, \cdots, 1) \in \Sigma_{1+p} : \ \sigma \in \tilde{\Lambda}_1\}, \\
E_1 = &\ \bigcup \{S_\sigma(D) : \ \sigma \in \Lambda_1^*\}.
\end{aligned}$$

Then Lemma 3.1 implies that $P(\tilde{\Lambda}_1) \geq \frac{1}{4}$, so that $P(\Lambda_1^*) \geq \frac{1}{4}w_1^p$. Moreover for $\sigma \in \Lambda_1^*$ we have

$$w_\sigma > \frac{\rho^d}{4\gamma}w_1^p = \frac{\rho^{d(1+p)}}{4\gamma}(\frac{w_1}{\rho^d})^p > \frac{\rho^{d(1+p)}}{\epsilon}m(D).$$

It follows that

$$m(E_1) \leq \rho^{d(1+p)}m(D) \cdot \#\{S_\sigma : \ \sigma \in \Lambda_1^*\} \leq \epsilon \sum_{\sigma \in \Lambda_1^*} w_\sigma \leq \epsilon P(\Lambda_1^*). \tag{3.1}$$

Suppose we have chosen $\Lambda_i^* \subseteq \Sigma_{1+ip}$, $E_i = \bigcup\{S_\sigma(D) : \ \sigma \in \Lambda_i^*\}$, $i = 1, \cdots, k-1$, such that

(i) $\Lambda_i^* \subseteq \Sigma_{1+ip}$ and $C_{\Lambda_i^*} \cap C_{\Lambda_j^*} = \emptyset$ for $i \neq j$;

(ii) $P(\Lambda_i^*) \geq \frac{1}{4}w_1^p$;

(iii) $m(E_i) \leq \epsilon P(\Lambda_i^*)$.

If $\sum_{i=1}^{k-1} P(\Lambda_i^*) \geq 1/2$, we stop the construction. Otherwise we let $\sigma|_n$ denote the first n coordinates of σ and define

$$\begin{aligned}
\Lambda_k = & \ \{\sigma \in \Sigma_{1+(k-1)p} : \ \sigma|_{1+ip} \notin \Lambda_i^*, \ 1 \leq i \leq k-1\}, \\
\tilde{\Lambda}_k = & \ \{\sigma \in \Lambda_k : \sum_{\sigma' \in [\sigma] \cap \Lambda} w_{\sigma'} > \frac{\rho^{d(1+(k-1)p)}}{4\gamma}\}, \\
\Lambda_k^* = & \ \{(\sigma, 1, \cdots, 1) \in \Sigma_{1+kp} : \ \sigma \in \tilde{\Lambda}_k\}, \\
E_k = & \ \bigcup\{S_\sigma(D) : \ \sigma \in \Lambda_k^*\}.
\end{aligned}$$

Then it is clear that $P(\Lambda_k) > \frac{1}{2}$ and $\Lambda_k^* \in \Sigma_{1+kp}$. That $C_{\Lambda_i^*} \cap C_{\Lambda_k^*} = \emptyset$ for $0 \leq i < k$ follows from the choice of Λ_k. (ii) is a direct consequence of the construction of Λ and Λ_k^* and an application of Lemma 3.1. The proof of (iii) is similar to (3.1).

In view of (ii), the process must stop at some finite step, say at k. Let $E = \bigcup_{i=1}^k E_i$. We recall that D satisfies $\bigcup_{j=1}^N S_j(D) \subseteq D$, and the closedness of D implies that $K \subseteq D$. Hence

$$\pi(C_{\Lambda_i^*}) = \bigcup_{\sigma \in \Lambda_i^*} S_\sigma(K) \subseteq \bigcup_{\sigma \in \Lambda_i^*} S_\sigma(D) = E_i.$$

This implies that $\pi\big(\bigcup_{i=1}^k C_{\Lambda_i^*}\big) \subseteq E$ and it follows that

$$\mu(E) = P(\pi^{-1}E) \geq P\big(\bigcup_{i=1}^k C_{\Lambda_i^*}\big) = \sum_{i=1}^k P\big(C_{\Lambda_i^*}\big) \geq \frac{1}{2}.$$

On the other hand (iii) implies that

$$m(E) \leq \sum_{i=1}^k m(E_i) \leq \epsilon \sum_{i=1}^k P\left(C_{\Lambda_i^*}\right) < \epsilon.$$

The singularity of μ is proved. $\square$

We have an interesting consequence of the above theorem.

