INVERTIBLE SUBSTITUTIONS WITH A COMMON PERIODIC POINT

HUI RAO AND ZHI-YING WEN

ABSTRACT. We characterize the invertible substitutions over a two-letter alphabet which share a common periodic point (or fixed point). The argument is geometrical.

1. Introduction

Let $A = \{1, 2\}$ be an alphabet. Let $A^* = \cup_{n \geq 0} A^n$ denote the free monoid over $\{1, 2\}$ endowed with the concatenation operation. A non-erasing homomorphism σ of the free monoid A^* is called a *substitution*.

An infinite word $s \in A^{\mathbb{N}}$ is a *fixed point* of the substitution σ if $\sigma(s) = s$; it is called a *periodic point* of σ if $\sigma^k(s) = s$ for some $k \geq 1$.

For a substitution σ, let $M_\sigma = (m_{ij})$ be its *incidence matrix*, where m_{ij} counts the number of occurrences of the letter i in $\sigma(j)$. We say σ is *unimodular* if $\det M_\sigma = \pm 1$; is *primitive* if M_σ is primitive, *i.e.*, M_σ^n has only positive entries for some $n \geq 1$.

A substitution is said *invertible* if it is an automorphism of the free group $\mathcal{F}$ generated by the alphabet A. Note that an invertible substitution is necessarily unimodular. There are a numerous paper on invertible substitutions, especially on invertible substitutions over two-letter alphabet. (See for example [11, 18, 5, 6, 16, 19].) An excellent survey can be found in Chapter 2 of [10].

Definition 1.1. *For two substitution σ and τ, we write $\sigma \sim \tau$ if they share a common periodic point; write $\sigma \cong \tau$ if they share a common fixed point.*

In general, it is difficult to characterize the substitutions with common periodic points or fixed points. In this paper, we give an answer to the question for invertible substitutions over two-letter alphabet.

It is easy to show that an invertible substitution σ over $\{1, 2\}$ is non-primitive if and only if M_σ has one of the following forms:

$$\begin{pmatrix} n & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

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In this case, the fixed points of the substitutions can only be 1^∞, 12^∞, 2^∞ or 12^∞, which is not interesting. So in the following, we study the primitive invertible substitutions. Our main result is following.

Theorem 1.2. *Let σ and τ be two primitive invertible substitutions over $\{1,2\}$. Then $\sigma \sim \tau$ if and only if there is a primitive invertible substitution γ such that $\sigma = \gamma^n,\ \tau = \gamma^m,\ m,n \geq 1$.*

As a corollary, we have

Corollary 1.3. *Let σ and τ be two primitive invertible substitutions over $\{1,2\}$. Then $\sigma \cong \tau$ if and only if there is a primitive invertible substitution γ possessing fixed point such that $\sigma = \gamma^n,\ \tau = \gamma^m,\ m,n \geq 1$.*

Proof. By Theorem 1.2, $\sigma = \gamma^m$ and $\tau = \gamma^n$. If γ does not possess a fixed point, then the word $\gamma(1)$ is initialed by 2 and $\gamma(2)$ is initialed by 1. To guarantee σ and τ having fixed points, m and n must be even numbers. Hence the corollary holds if we replace γ by γ^2. $\square$

We will use a geometrical method to prove the above results, where the notion of *Rauzy fractal* plays a central role. We have seek for a combinatorial proof but did not succeed. It would be interesting to know a combinatorial proof.

Rauzy fractals have many applications in number theory (see for instance [15, 8, 14, 3]), and the present paper gives a new one. For general theory of Rauzy fractal, we refer to [2, 8].

2. Some known results concerning invertible substitutions

2.1. **Frequency of a substitution, generating matrix.** Let σ be a primitive unimodular substitution over $\{1,2\}$. Let β be the maximal eigenvalue of its incidence matrix M_σ. Its algebraic conjugate β' is also an eigenvalue of M_σ. By Perron-Frobenius' theorem, we have $\beta > 1$. Now $\beta\beta' = \det M = \pm 1$ implies $|\beta'| < 1$. Therefore β is a Pisot number and the substitution σ is said to be of *Pisot type*.

