ON SUBSTITUTION INVARIANT STURMIAN WORDS: AN APPLICATION OF RAUZY FRACTALS

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ABSTRACT. Sturmian words are infinite words that have exactly $n+1$ factors of length n for every positive integer n. A Sturmian word $s_{\alpha,\rho}$ is also defined as a coding over a two-letter alphabet of the orbit of the point ρ under the action of the irrational rotation $R_\alpha: \ x \mapsto x+\alpha \pmod 1$. Yasutomi characterized in [34] all the pairs (α, ρ) such that the Sturmian word $s_{\alpha,\rho}$ is a fixed point of some non-trivial substitution. By investigating the Rauzy fractals associated with invertible substitutions, we give an alternative geometric proof of Yasutomi's characterization.

1. Introduction

1.1. **Sturmian words and substitution invariance.** *Sturmian words* are infinite words over a binary alphabet, say, $\{1,2\}$, that have exactly $n+1$ factors of length n for every positive integer n. Sturmian words can also be defined in a constructive way as follows. Let $0 < \alpha < 1$. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x+\alpha$. For a given real number α, we introduce the following two partitions of $\mathbb{T}^1$:

$$\underline{I}_1 = [0, 1-\alpha), \quad \underline{I}_2 = [1-\alpha, 1); \quad \overline{I}_1 = (0, 1-\alpha], \quad \overline{I}_2 = (1-\alpha, 1].$$

Tracing the orbit of $R_\alpha^n(\rho)$, we define two infinite words:

$$\underline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \underline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \overline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \overline{I}_2. \end{cases}$$

It is well known ([13, 25]) that an infinite word is a Sturmian word if and only if it is equal either to $\overline{s}_{\alpha,\rho}$ or to $\underline{s}_{\alpha,\rho}$ for some irrational number α. The word $\underline{s}_{\alpha,\rho}$ is called *lower Sturmian word* whereas the word $\overline{s}_{\alpha,\rho}$ is called *upper Sturmian word. The notation $s_{\alpha,\rho}$ stands in*

Date: September 8, 2005.
1991 *Mathematics Subject Classification.* Primary 37B10; Secondary 11J70, 68R15.

all that follow indifferently for $\overline{s}_{\alpha,\rho}$ or for $\underline{s}_{\alpha,\rho}$ when there is no need to distinguish between both. A detailed description of Sturmian words can be found in [23], see also [28].

Let $\{1,2\}^*$ be the free monoid over $\{1,2\}$ endowed with the concatenation operation. A non-erasing homomorphism σ of the free monoid $\{1,2\}^*$ is called a *substitution*. An infinite word $s \in \{1,2\}^{\mathbb{N}}$ is a *fixed point* of the substitution σ if $\sigma(s) = s$; it is called a *periodic point* of σ if $\sigma^k(s) = s$ for some $k \geq 1$.

It is well known that the famous Fibonacci word, that is, the fixed point of the Fibonacci substitution $1 \mapsto 12, 2 \mapsto 1$, is a Sturmian word. It is thus natural to ask when a Sturmian sequence is a fixed point of some non-trivial substitution. More precisely, we want to know

Question 1. For which α and ρ, is the Sturmian word $\underline{s}_{\alpha,\rho}$ (resp. $\overline{s}_{\alpha,\rho}$) a fixed point of some non-trivial substitution?

By non-trivial substitution, we mean here a substitution that is distinct from the identity. In all that follows, we say that a Sturmian word is *substitution invariant* if it is a fixed point of a non-trivial substitution.

There is a large literature devoted to Question 1. The first step has been made in [14] (Theorem A below). When $\rho = \alpha$, we have $\underline{s}_{\alpha,\alpha} = \overline{s}_{\alpha,\alpha}$ since α is an irrational number. We thus denote this word by $s_{\alpha,\alpha}$. It is usually called the *characteristic* word of α. For a number x in a quadratic field, we denote by x' the conjugate of x in this field.

Theorem A. (Crisp and al. [14]) *Let $0 < \alpha < 1$ be an irrational number. Then the following two conditions are equivalent:*

(i) the characteristic word $s_{\alpha,\alpha}$ is substitution invariant.