Theorem 3.3 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. If μ is absolutely continuous, then the density function $f = D\mu$ will be bounded, i.e., $f \in L^\infty(\mathbb{R}^d)$. Moreover f satisfies*

$$f(x) = \sum_{j=1}^{N} c_j f \circ S_j^{-1}(x), \;\; x \in \mathbb{R}^d.$$

where $c_j = |\det A_j| w_j$.

Proof. We need only show that $f \in L^\infty(\mathbb{R}^d)$. Suppose otherwise, then for any large $M > 0$, the Lebesgue density theorem implies that there exists a ball $B_{\rho^n}(x)$ such that

$$\frac{\mu(B_{\rho^n}(x))}{\rho^{dn}} = \frac{1}{\rho^{dn}} \int_{B_{\rho^n}(x)} f(t)dt > M. \tag{3.2}$$

If we iterate $\{S_j\}_{j=1}^N$ for n times, we obtain the family $\mathcal{A}_n$ and we will use it to form a new IFS. For each $S = S_\sigma \in \mathcal{A}_n$, the associated weight is

$$w_S = \sum \{w_{\sigma'} \; : \; \sigma' \in [\sigma]\}.$$

The corresponding self-similar identity is $\mu = \sum_{S\in\mathcal{A}_n} w_S \; \mu \circ S^{-1}$. By (3.2) we have

$$\sum_{S\in\mathcal{A}_n} w_S \; \mu \circ S^{-1}(B_{\rho^n}(x)) > M\rho^{dn}. \tag{3.3}$$

Now we fix a basic region D, then μ is supported by D. Hence $S(D) \bigcap B_{\rho^n}(x) = \emptyset$ implies that $D \bigcap S^{-1}(B_{\rho^n}(x)) = \emptyset$ and thus $\mu(S^{-1}(B_{\rho^n}(x))) = 0$. In view of Corollary 2.6, $\#\{S \in \mathcal{A}_n : S(D) \cap B_{\rho^n}(x) \neq \emptyset\} < c_1$, we see that the sum on the left of (3.3) has at most c_1 nonzero terms, and there exists at least one w_S such that $w_S > \frac{M}{c_1}\rho^{dn}$. We choose M such that $\frac{M}{c_1} > 1$. Then Theorem 3.2 implies that μ is singular. This is a contradiction. □

Corollary 3.4 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. Then the self-similar measure μ is absolutely continuous if and only if the L^2-density of μ exists.*

4 The transition matrix

Hardy and Littlewood [HL] proved that for a probability measure μ,

$$\overline{\lim_{h\to 0}} \frac{1}{h^{2d}} \int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx < \infty$$

if and only if μ is absolutely continuous and $D\mu \in L^2$. There is no similar criterion for the absolute continuity of μ (with $D\mu \in L^1$). However, by using Corollary 3.4, we have

Proposition 4.1 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC. Let β be such that*

$$0 < \overline{\lim_{h\to 0}} \frac{1}{h^{2\beta}} \int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx < \infty. \tag{4.1}$$

Then μ is absolutely continuous if and only if $\beta = d$.

In the following we will study the expression (4.1) and the exponent β through certain transition matrix to be defined. Note that the L^2-*dimension* (also called the *correlation dimension*) of μ is defined as

$$\begin{aligned}\underline{\dim}_2(\mu) &= \underline{\lim}_{h\to 0} \frac{\log \int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx}{\log h} - d \\ &= \sup\{\alpha : \overline{\lim}_{h\to 0} \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx < \infty\},\end{aligned}$$

and it follows that $\underline{\dim}_2(\mu) = 2\beta - d$ [St]. The limit expression in (4.1) corresponding to the β is called the *upper mean quadratic variation* of μ [LW]. Similarly we can define $\overline{\dim}_2(\mu)$ and $\dim_2(\mu)$.

Let $\mathcal{S}$ denote the set of maps $S = S_\sigma^{-1} S_{\sigma'}$ for $(\sigma, \sigma') \in \bigcup_{n=1}^\infty (\Sigma_n \times \Sigma_n)$. We will consider $\mathcal{S}$ as a state space and define an (infinite) transition matrix on $\mathcal{S}$ as follows. For $S \in \mathcal{S}$, let

$$T(S) = \sum_{S' \in \mathcal{S}} w_{(S,S')} S'$$

where

$$w_{(S,S')} = \sum_{i,j} \{w_i w_j : S_i^{-1} \circ S \circ S_j = S'\}.$$

This amounts to saying that the transition from S to S' has weight $w_{(S,S')}$. Also note that $\sum_{S'} w_{(S,S')} = (\sum_i w_i)^2 = 1$. It follows that T defines a Markov matrix on $\mathcal{S}$.