It is well-known that the frequencies of occurrences of letters exist in periodic points of primitive substitutions (see [13]). Let $1-\alpha$ and α, $0 \leq \alpha \leq 1$, be the frequencies of the letters 1 and 2 respectively. We shall call α the *frequency* of σ. It is obvious that $\sigma \sim \tau$ implies that σ and τ have the same frequency. We shall denote $\mathcal{I}_p^2(\alpha)$ the collection of primitive invertible substitutions with frequency α.

It is seen that $(1-\alpha, \alpha)$ is an *expanding eigenvector*, that is, an eigenvector of M associated with the expanding eigenvalue β. The real number α is quadratic; the vector $(1-\alpha', \alpha')$ is an eigenvector associated with the eigenvalue β'. Still by Perron-Frobenius' theorem, the coordinates $1-\alpha'$, α' cannot be both positive, hence $\alpha'(1-\alpha') \leq 0$. A quadratic number α with $0 < \alpha < 1$ and $\alpha' \notin [0,1]$ is called a *Sturm number* according to [1]. Hence we have proved that the set $\mathcal{I}_p^2(\alpha)$ is empty if α is not a Sturm number. For a Sturm number α, the set $\mathcal{I}_p^2(\alpha)$ have been well understood.

Proposition 2.1. *Let α be a Sturm number. Then there is a non-negative primitive unimodular matrix $M(\alpha)$, which we call the generating matrix, such that $\sigma \in \mathcal{I}_p^2(\alpha)$ if and only if $\sigma \in \mathcal{I}_p^2$ and $M_\sigma = (M(\alpha))^k$ for some $k \geq 1$.*

The above proposition is essentially contained in Wen, Wen and Wu [19]. Given a Sturm number α, Berthé and Rao [4] give an explicit construction of the generating matrix $M(\alpha)$.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix. Séébold [16] proved that

Proposition 2.2. *Let M be a unimodular non-negative integral matrix. The number of invertible substitutions with incidence matrix M is equal to $a+b+c+d-1$.*

Various characterizations of these $a+b+c+d-1$ substitutions can be found in the book [10].

2.2. **Sturmian words.** *Sturmian words* are infinite words over a binary alphabet, say, $\{1,2\}$, that have exactly $n+1$ factors of length n for each $n \geq 0$. Sturmian words can be defined constructively in terms of rotation.

Let $0 < \alpha < 1$. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha(x) := x + \alpha$. For a real number ρ, we introduce two partitions of $\mathbb{T}^1$ as follows:

$$\underline{I}_1 = [0, 1-\alpha), \quad \underline{I}_2 = [1-\alpha, 1); \quad \overline{I}_1 = (0, 1-\alpha], \quad \overline{I}_2 = (1-\alpha, 1].$$

Tracing the orbit of $R_\alpha^n(\rho)$, we define two infinite words as follows.

$$\underline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \underline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \overline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \overline{I}_2. \end{cases}$$

It is a folklore ([12]) that an infinite word is a Sturmian word if and only if it is in the form $\overline{s}_{\alpha,\rho}$ or $\underline{s}_{\alpha,\rho}$ and α is an irrational number. We shall call the word $\underline{s}_{\alpha,\rho}$ the *lower Sturmian word* whereas the word $\overline{s}_{\alpha,\rho}$ the *upper Sturmian word*.

It is well-known that the periodic points of a primitive invertible substitution are Sturmian words (see [10]).

3. **Rauzy fractals of invertible substitutions**

Let us first give an alternative definition of the Rauzy fractals of substitutions over two letters. For the original definition, see for example [2, 9].

Let σ be a primitive and unimodular substitution over $\{1,2\}$. Let $s = s_0s_1s_2\ldots$ be a periodic point of σ. Let α be the frequency of σ.