(ii) α is a quadratic irrational with $\alpha' \notin [0,1]$.

A quadratic number α with $0 < \alpha < 1$ and $\alpha' \notin [0,1]$ is called a *Sturm number* according to [2]. Let us note that the simplification of Condition (ii) in Theorem A to its present form is due to [2]. Furthermore, the expression of the substitutions which fix $s_{\alpha,\alpha}$ can be explicitly obtained from the continued fraction expansion of α (see [14]).

For more results on the homogeneous case (that is, the case $\rho = \{n\alpha\}$ for $n \in \mathbb{Z}$, where $\{x\}$ denotes the fractional part of x), see for instance [7, 8, 11, 16, 21, 23]; for results in the non-homogeneous case, see [22, 26, 6]. Some variants of Question 1 are also considered in [27, 9].

Yasutomi has given a complete answer to Question 1 in [34]. Its characterization involves the conjugates of the quadratic real number

x and can be compared to Galois' theorem for classical continued fractions describing numbers having a purely periodic continued fraction expansion.

Theorem 1 (Yasutomi [34]). *Let $0 < \alpha < 1$ and $0 \le \rho \le 1$. Then $s_{\alpha,\rho}$ is substitution invariant if and only if the following two conditions are satisfied:*
(i) α is an irrational quadratic number and $\rho \in \mathbb{Q}(\alpha)$;
(ii) $\alpha' > 1,\ 1-\alpha' \le \rho' \le \alpha'$ or $\alpha' < 0,\ \alpha' \le \rho' \le 1-\alpha'$.

Remark 1. Let us notice the symmetry between both cases in Assertion (ii) of Theorem 1. Indeed let $E : 1 \mapsto 2, 2 \mapsto 1$ be the substitution exchanging letters; then $s_{\alpha,\rho}$ is substitution invariant if and only if so does $E(\underline{s}_{\alpha,\rho}) = \underline{s}_{1-\alpha,1-\rho}$ (respectively $\overline{s}_{1-\alpha,1-\rho}$); furthermore, (α, ρ) satisfies $\alpha' > 1,\ 1-\alpha' \le \rho' \le \alpha'$ if and only if $(1-\alpha, 1-\rho)$ satisfies $\alpha' < 0,\ \alpha' \le \rho' \le 1-\alpha'$.

As a corollary of Theorem 1, we easily obtain:

Corollary 1. *Let α be a Sturm number. Then*
(i) for any $\rho \in \mathbb{Q} \cap (0,1)$, $\underline{s}_{\alpha,\rho} = \overline{s}_{\alpha,\rho}$ is substitution invariant.
(ii) The sturmian word $s_{\alpha,\{n\alpha\}}$ is substitution invariant if and only if $n = -1, 0, 1$. In total we obtain exactly five substitution invariant Sturmian words

$$\{21s_{\alpha,\alpha}, 12s_{\alpha,\alpha}, 2s_{\alpha,\alpha}, 1s_{\alpha,\alpha}, s_{\alpha,\alpha}\}$$

in the homogeneous case.

Let us note condition (ii) is also proved in [34] and in [16].

Proof. (i) Since ρ is a rational number, we have $\rho' = \rho$. Hence the condition of Condition (ii) of Theorem 1 is fulfilled if $\alpha' > 1$ or $\alpha' < 0$.

(ii) Let us first assume that $\alpha' > 1$. Let $n, p \in \mathbb{Z}$ such that $\rho = \{n\alpha\} = n\alpha - p$.

For $n = -1, 0, 1$, we have $\rho = 1-\alpha, 0, \alpha$ respectively, so that $\rho' = 1-\alpha', 0, \alpha'$. Hence $\rho' \in [1-\alpha', \alpha']$. Therefore $\overline{s}_{\alpha,\rho}$ and $\underline{s}_{\alpha,\rho}$ are substitution invariant.

For $n \ge 2$, $\rho' = n\alpha' - p > \alpha'$ since $p = [n\alpha] \le n-1$; for $n \le -2$, $\rho' = n\alpha' - p < 1-\alpha'$ since $p = [n\alpha] \ge n\alpha - 1 \ge n$. Therefore, $\overline{s}_{\alpha,\rho}$ and $\underline{s}_{\alpha,\rho}$ are not substitution invariant.