For a fixed β and for any $S \in \mathcal{S}$, we define

$$\Phi_S(h) = \frac{1}{h^{2\beta}} \int_{\mathbb{R}^d} \mu(B_h(Sx))\mu(B_h(x))dx.$$

We use $\mathbf{\Phi}(h)$ to denote the vector $\{\Phi_S(h)\}_{S\in\mathcal{S}}$ and let $\langle \mathcal{S} \rangle$ denote the linear space spanned by $\mathcal{S}$. Then $\Phi_S(h) = \langle \mathbf{\Phi}(h), S\rangle$ and for any $v \in \langle \mathcal{S} \rangle$,

$$\langle \mathbf{\Phi}(h), v\rangle = \sum_S v_S \Phi_S(h).$$

For convenience we also write the above expression as $\Phi_v(h)$. The following is the main purpose for defining the Markov matrix T and the consideration of $\Phi_S(h)$.

Proposition 4.2 *For $S \in \mathcal{S}$, $\Phi_S(h) = \rho^{d-2\beta}\Phi_{TS}(\frac{h}{\rho})$.*

Proof. By substituting $\mu = \sum_j w_j \mu \circ S_j^{-1}$ into $\Phi_S(h)$, we have

$$\begin{aligned}\Phi_S(h) &= \frac{1}{h^{2\beta}} \sum_{i,j} w_i w_j \int \mu(B_{\frac{h}{\rho}}(S_i^{-1}S(x)))\mu(B_{\frac{h}{\rho}}(S_j^{-1}(x)))dx \\ &= \frac{\rho^d}{h^{2\beta}} \sum_{i,j} w_i w_j \int \mu(B_{\frac{h}{\rho}}(S_i^{-1}SS_j(x)))\mu(B_{\frac{h}{\rho}}(x))dx \\ &= \frac{\rho^d}{h^{2\beta}} \sum_{S'} w_{(S,S')} \int \mu(B_{\frac{h}{\rho}}(S'(x)))\mu(B_{\frac{h}{\rho}}(x))dx \\ &= \rho^{d-2\beta}\Phi_{TS}(\frac{h}{\rho}).\end{aligned}$$

□

We recall that supp μ is contained in a ball of radius $\frac{\rho}{1-\rho}\max|d_j|$. We also recall that in the definition of WSC we can take $x_0 = 0$ without loss of generality.

Definition 4.3 *Let $S_j(x) = A_j(x + d_j), j = 1, \cdots, N$, with $A_j = \rho R_j$ as before. Let $\tilde{C} = \frac{2\rho}{1-\rho}\max_j |d_j|$ and let*

$$\tilde{\mathcal{S}} = \{S \in \mathcal{S} : |S(0)| \le \tilde{C}\}.$$

We say that $\{S_j\}_{j=1}^N$ satisfies the weak separation condition(WSC*) if $\tilde{\mathcal{S}}$ is a finite set.*

We remark that from the definition of WSC*, the set $\{S(0) : S \in \tilde{\mathcal{S}}\}$ is a finite set, so there exists $a > 0$ such that for $S, S' \in \tilde{\mathcal{S}}$ with $S(0) \ne S'(0)$, then $|S(0) - S'(0)| \ge a > 0$. This makes WSC* stronger than the WSC, which only requires that $|S(0)| \ge a$ for any $S \in \tilde{\mathcal{S}}$ with $S(0) \ne 0$ (see (2.2)).

We also remark that all the examples in Section 2 satisfy the WSC*. We will discuss this in more detail at the end of the section.

Proposition 4.4 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC*. Then*
(i) T maps $\mathcal{S} \setminus \tilde{\mathcal{S}}$ to itself;
(ii) There exists $h_0 > 0$ such that for $0 < h < h_0$ and $S \in \mathcal{S} \setminus \tilde{\mathcal{S}}$, $\Phi_S(h) = 0$.