We define an oriented walk on the real line as follows. Starting from the origin, in the n-th step, if $s_{n-1} = 1$, we move to the right side with length α; if $s_{n-1} = 2$, we move to the left side of length $1-\alpha$. Taking a closure of the orbit, we obtain

$$X = \text{closure } \{|s_0 s_1 \dots s_{n-1}|_1 \cdot \alpha + |s_0 s_1 \dots s_{n-1}|_2 \cdot (\alpha - 1);\ n \geq 0\},$$

where $|s_0 s_1 \dots s_n|_j$ denotes the occurrences of letter j in the word $s_0 s_1 \dots s_n$. Furthermore, we define

$$\begin{aligned} X_1 &= \text{closure } \{|s_0 s_1 \dots s_{n-1}|_1 \cdot \alpha + |s_0 s_1 \dots s_{n-1}|_2 \cdot (\alpha - 1);\ s_n = 1,\ n \geq 0\}, \\ X_2 &= \text{closure } \{|s_0 s_1 \dots s_{n-1}|_1 \cdot \alpha + |s_0 s_1 \dots s_{n-1}|_2 \cdot (\alpha - 1);\ s_n = 2,\ n \geq 0\} \end{aligned} \tag{3.1}$$

We shall call X the *Rauzy fractal* of σ, and we call X_1, X_2 in formula (3.1) the *partial Rauzy fractals* of σ. Particularly, formula (3.1) is well defined for any Sturmian word s, and we also call X_1, X_2 the Rauzy fractals of the Sturmian word s.

The Rauzy fractals defined above are affine images of the original Rauzy fractals([3]). Also the definition of the Rauzy fractals do not depend on the choice of the particular periodic point ([9]).

Obviously the Rauzy fractals of a Sturmian word are intervals, and the length of X is 1. The periodic points of a primitive invertible substitutions are Sturmian, and hence the associated Rauzy fractals are intervals, and the length of X is 1. (Actually it is shown that ([7, 3]), if σ is a primitive unimodular substitution over $\{1, 2\}$, then the Rauzy fractals are intervals if and only if σ is invertible.)

So the Rauzy fractals X_1,X_2 are intervals with length $1-\alpha$ and α respectively. Let us denote by $h = h_\sigma$ the intersection $X_1 \cap X_2$, then X_1, X_2 have the form

$$X_1 = [-1 + \alpha + h, h], \quad X_2 = [h, \alpha + h].$$

Lemma 3.1. *Let $\sigma, \tau \in \mathcal{I}_p^2$. Then $\sigma \sim \tau$ if and only if they have the same Rauzy fractals; in other words, if and only if $h_\sigma = h_\tau$.*

Proof. It is shown in [3] that if the Rauzy fractals of σ are intervals, then they can be obtained from at most two Sturmian sequences by the above oriented walk, and they are all periodic points of σ. The lemma is proved. □

From this lemma, it is seen that the Rauzy fractal is a suitable tool to handle the problem that when two substitutions share a common periodic point.

4. Characterization of Rauzy fractals stepped-surface

4.1. **The stepped-surface.** Denote by V be the expanding eigenspace of the matrix M_σ corresponding to the eigenvalue β, and V' the contractive eigenspace corresponding to β'. Then V and V' are generated by the vectors $\vec{v} = (1-\alpha, \alpha)$ and $\vec{v}' = (1-\alpha', \alpha')$ respectively. According to the direct sum $V \oplus V' = \mathbb{R}^2$, two natural projections are defined:

$$\pi : \ \mathbb{R}^2 \to V' \quad \text{and} \quad \pi' : \ \mathbb{R}^2 \to V.$$

We denote the right side of V' (including V') by $(V')^+$, that is

$$(V')^+ = \{x \in \mathbb{R}^2;\ \pi'(x) \geq 0\}.$$

Let us consider the unit segment connecting two integer points in $\mathbb{R}^2$. Two such segments are neighbor if they belong to one line and they share an endpoint. So a segment has two neighbors.

Let S be the collection of unit segments in $(V')^+$ which has a neighbor intersecting V'. Let $\bar{S}$ be the union of segments in S. Then $\bar{S}$ is a broken line approximating V'. We shall call $\bar{S}$ the *stepped-surface* of V'. See Figure 1.

The notion of stepped-surface was introduced by Arnoux and Ito [2], to handle the set equations of the Rauzy fractals.