We deduce the case $\alpha' < 0$ by applying Remark 1. □

1.2. **Invertible substitutions.** Let σ be a substitution over $\{1,2\}$ and let $M_\sigma = (m_{ij})$ be its *incidence matrix*, where m_{ij} counts the number of occurrences of the letter i in $\sigma(j)$. We assume that $\det M_\sigma =$

± 1 (the substitution is said *unimodular*) and M_σ is *primitive* (M_σ^n has only positive entries for some non-negative integer n).

A substitution is said *invertible* if it is an automorphism of the free group generated by the alphabet $\{1,2\}$. Note that if σ is an invertible substitution, then its incidence matrix is unimodular.

Theorem B ([33]). *Every invertible substitution over $\{1,2\}$ is a composition of the following three invertible substitutions:*

$$1 \mapsto 2, 2 \mapsto 1;\ 1 \mapsto 12, 2 \mapsto 1;\ 1 \mapsto 21, 2 \mapsto 1. \tag{1}$$

Question 1 is related to invertible substitution according to the following well-known result (see for instance [23]).

Theorem 2. *A Sturmian word is substitution invariant if and only if it is a fixed point of some primitive and invertible substitution.*

Let us illustrate the main idea of the proof of Theorem 1 in [34]. According to the three substitutions in Theorem B, S. Ito and S. Yasutomi [21] define three transformations from $[0,1]^2$ to $[0,1]^2$, namely:

$$T_1(\alpha,\rho) = (\frac{\alpha}{1+\rho}, \frac{\rho}{1+\alpha}), \quad T_2(\alpha,\rho) = (\frac{1}{2-\alpha}, \frac{\rho}{2-\alpha}),$$
$$T_3(\alpha,\rho) = (1-\alpha, 1-\rho).$$

Then it is proved that a Sturmian sequence $s_{\alpha,\rho}$ is substitution invariant if and only if there exists a sequence $S_1,\ldots,S_n$ with $S_i \in \{T_1,T_2,T_3\}$ such that $(\alpha,\rho) = S_1 \circ \cdots \circ S_n(\alpha,\rho)$. Since there are three transformations, the task of determining such (α,ρ) is tedious. Yasutomi's original proof of Theorem 1 in [34] is somewhat technique and lengthy.

1.3. **Rauzy fractals: an alternative proof of Theorem 1.** Rauzy fractals (first introduced in [30] in the Tribonacci case) are compact attractors of a graph-directed iterated function system associated with a substitution of Pisot type.

Let us first describe an intuitive approach to Rauzy fractals for two-letter substitutions. We give a more formal definition in Section 2. Let σ be a primitive and unimodular substitution over $\{1,2\}$. Let $s = s_0s_1s_2\ldots$ be a periodic point of σ. Let $(1-\alpha,\alpha)$ be the eigenvector of M_σ corresponding to the Perron-Frobenius eigenvalue. We shall call α the *characteristic length* of the matrix M_σ or of the substitution σ, according to the context.

We define an oriented walk on the real line as follows. We start from the origin; in the n-th step, if $s_{n-1} = 1$, we move to the right side by

α; if $s_{n-1} = 2$, we move to the left side by $1 - \alpha$. Taking the closure of the orbit of the origin under this transformation, we obtain

$$X = \text{closure } \{|s_0 s_1 \dots s_{k-1}|_1 \cdot \alpha + |s_0 s_1 \dots s_{k-1}|_2 \cdot (\alpha - 1);\ k \geq 0\},$$

where $|s_0 s_1 \dots s_{n-1}|_j$ denotes the number of occurrences of the letter j in the word $s_0 s_1 \dots s_n$. Furthermore, we define

$$\begin{aligned} X_1 = \text{closure } \{|s_0 s_1 \dots s_{k-1}|_1 \cdot \alpha + & \ |s_0 s_1 \dots s_{k-1}|_2 \cdot (\alpha - 1); \\ & \ k \geq 0, s_k = 1\}, \\ X_2 = \text{closure } \{|s_0 s_1 \dots s_{k-1}|_1 \cdot \alpha + & \ |s_0 s_1 \dots s_{k-1}|_2 \cdot (\alpha - 1); \\ & \ k \geq 0, s_k = 2\}. \end{aligned} \tag{2}$$