Proof. (i) We let $S = S_\sigma^{-1} S_{\sigma'} \in \mathcal{S} \setminus \tilde{\mathcal{S}}$, then $|S(0)| > \tilde{C} + \delta$ for some $\delta > 0$. It follows that

$$\begin{aligned} |S_i^{-1} S S_j(0)| &\ge \rho^{-1}|S S_j(0)| - |d_i| \ge \rho^{-1}|S(0) + \rho R_\sigma^{-1} R_{\sigma'}(R_j d_j)| - \max_k |d_k| \\ &\ge \frac{2\rho}{(1-\rho)} \max_k |d_k| + \frac{\delta}{\rho} = \tilde{C} + \frac{\delta}{\rho}. \end{aligned}$$

(ii) In view of the above fact that $|S(0)| > \tilde{C} + \delta$ implies $|S_i^{-1} S S_j(0)| > \tilde{C} + \rho^{-1}\delta$ and the hypothesis that $\tilde{\mathcal{S}}$ is a finite set, we can choose h_0 small enough so that for $S \notin \tilde{\mathcal{S}}$, $|S(0)| > \tilde{C} + 2h_0$. We claim that for such S, $\mu(B_h(Sx))\mu(B_h(x)) = 0$ for all x. Otherwise $\mu(B_h(Sx)) \ne 0$ and $|x| \le \frac{\tilde{C}}{2} + h$. It follows that

$$|S(x)| = |S(0) + R_\sigma^{-1} R_{\sigma'}(x)| > (\tilde{C} + 2h_0) - (\frac{\tilde{C}}{2} + h_0) > \frac{\tilde{C}}{2} + h_0$$

and $\mu(B_h(Sx)) = 0$ for $h \le h_0$. This is a contraction and (ii) follows from the claim.
□

From the above proposition we can write T as

$$T = \begin{pmatrix} \tilde{T} & 0 \\ Q & T' \end{pmatrix}$$

where $\tilde{T}$ is a sub-Markov matrix on the states $\tilde{\mathcal{S}}$ (since the sum of each column of T is 1, the sum of each column of $\tilde{T}$ is ≤ 1). $\tilde{T}$ is a finite matrix by the WSC*.

Let λ be an eigenvalue of $\tilde{T}$ and v a corresponding eigenvector. By Propositions 4.2 and 4.4, we have

$$\begin{aligned}\Phi_v(h) =&\rho^{d-2\beta}\Phi_{Tv}(\frac{h}{\rho}) = \rho^{d-2\beta}\big(\Phi_{\tilde{T}v}(\frac{h}{\rho}) + \Phi_{Qv}(\frac{h}{\rho})\big)\\ =&\rho^{d-2\beta}\big(\Phi_{\tilde{T}v}(\frac{h}{\rho}) + 0\big) = \rho^{d-2\beta}\Phi_{\lambda v}(\frac{h}{\rho}) = \lambda\rho^{d-2\beta}\Phi_v(\frac{h}{\rho}).\end{aligned}$$

If the matrix $\tilde{T}$ is irreducible, the Perron-Frobenius theorem implies that the maximal eigenvalue $\lambda_{\max}$ is positive and that all the coordinates of the eigenvector v is positive. If we take β such that $\lambda_{\max}\rho^{d-2\beta} = 1$, then $\Phi_v(h) = \Phi_v(\frac{h}{\rho})$. It is easy to show that $\Phi_v(h) > 0$ and $0 < \underline{\lim}_{h\to 0}\Phi_v(h) \leq \overline{\lim}_{h\to 0}\Phi_v(h) < \infty$. From Proposition 4.1 we see that the absolute continuity criterion is $\lambda_{\max} = \rho^d$.

However the matrix $\tilde{T}$ is not always irreducible. Hence we cannot guarantee that $\Phi_v(h) > 0$ and we will need a little more elaborate work. We need to pick up the essential part of $\tilde{T}$ first. Let I be the identity map in $\tilde{\mathcal{S}}$. Let $\mathcal{S}_I$ be the $\tilde{T}$-irreducible component of $\tilde{\mathcal{S}}$ that contains I, i.e.,

$$S \in \mathcal{S}_I \text{ if and only if there exist } m, n \geq 1 \text{ such that } w^{(m)}_{(I,S)}, w^{(n)}_{(S,I)} > 0$$

where $w^{(n)}_{(S,S')}$ denotes the (S, S') entry of $\tilde{T}^n$. Let T_I be the truncated square matrix of $\tilde{T}$ on $\mathcal{S}_I$, then T_I is irreducible and is a finite matrix by the WSC*.