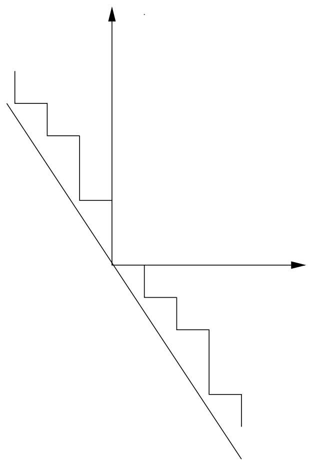

FIGURE 1. The stepped-surface.

4.2. **A tiling associated with the stepped-surface.** Projecting the segments of the stepped-surface S to V', we obtain a tiling $\mathcal{J}'$ of V':

$$\mathcal{J}' = \{\pi(\mathbf{s});\ \mathbf{s} \in S\}.$$

The prototiles of $\mathcal{J}'$ consists of two kind of segments with length $|\pi(\vec{e}_1)|$ and $|\pi(\vec{e}_2)|$ respectively, where $\vec{e}_1, \vec{e}_2$ are the canonical basis of $\mathbb{R}^2$.

Let ϕ be the linear transformation which mapping V' to the real line $\mathbb{R}$ such that $\phi \circ \pi(\vec{e}_1) > 0$, and $(|\phi \circ \pi(\vec{e}_1)|, |\phi \circ \pi(\vec{e}_2)|) = (1-\alpha, \alpha)$. Then $\mathcal{J} = \phi(\mathcal{J}')$ is a tiling of the real line consisting of two kind of segments with length $1-\alpha$ and α respectively.

Let $G = \{g_k;\ k \in \mathbb{Z}\}$ be the end points of tiles in $\mathcal{J}$, where g_k is increasing.

Projecting the integer points on $\bar{S}$ to V by π', one can regard the projection points as the orbit of a rotation on the torus $V \pmod{\pi'(\vec{e_1} + \vec{e_2})}$ with angle $\pi'(\vec{e}_2)$. Hence it is not difficult to show that ([3])

Theorem 4.1. *If α is a Sturm number, then*

$$G = \{g \in \mathbb{Z}[\alpha];\ 0 \le g' < 2\alpha' - 1\} \text{ when } \alpha' > 1,$$
$$G = \{g \in \mathbb{Z}[\alpha];\ 2\alpha' - 1 < g' \le 0\} \text{ when } \alpha' < 0,$$

where $\mathbb{Z}[\alpha] := \{m\alpha + n;\ m, n \in \mathbb{Z}\}$.

4.3. **Invertible substitutions with a given incidence matrix.** Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix, then there are $a+b+c+d-1$ invertible substitutions with incidence matrix M (Proposition 2.2). Let us denote them by σ_k and denote by h_k the intersection point of the partial Rauzy fractals of σ_k, $1 \le k \le a+b+c+d-1$. We arrange σ_k in the order such that h_k is increasing.

By the connectedness and the self-similarity of the Rauzy fractals of invertible substitutions, [3] determined the intersections h_k in terms of the set G.

Theorem 4.2. *Let σ_k, $1 \le k \le a+b+c+d-1$, be the invertible substitutions with the incidence matrix M, let β be the maximal eigenvalue of M. Then*

(i) The values h_k are given by

$$h_k = \begin{cases} \frac{g-k+a+b}{\beta-1}, & \text{if } \det M = 1 \\ \frac{g-k+c+d}{\beta+1}, & \text{if } \det M = -1. \end{cases}$$

(ii) Another characterization of h_k is

$$\{h_k;\ 1 \le k \le a+b+c+d-1\} = \begin{cases} \frac{G}{\beta-1} \cap [-\alpha, 1-\alpha], & \text{if } \det M = 1 \\ \frac{G}{\beta+1} \cap [-\alpha, 1-\alpha], & \text{if } \det M = -1. \end{cases}$$

As a direct consequence of item (i), if two primitive invertible substitutions have the same incidence matrix and share a periodic point, then they must coincide.