We shall show in Section 2 that $X = X_1 \cup X_2$ is an affine image of the so-called Rauzy fractal. By abuse of language, we call X_1, X_2 in (2) the Rauzy fractals of σ; in particular, (2) is well defined for any Sturmian word s, and we also call X_1, X_2 the Rauzy fractals of the Sturmian word s.

One easily checks that the Rauzy fractals of the Sturmian word $s_{\alpha,\rho}$ are intervals equal to $X_1 = [-\rho, 1 - \alpha - \rho]$, $X_2 = [1 - \alpha - \rho, 1 - \rho]$. The periodic points of an invertible substitution are Sturmian (see Lemma 4), and hence the associated Rauzy fractals are intervals. Furthermore, one has

Theorem 3 ([12])**.** *Let σ be a primitive unimodular substitution over $\{1, 2\}$. Then the Rauzy fractals X_1, X_2 and $X_1 \cup X_2$ are intervals if and only if σ is invertible.*

A simple proof of this result is given in Section 3.

Since Theorem 1 is an elementary and important result, it is worth giving a proof that is more transparent and accessible. This is the main purpose of the present paper. Let us note that a geometric proof based on the use of cut-and-project schemes has also been given in [4].

Let us give a sketch of our proof. By Theorem 2 and Theorem 3, *a Sturmian word is substitution invariant if and only if it is a fixed point of some primitive substitution with connected Rauzy fractals.* The principal idea used here is to study the Rauzy fractals associated with invertible primitive substitutions following the approach of [15].

Let σ be an invertible substitution with characteristic length α. Then α is a Sturm number, and the Rauzy fractals X_1,X_2 are intervals with length $1 - \alpha$ and α respectively. Suppose $s = s_{\alpha,\rho}$ is a periodic point of σ, then $\rho = h + 1 - \alpha$, where $\{h\} = X_1 \cap X_2$.

Let L be the line $y = \frac{1-\alpha'}{\alpha'}$, where α' is the algebraic conjugate of α. A broken line in $\mathbb{R}^2$, the so-called stepped surface, is associated with the line L, defined as a discretization of L.

The sets X_1,X_2 have a self-similar structure, and the set equation is controlled by the stepped surface of L (see Lemma 5 and Theorem 4). Hence, by connectedness and self-similarity of the Rauzy fractals, we express the intersection $X_1 \cap X_2$ in terms of the stepped surface (see Theorem 5).

Then we show that the stepped surface is associated with the rotation R_γ with $\gamma = \frac{\alpha'-1}{2\alpha'-1}$, which may be considered as the *dual rotation* of R_α. An algebraic characterization of the stepped surface is obtained (see Theorem 7). This allows us to get an algebraic description of the intersection set $X_1 \cap X_2$ for an invertible substitution σ, which yields a proof of Theorem 1.

Let us note that Rauzy fractals have numerous applications in number theory, ergodic theory, dynamical systems, fractal geometry and tiling theory (see for instance [3, 18, 19, 20, 30, 32], and Chap. 7 in [28]). The present paper contains a new application of Rauzy fractals to Sturmian words.

This paper is organized as follows. We first recall in Section 2 some basic facts on Rauzy fractals. We then discuss in Section 3 the connectedness of Rauzy fractals for a two-letter alphabet. Theorem 3 is proved in this section. In Section 4, we study the set equations of Rauzy fractals, especially in the invertible case. The intersection set $X_1 \cap X_2$ for invertible substitutions is determined in Section 5. In Section 6, an algebraic characterization of the stepped surface is given. Theorem 1 of Yasutomi is proved in Section 7.

2. Rauzy fractals

In this section we recall some basic facts on Rauzy fractals. We present here all the definitions in the case of a two-letter alphabet since it is enough for our purpose. Let us note that the notation, which is adapted from [18], is slightly different from [3].