Theorem 4.5 *Suppose $\{S_j\}_{j=1}^N$ satisfies the WSC*. Let $\lambda_{\max}$ be the maximal eigenvalue of T_I and let*

$$\beta = \frac{1}{2}\big(d + |\frac{\log\lambda_{\max}}{\log\rho}|\big).$$

Then the mean quadratic variation satisfies

$$0 < \underline{\lim}_{h\to 0}\ \frac{1}{h^{2\beta}}\int_{\mathbb{R}^d}|\mu(B_h(x))|^2dx \ \leq\ \overline{\lim}_{h\to 0}\ \frac{1}{h^{2\beta}}\int_{\mathbb{R}^d}|\mu(B_h(x))|^2dx < \infty.$$

The L^2-dimension of μ is hence given by $\dim_2(\mu) = |\frac{\log\lambda_{\max}}{\log\rho}|$.

As a direct consequence we have

Corollary 4.6 *Suppose $\{S_j\}_{j=1}^N$ satisfies the hypotheses of Theorem 4.5. Then μ is absolutely continuous if and only if $\lambda_{\max} = \rho^d$.*

The theorem is a direct consequence of the following Lemmas 4.8, 4.9. We first observe that the (I, I) entry of T_I^n is

$$w^{(n)}_{(I,I)} = \sum_{\sigma,\sigma'\in\Sigma_n}\{w_\sigma w_{\sigma'} : S_\sigma^{-1}S_{\sigma'} = I\}$$

and $w^{(n)}_{(I,I)} = \sum_{S\in\mathcal{A}_n} w_S^2$. Since T_I is a nonnegative irreducible matrix, it is well known that there exist $a_1, a_2 > 0$ such that

$$a_1\lambda^n_{\max} \le w^{(n)}_{(I,I)} \le a_2\lambda^n_{\max}\ . \tag{4.2}$$

Proposition 4.7 *Under the assumption of Theorem 4.5, we have* $\lambda_{\max} \ge \rho^d$.

Proof. By Proposition 2.5, we have $\#\mathcal{A}_n \le \gamma\rho^{-dn}$. Since $\sum_{S\in\mathcal{A}_n} w_S = 1$, the Cauchy-Schwartz inequality implies that

$$w^{(n)}_{(I,I)} = \sum_{S\in\mathcal{A}_n} w_S^2 \ge \gamma^{-1}\rho^{dn}.$$

Hence by (4.2) we conclude that $\lambda_{\max} \ge \rho^d$. □

Lemma 4.8 *Under the assumption of Theorem 4.5 and let D be a basic region, then there exists $c > 0$ such that*

$$\int_{\mathbb{R}^d} |\mu(B_{|D|\rho^n}(x))|^2 dx \ge c\rho^{nd}\lambda^n_{\max}\ .$$

Proof. Let $h = |D|\rho^n$. For $S \in \mathcal{A}_n$, if $x \in S(D)$, then $S(D) \subset B_h(x)$. It follows that

$$\int_{S(D)} |\mu(B_h(x))|^2 dx \ge \int_{S(D)} |\mu(S(D))|^2 dx \ge \int_{S(D)} w_S^2 dx = m(D)\rho^{dn} w_S^2.$$

Since the WSC* implies the WSC, we see that each point is covered by at most γ of the $S(D)$ (Proposition 2.4). Hence

$$\int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx \ge \frac{1}{\gamma}\sum_{S\in\mathcal{A}_n}\int_{S(D)} |\mu(S(D))|^2 dx \ge \frac{m(D)}{\gamma}\rho^{dn}\sum_{S\in\mathcal{A}_n} w_S^2 = c\rho^{nd} w^{(n)}_{(I,I)}$$

where $c = m(D)/\gamma$. This together with (4.2) implies the lemma. □

Lemma 4.9 *Under the same assumption as in Theorem 4.5 and for $\rho^{n+1} \le h < \rho^n$, we have*

$$\int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx < c\rho^{dn}\lambda^n_{\max},$$

for some $c > 0$ independent of n.