Tan and Wen [17] also give an alternative method to determine the value h for a given invertible substitution. In [3], Theorem 4.2 are designed to prove a theorem of Yasutomi [20] concerning substitution invariant Sturmian sequences.

5. Proof of Theorem 1.2

Now we are in the position to proof our main theorem.

Proof. Let σ, τ be two primitive invertible substitution over $\{1, 2\}$ such that $\sigma \sim \tau$. Let α be their common frequency, which must be a Sturm number. Let $M = M(\alpha)$ be the generating matrix of α, let β be the maximal eigenvalue of β. The assumption $\sigma \sim \tau$ implies that they have the same Rauzy fractal (Lemma 3.1).

First, by Proposition 2.1, there are integers $m, n \ge 1$ such that $M_\sigma = M^m, M_\tau = M^n$. If $m = n$, then by Theorem 4.2 (i), $\sigma = \tau$. The theorem is true.

Hence in what follows we assume that $m > n$ without loss of generality. We shall show that there is a primitive invertible substitution γ such that $\gamma \sim \sigma \sim \tau$ and $M_\gamma = M^{m-n}$.

Let us first deal with the case $\det M_\sigma = \det M_\tau = 1$. By Theorem 4.2 (ii), we have

$$h_\sigma = \frac{g_1}{\beta^m - 1},\ h_\tau = \frac{g_2}{\beta^n - 1}, \tag{5.1}$$

for some $g_1, g_2 \in G$. Our assumption $\sigma \sim \tau$ implies that $h_\sigma = h_\tau$.

Since $m > n$, from $h'_\sigma = h'_\tau$ we infer that

$$g'_1 = \frac{(\beta')^m - 1}{(\beta')^n - 1} g'_2,$$

which implies that $|g'_1| > |g'_2|$. (Remember that $|\beta'| < 1$.) By (5.1) we have

$$h_\sigma = \frac{g_1 - \beta^{m-n} g_2}{\beta^{m-n} - 1}.$$

We assert that $g_1 - g_2\beta^{m-n} \in G$.

If $\alpha' > 1$, then $0 \le g'_2 < g'_1 < 2\alpha' - 1$ by Theorem 4.1. Hence $0 \le (\beta')^{m-n} g_2 \le g'_2$. For in case that $\beta' < 0$, $\det M_\sigma = (\beta\beta')^m = 1$ implies that $(\beta')^m > 0$ and so that m is an even number; likewise n is also an even number.

So $0 \le g'_1 - g'_2\beta^{m-n} \le 2\alpha' - 1$, and it follows that $g_1 - g_2\beta^{m-n} \in G$ by Theorem 4.1.

If $\alpha' < 0$, then $2\alpha' - 1 < g'_1 < g'_2 \le 0$. Hence $g'_2 \le (\beta')^{m-n} g_2 \le 0$ since $m - n$ is an even number in case that $\beta' < 0$. So $2\alpha' - 1 < g'_1 - g'_2\beta^{m-n} \le 0$, and it follows that $g_1 - g_2\beta^{m-n} \in G$.

On the other hand $-\alpha \le h \le 1 - \alpha$ since h is the intersection the partial Rauzy fractals of σ. By Theorem 4.2 (ii), there is a primitive invertible substitution γ with incidence matrix M^{m-n} such that $h_\gamma = h_\sigma = h_\tau$.

The cases $(\det M_\sigma, \det M_\tau) = (1, -1), (-1, 1), (-1, -1)$ can be proved in the same manner.

Repeating the above argument, we conclude that there is a primitive invertible substitution θ with incidence matrix M^l, $l = \gcd\{m, n\}$, such that $h_\theta = h_\sigma = h_\tau$.

Let $p = m/l$, $q = n/l$. Then θ^p and σ have the same incidence matrix and share a common periodic point. Hence $\sigma = \theta^p$. Likewise $\tau = \theta^q$. The theorem is proved. $\square$

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H. Rao, Department of Mathematics, Tsinghua University, Beijing, China
E-mail address: hrao@math.tsinghua.edu.cn

Zhi-ying Wen, Department of Mathematics, Tsinghua University, Beijing, China
E-mail address: wenzy@mail.tsinghua.edu.cn