2.1. **Sturm numbers.** Let σ be a primitive unimodular substitution over $\{1, 2\}$. Let β be the maximal eigenvalue of its incidence matrix M_σ. Its algebraic conjugate β' is also an eigenvalue of M_σ. By Perron-Frobenius' theorem, we have $\beta > 1$. Now $\beta\beta' = \det M = \pm 1$ implies $|\beta'| < 1$. Therefore β is a Pisot number and the substitution σ is said to be of *Pisot type*.

It is well-known that the densities of letters exist in fixed points of primitive substitutions (see [29]). Furthermore the vector of densities of the letters 1 and 2 that we denote $(1-\alpha, \alpha)$, with $0 \leq \alpha \leq 1$, is easily seen to be an *expanding eigenvector*, that is, an eigenvector associated with the expanding eigenvalue β. The real number α is quadratic; the vector $(1-\alpha', \alpha')$ is an eigenvector associated with the eigenvalue β'. Still by Perron-Frobenius' theorem, the coordinates $1-\alpha'$, α' cannot be both positive, hence $\alpha'(1-\alpha') \leq 0$, which implies that $\alpha' \notin]0,1[$. Hence α is a Sturm number.

Conversely, any Sturm number is the characteristic length of a primitive unimodular matrix M of size 2×2. Indeed if α is a Sturm number, then $s_{\alpha,\alpha}$ is a fixed point of an invertible primitive substitution σ following Theorem A, and hence α is the characteristic length of M_σ. We thus have proved the lemma below. Let us note that in all that follows all *primitive matrices are assumed to be square matrices with non-negative integral entries.*

Lemma 1. *A number α is a Sturm number if and only if there exists a primitive unimodular matrix M of size 2×2 such that $(1-\alpha, \alpha)$ is an expanding eigenvector of M. In consequence, if the Sturmian word $s_{\alpha,\rho}$ is substitution invariant, then this implies that α is a Sturm number.*

2.2. **Definition of Rauzy fractals.** Let $\vec{e}_1, \vec{e}_2$ be the canonical basis of $\mathbb{R}^2$. Let $f:\ \{1,2\}^* \mapsto \mathbb{Z}^2$ be the *Parikh map*, also called *abelianization homomorphism*, defined by $f(w) = |w|_1\vec{e}_1 + |w|_2\vec{e}_2$, where $|w|_i$ denotes the number of occurrences of the letter i in w.

Denote by V be the expanding eigenspace of the matrix M_σ corresponding to the eigenvalue β, and V' the contracting eigenspace corresponding to β'. The expanding subspace is generated by the vector $\vec{v} = (1-\alpha, \alpha)$, therefore the contracting subspace is generated by the vector $\vec{v}' = (1-\alpha', \alpha')$. Then $V \oplus V' = \mathbb{R}^2$ is a direct sum decomposition of $\mathbb{R}^2$. According to this direct sum, two natural projections are defined:

$$\pi:\ \mathbb{R}^2 \to V' \quad \text{and} \quad \pi':\ \mathbb{R}^2 \to V.$$

We define the *Rauzy fractal* associated with σ as the closure of the projection according to π of the vertices of the broken line (illustrated in Figure 1) obtained by applying the map f to the prefixes of a given periodic point of σ.

More precisely, let $s = (s_k)_{k\geq 0}$ be a periodic point of σ. We first define

$$Y = \{f(s_0 \dots s_{k-1});\ k \geq 0\},$$

where the notation $s_0 \dots s_{k-1}$ stands for the empty word when $k = 0$. We then divide Y into two parts:

$$Y_1 = \{f(s_0 \dots s_{k-1});\ s_k = 1\},\ Y_2 = \{f(s_0 \dots s_{k-1});\ s_k = 2\}.$$

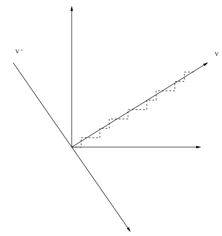

Figure 1. The broken line.