Proof. Let Q_j be a ρ^n-mesh cube and let $\tilde{Q}_j = \bigcup_{x\in Q_j} B_h(x)$. Let $D(\supseteq K)$ be a basic region as before. Then by using $\mu = \sum_{S\in\mathcal{A}_n} w_S\ \mu\circ S^{-1}$, we have for $x \in Q_j$,

$$\mu(B_h(x)) \le \sum_{S\in\mathcal{A}_n}\{w_S :\ D\cap S^{-1}(B_h(x)) \ne \emptyset\} \le \sum_{S\in\mathcal{A}_n}\{w_S :\ S(D)\cap\tilde{Q}_j \ne \emptyset\}.$$

By Corollary 2.6, $\#\{S \in \mathcal{A}_n : S(D) \cap \tilde{Q}_j \neq \emptyset\} < c_1$. It follows that

$$
\begin{aligned}
&\int_{Q_j} |\mu(B_h(x))|^2 dx \\
&\leq \Big(\sum_{S \in \mathcal{A}_n} \{w_S : \ S(D) \cap \tilde{Q}_j \neq \emptyset\}\Big)^2 \cdot m(Q_j) \\
&\leq c_1 \sum_{S \in \mathcal{A}_n} \{w_S^2 : \ S(D) \cap \tilde{Q}_j \neq \emptyset\} \cdot m(Q_j) \qquad \text{(Cauchy-Schwartz inequality)} \\
&= c_2 \rho^{dn} \sum_{S \in \mathcal{A}_n} \{w_S^2 : \ S(D) \cap \tilde{Q}_j \neq \emptyset\}.
\end{aligned}
$$

For each $S \in \mathcal{A}_n$, it is clear that $\#\{Q_j : \ S(D) \cap \tilde{Q}_j \neq \emptyset\} < c_3$ for some fixed c_3. Summing both sides of the above expressions, we obtain

$$
\int_{\mathbb{R}^d} |\mu(B_h(x))|^2 dx \leq c_2 c_3 \rho^{dn} \sum_{S \in \mathcal{A}_n} w_S^2 = c_4 \rho^{dn} w_{(I,I)}^{(n)}.
$$

The lemma now follows from (4.2).

□

To conclude this section we will discuss the examples in Section 2 in regard to the WSC*. We show that the state space $\tilde{\mathcal{S}}$ and the matrix $\tilde{T}$ can easily be implemented after some concrete identifications.

Let $\{S_j\}_{j=1}^N$ be an IFS on $\mathbb{R}$ with $S_j(x) = \rho(x + d_j)$, where $0 < \rho < 1$. Without loss of generality we assume that $0 = d_1 < d_2 \cdots < d_N$. We can prove by induction that the state $S = S_\sigma^{-1} S_{\sigma'} \in \mathcal{S}$ has the form

$$
Sx = x + s, \ \ x \in \mathbb{R}
$$

for some $s \in \mathbb{R}$. We can represent the map S by the translation number s. We construct the set $\mathcal{S}$ inductively, starting from $s = 0$, by letting

$$
s' = \rho^{-1} s + d_j - d_i, \quad 1 \leq i, j \leq N. \tag{4.3}
$$

The set $\tilde{\mathcal{S}}$ can be obtained by keeping those s' with $|s'| \leq \tilde{C} = \frac{2\rho}{1-\rho} d_N$. The matrix T will send s into the states s' in (4.3) with weight

$$
w_{(s,s')} = \sum \{w_i w_j : \rho^{-1} s + d_j - d_i = s'\}.
$$

It can be checked from this that Examples 2.1-3 have the WSC* (for Example 2.2, we need to use Garsia's lemma again). For the numerical examples the reader can check on [L], [LN3], [FLN]. In all those cases $\tilde{T} = T_I$ and $\tilde{T}$ can be reduced further to smaller size by the symmetry of the $\tilde{\mathcal{S}}$.

For the IFS in Example 2.4, $S_j(x) = A(x + d_j)$ in $\mathbb{R}^d$, the maps $S \in \mathcal{S}$ still have the form $Sx = x + s$. The WSC* is clear and the construction of the set $\tilde{\mathcal{S}}$ and the map $\tilde{T}$ is the same as the above one dimensional case.

For the IFS in Example 2.5, we can prove by induction that each $S = S_\sigma^{-1}S_{\sigma'} \in \mathcal{S}$ can be represented as

$$S(x) = \gamma(x) + s.$$

where $\gamma = A_\sigma^{-1}A_{\sigma'} \in \Gamma$ and $s = S_\sigma^{-1}S_{\sigma'}(0) \in \mathbb{Z}^d$. Hence each $S \in \mathcal{S}$ can be represented by (γ, s). It is immediate to see that the IFS satisfies the WSC* since Γ is a finite set and $s \in \mathbb{Z}^d$.

For the construction of $\tilde{\mathcal{S}}$ and $\tilde{T}$, we first define $\gamma^\#$ by $\gamma A = A\gamma^\#$ and then observe that

$$\begin{aligned} S_i^{-1}SS_j(0) &= A_i^{-1}(\gamma A_j d_j + s) - d_i \\ &= (\gamma_i^{-1}\gamma\gamma_j)^\# d_j + A_i^{-1}s - d_i \\ &= (\gamma_i^{-1}\gamma\gamma_j)^\# d_j + S_i^{-1}(s). \end{aligned}$$

Hence the set $\mathcal{S}$ can be construct inductively as follow, starting from $(\gamma, s) = (I, 0)$,

$$(\gamma', s') = \big((\gamma_i^{-1}\gamma\gamma_j)^\#,\ (\gamma_i^{-1}\gamma\gamma_j)^\# d_j + S_i^{-1}(s)\big), \quad 1 \le i, j \le N.$$

The set $\tilde{\mathcal{S}}$ is obtained by choosing the $(\gamma, s) \in \mathcal{S}$ such that $(\gamma_i^{-1}\gamma\gamma_j)^\# d_j + S_i^{-1}(s) \le \tilde{C}$ in the construction. The map T will send (γ, s) into the above states with weight

$$w_{((\gamma,s),\ (\gamma',s'))} = \sum_{i,j}\{w_i w_j :\ (\gamma_i^{-1}\gamma\gamma_j)^\# = \gamma',\ (\gamma_i^{-1}\gamma\gamma_j)^\# d_j + S_i^{-1}(s) = s'\}.$$

5 The case of PV numbers.

Erdös proved that if $X = \sum_{i=1}^\infty \rho^n X_n$ where $\{X_n\}_{n=1}^\infty$ are i.i.d. Bernoulli random variables and $1 < \rho^{-1} < 2$ is a PV number, then the distributional measure μ is singular [S]. In our present notation, μ is the self-similar measure

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}$$

where $S_1 x = \rho x,\ S_2 x = \rho x + 1$. In the following we will extend Erdös' theorem to $S_j(x) = \rho x + b_j$, $1 \le j \le N$, where $\rho^{-1} > 1$ is a PV number and b_j are rationals, and the probability weights are arbitrary (even negative). We need two technical lemmas.

For fixed $(k_0, n_0) \in \mathbb{Z} \times \mathbb{Z}$, consider the following equation

$$k\beta^n + k_0\beta^{n_0} \in \mathbb{Z}. \tag{5.1}$$

If $k, n, p \in \mathbb{Z}$ satisfy $k\beta^n + k_0\beta^{n_0} = p$, we say that (k, n, p) is a solution of (5.1); we also say that (k, n) is a solution of (5.1) if (k, n, p) is a solution for some $p \in \mathbb{Z}$.

Lemma 5.1 *Suppose $\beta > 1$ is an algebraic number and at least one of its conjugate roots has modulus < 1. Then for any $(k_0, n_0) \in \mathbb{Z} \times \mathbb{Z}$ with $k_0, n_0 \neq 0$, the number of solutions of (5.1) is finite.*

Proof. We first observe that $n \neq 0$, for otherwise $\beta^{n_0} = (p-k)/k_0$ will imply that β and its conjugates are multiple of unity. This contradicts that $\beta > 1$ and one of its conjugates has modulus < 1. Next, if (k, n, p) and (k', n', p) are two distinct solutions of (5.1), then $\beta^{n-n'} = k/k'$ which is impossible by the same reason.

Let λ be a conjugate of β such that $|\lambda| < 1$. If equation (5.1) has infinitely many solutions (for different p), then we have solutions (k, n, p) with $|p|$ arbitrarily large. Hence

$$|\frac{\beta}{\lambda}|^n = \frac{|k\beta^n|}{|k\lambda^n|} = \frac{|p - k_0\beta^{n_0}|}{|p - k_0\lambda^{n_0}|} \to 1,$$

when $p \to \infty$. This contradicts the fact that $|\frac{\beta}{\lambda}|^n > |\frac{\beta}{\lambda}| > 1$ when $n > 0$, and $|\frac{\beta}{\lambda}|^n < |\frac{\beta}{\lambda}| < 1$ when $n < 0$. □

Lemma 5.2 *Let $Q(\xi) = \sum_{j=1}^N c_j e^{2\pi i b_j \xi}$ be a trigonometric polynomial with $\sum_{j=1}^N c_j \neq 0$, $c_j \in \mathbb{R}$ and b_j rationals. Let B be such that $B_j = Bb_j, 1 \leq j \leq N$, are integers and let β be an algebraic number as in Lemma 5.1. Then there exists $m \in \mathbb{Z}$ such that $Q(mB\beta^k) \neq 0$ for all $k \in \mathbb{Z}$.*