Projecting Y_1, Y_2 onto the contracting eigenspace P and taking its closure, we get

$$\vec{X}_1 = \overline{\pi(Y_1)},\ \vec{X}_2 = \overline{\pi(Y_2)}.$$

We call $\vec{X}_1$ and $\vec{X}_2$ the *Rauzy fractals* of the substitution σ. It is shown that the Rauzy fractals are independent of the choice of the periodic point in the definition, according to [18].

Clearly the Rauzy fractals $\vec{X}_1$ and $\vec{X}_2$ are one-dimensional objects. An easy computation shows that

$$X_1 = \phi(\vec{X}_1), \quad X_2 = \phi(\vec{X}_2),$$

where X_1, X_2 are defined in (2) and ϕ is the linear map defined by

$$\phi:\ V' \to \mathbb{R},\ \phi(\frac{x\vec{v}'}{\alpha - \alpha'}) = x.$$

By abuse of language, we also call X, X_1 and X_2 the *Rauzy fractals* of the substitution σ.

Barge and Diamond showed in [5] that every Pisot substitution over a two-letter alphabet satisfies a certain combinatorial condition, called the *strong coincidence condition*. Thanks to this, one can show that

Lemma 2 ([18]). *Let σ be a primitive Pisot substitution over two letters. Then*

$$\mu(X_1) = 1 - \alpha, \quad \mu(X_2) = \alpha,$$

where μ is the Lebesgue measure and α is the characteristic length of σ.

3. Connectedness of Rauzy fractals

3.1. **Proof of Theorem 3.** It is in general hard to decide whether the Rauzy fractals are connected (see for instance [1, 12]). However, in the two-letter case we have a complete characterization given by Theorem 3. We provide an elementary proof of this folklore result. We first need the following lemmas.

Lemma 3 ([24, 33, 7])**.** *Let σ be a non-trivial substitution over $\{1,2\}$. The following three conditions are equivalent:*

(i) σ is primitive invertible;

(ii) for any Sturmian word s, $\sigma(s)$ is still a Sturmian word;

(iii) there exists a Sturmian word s such that $\sigma(s)$ is a Sturmian word.

The equivalence between (i) and (ii) is due to [24] and [33], the equivalence with (iii) is proved in [7]. For more details, see [23].

Lemma 4. *Let σ be a non-trivial substitution over $\{1,2\}$. The following statements are equivalent:*

(i) σ is a primitive invertible substitution;

(ii) all periodic points of σ are Sturmian words;

(iii) there exists a periodic point of σ which is a Sturmian word.

Proof. (i) $\Rightarrow$ (ii). Suppose σ is a primitive invertible substitution. Let s be a periodic point, say, $\sigma^k(s) = s$, with $k \geq 1$. Let s' be any Sturmian word with the same initial letter as s. Then by Lemma 3, $(\sigma^k)^n(s')$ is a Sturmian word for any $n \geq 1$. But the sequence of words $(\sigma^k)^n(s')_{n\geq 1}$ converges to s. Hence s has at most $n+1$ factors of length n. Since σ is both unimodular and primitive, we infer that the density of letter 1 in s is irrational. Hence s is aperiodic, and it is a Sturmian word.

(ii) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$ (i). Let s be a Sturmian periodic point of σ, say, $\sigma^k(s) = s$, with $k \geq 1$. The condition (iii) of Lemma 3 is fulfilled and σ^k is invertible. So σ is also invertible. □

Proof of Theorem 3. Let σ be a primitive unimodular substitution over $\{1,2\}$. We first assume that the Rauzy fractals of σ, namely X_1, X_2, and $X = X_1 \cup X_2$, are intervals. Let $s = (s_k)_{k\geq 0}$ be a periodic point of σ which defines X_1 and X_2. Then according to Lemma 2, $|X_1| = 1-\alpha$ and $|X_2| = \alpha$. Hence s is the coding of the orbit of an irrational rotation so that it is a Sturmian word. We thus deduce that σ is invertible from Lemma 4.

Conversely, if σ is primitive invertible, then it has at least one periodic point s and it is Sturmian. Hence the Rauzy fractals are intervals, according to Section 1. □

Corollary 2. *Let σ be a primitive invertible substitution. Then there exists $h \in \mathbb{Z}$ such that the Rauzy fractals satisfy*

$$X_1 = [-1+\alpha+h, h], \quad X_2 = [h, \alpha+h],$$

where α is the characteristic length of σ.