Proof. Let $\tilde{Q}(x) = \sum_{j=1}^N c_j x^{B_j}$ and let $\{x_1, \cdots, x_s\}$ be the roots of $\tilde{Q}(x)$ with modulus 1 that can be written as $e^{2\pi i k_\ell \beta^{n_\ell}}, 1 \leq \ell \leq s$. Then $\sum_{j=1}^N c_j \neq 0$ implies that k_0 and $n_\ell \neq 0$ for all $1 \leq l \leq s$. Note that $Q(mB\beta^k) = 0$ if and only if $e^{2\pi i m\beta^k}$ is a root of $\tilde{Q}$. It follows that

$$m\beta^k - k_\ell \beta^{n_\ell} \in \mathbb{Z} \tag{5.2}$$

for some $1 \leq \ell \leq s$. By Lemma 5.1 there are only finitely many m for (5.2) to hold. Hence we can choose $m \in \mathbb{Z}$ such that $Q(mB\beta^k) \neq 0$ for all $k \in \mathbb{Z}$. □

Theorem 5.3 *Let ρ^{-1} be an irrational PV number, b_j be rationals, and let*

$$S_j x = \rho x + b_j, \quad j = 1, \cdots, N.$$

Then for any set of probability weights $\{w_j\}_{j=1}^N$ the self-similar measure μ is singular.

Proof. By (1.1), the Fourier transformation $\Phi(\xi) = \hat{\mu}(\xi)$ satisfies

$$\Phi(\xi) = Q(\xi)\Phi(\rho\xi) = \prod_{j=0}^{\infty} Q(\rho^j \xi),$$

where $Q(\xi) = \sum_{j=1}^N w_j e^{2\pi i b_j \xi}$. Note that $\Phi(0) = 1$ and the product converges to Φ uniformly on compact subsets of $\mathbb{R}$. We will show that $|\Phi(\xi)|$ does not converge to 0 as $|\xi| \to \infty$. The Riemann-Lebesgue lemma will imply that μ is singular.

Let $\beta = \rho^{-1}$. We assume without loss of generality that there exists B such that $B_j = Bb_j$ are integers and $Q(B\beta^k) \neq 0$ for all $k \in \mathbb{Z}$ (Lemma 5.2). Let $C = \prod_{n=0}^{\infty} Q(B\beta^{-n})$, then $C \neq 0$. For $k > 0$,

$$|\Phi(B\beta^k)| = \prod_{n=0}^{\infty} |Q(B\beta^{k-n})| = C \prod_{n=1}^{k} |Q(B\beta^n)| \geq C \prod_{n=1}^{\infty} |Q(B\beta^n)|. \tag{5.3}$$

Now we recall a well known property of the PV number [S]: there exists $0 < \theta < 1$ such that $|[\beta^n]| < \theta^n$ for large n, where $[x]$ denotes the distance from x to the nearest integer. We can choose n_0 such that for $n > n_0$, $2B\theta^n < \frac{1}{2}$. Let $C' = \prod_{n=1}^{n_0} |Q(B\beta^n)| > 0$. Then

$$\begin{aligned}
\prod_{n=1}^{\infty} |Q(B\beta^n)| &= C' \prod_{n=n_0+1}^{\infty} |Q(B\beta^n)| \\
&= C' \prod_{n=n_0+1}^{\infty} |\sum_{n=1}^{N} w_j e^{2\pi i B_j [\beta^n]}| \\
&\geq C' \prod_{n=n_0+1}^{\infty} \min_{1\leq j\leq N} |\text{Re } e^{2\pi i B_j \theta^n}| \\
&\geq C' \prod_{1\leq j\leq N} \prod_{n=n_0+1}^{\infty} \cos(2\pi B_j \theta^n).
\end{aligned}$$

The product is a positive constant and if we put it back to (5.3), we have $\Phi(B\beta^k) \not\to 0$ as $k \to \infty$. □

Note that in the proof we have not use the property of the positive weights. The proof can hence be applied directly to conclude that

Corollary 5.4 *Let the IFS be as in Theorem 5.3 and let* $\sum_{j=1}^{N} c_j = \rho^{-1}$. *Then the functional equation*

$$f = \sum_{j=1}^{N} c_j f \circ S_j^{-1}$$

has no L^1*-solution.*

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