3.2. **Upper and lower Sturmian sequences.** In this subsection we show that $\underline{s}_{\alpha,\rho}$ is substitution invariant if and only if $\overline{s}_{\alpha,\rho}$ is also substitution invariant.

Proposition 1. *Let $0 < \alpha < 1$ be an irrational number and $0 < \rho < 1$. Let σ be a non-trivial substitution. Then $\underline{s}_{\alpha,\rho}$ is a fixed point of σ if and only $\overline{s}_{\alpha,\rho}$ is also a fixed point of σ.*

Proof. Suppose $\underline{s}_{\alpha,\rho} = s_0 s_1 s_2 \dots$ is a fixed point of a non-trivial substitution σ. According to Lemma 3, σ is primitive invertible. Let $X_1 = [-\rho, 1-\alpha-\rho]$ and $X_2 = [1-\alpha-\rho, 1-\rho]$ be the associated Rauzy fractals of $s = \underline{s}_{\alpha,\rho}$.

If the orbit of the oriented walk of s does not contain $X_1 \cap X_2$, then $\underline{s}_{\alpha,\rho} = \overline{s}_{\alpha,\rho}$. In this case, there is nothing to prove.

Therefore we assume that $\underline{s}_{\alpha,\rho} \neq \overline{s}_{\alpha,\rho}$. Then the orbit of the oriented walk of s must contain h, i.e., there exists a nonnegative integer n such that $f(s_0 s_1 \dots s_{n-1}) = h$. One has either

$$\begin{aligned} \underline{s}_{\alpha,\rho} &= s_0 \dots s_{n-1} 21 s_{n+2} \dots = s_0 \dots s_{n-1} 21 s_{\alpha,\alpha}, \\ \overline{s}_{\alpha,\rho} &= s_0 \dots s_{n-1} 12 s_{n+2} \dots = s_0 \dots s_{n-1} 12 s_{\alpha,\alpha} \end{aligned} \tag{3}$$

or

$$\begin{aligned} \underline{s}_{\alpha,\rho} &= 1 s_{\alpha,\alpha}, \\ \overline{s}_{\alpha,\rho} &= 2 s_{\alpha,\alpha}. \end{aligned} \tag{4}$$

Let us denote $s' = \overline{s}_{\alpha,\rho}$. We assume that we are in case (3), the case (4) can be handled in the same way.

We claim that *the Rauzy fractals of $\sigma(s')$ are also the intervals X_1, X_2.* Indeed, by Lemma 3, $\sigma(s')$ is Sturmian so that the Rauzy fractals of $\sigma(s')$ are intervals. It is shown in [15] (as a consequence of Theorem B) that if σ is invertible, then there exist two words w and u such that either $\sigma(12) = w12u$, $\sigma(21) = w21u$, or $\sigma(12) = w21u$, $\sigma(21) = w12u$. Hence there exist a finite word w and an infinite word t such that

$$\sigma(s) = w12t,\ \sigma(s') = w21t \ \text{ or } \ \sigma(s) = w21t, \ \ \sigma(s') = w12t. \tag{5}$$

Hence $X(\sigma(s))$ (which is equal to $X(s)$) and $X(\sigma(s'))$ differ at one point at most. Therefore $X(s)$ and $X(\sigma(s'))$ coincide since they are intervals. Our claim is proved.

By (3) and (5), one has $X_1 \cap X_2 = \{f(s_0 s_1 \dots s_{n-1})\} = \{f(w)\}$. Since $s_0 s_1 \dots s_{n-1}$ is a prefix of w and α is irrational, we deduce from $f(s_0 s_1 \dots s_{n-1}) = f(w)$ that $w = s_0 s_1 \dots s_{n-1}$. This is possible only if $s_0 s_1 \dots s_{n-1}$ is the empty word. Again by (3) and (5), we either have

$$s = \sigma(s) = 12t, \ \ \sigma(s') = 21t = s' \ \text{ or } \ \sigma(s) = 21t, \ \ \sigma(s') = 12t = s'$$

Hence s' is a fixed point of σ. □

Corollary 3. *Let $0 < \alpha < 1$ be an irrational number and $0 \leq \rho \leq 1$. Then $\underline{s}_{\alpha,\rho}$ is a periodic point of a non-trivial substitution σ if and only $\overline{s}_{\alpha,\rho}$ is also a periodic point of σ.*

Proof. Let $\underline{s}_{\alpha,\rho}$ be a periodic point that is not a fixed point of σ. Since we are on a two-letter alphabet, then there exist two words w, u such that σ satisfies $\sigma(1) = 2w$ and $\sigma(2) = 1u$. Hence $\underline{s}_{\alpha,\rho}$ is a fixed point of σ^2, so that $\overline{s}_{\alpha,\rho}$ is also a fixed point of σ^2 by Proposition 1, meaning that $\overline{s}_{\alpha,\rho}$ is a periodic point of σ. □

4. Self-similarity of Rauzy fractals

In this section, we discuss the self-similar structure of the Rauzy fractals X_1 and X_2; special attention is payed to the case σ invertible. The stepped surface is shown to play an important role.

4.1. **Set equations of Rauzy fractals.**

Let σ be a primitive substitution over $\{1, 2\}$ and let β be the Perron-Frobenius eigenvalue of M_σ.

It is well-known ([3],[32], [18]) that $\vec{X}_1$ and $\vec{X}_2$, and thus X_1 and X_2, have a self-similar structure, that is, both $\frac{1}{\beta'}X_1$ and $\frac{1}{\beta'}X_2$ are unions of translated copies of X_1 and X_2. In oder to describe the corresponding set equations, we introduce the following notation: let D_1 (resp. D_2) be the set of those $(a, i) \in \mathbb{R}^2 \times \{1, 2\}$ such $X_i + a \subset \frac{1}{\beta'}X_1$ (resp. $X_i + a \subset \frac{1}{\beta'}X_2$), that is,

$$\frac{1}{\beta'}X_1 = \bigcup_{(a,i)\in D_1} X_i + a, \quad \frac{1}{\beta'}X_2 = \bigcup_{(b,i)\in D_2} X_i + b.$$

For the explicit form of D_1, D_2, we refer to [3].

4.2. **The stepped surface.**

Recall that V' is the contracting eigenline of M_σ. We denote the upper closed half-plan delimited by V' as $(V')^+$, and the lower open half-plane delimited by V' as $(V')^-$. We define

$$S = \{[z, i^*]; \quad z \in \mathbb{Z}^2, z \in V^+ \text{ and } z - \vec{e}_i \in V^-\},$$

where the notation $[z, i^*]$, for $z \in \mathbb{Z}^n$ and $i^* \in \{1^*, 2^*\}$ endows the point z in $\mathbb{Z}^n$ with color $i^* = 1^*, 2^*$. Intuitively S consists of the the collection of those colored points $[z, i^*]$ which are close to the contracting eigenline V'.

We now define $\overline{[z,1^*]}$ (resp. $\overline{[z,2^*]}$) as the closed line segment from z to $z+\vec{e}_2$ (resp. to $z+\vec{e}_1$) (see Figure 2). Then the *stepped surface* $\overline{S}$ of V' is defined as the broken line consisting of the following segments

$$\overline{S} = \bigcup_{[z,i^*]\in S} \overline{[z,i^*]}.$$

A piece of a stepped surface is depicted in Figure 3. By abuse of language, the formal set S will also be called the stepped surface of V'.

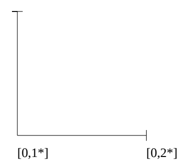

FIGURE 2. The segments $\overline{[0,1^*]}$ and $\overline{[0,2^*]}$.

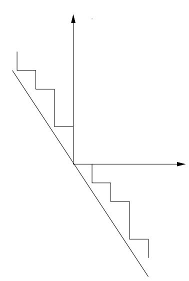

FIGURE 3. A piece of the stepped surface for $1 \mapsto 12112$, $2 \mapsto 121$.

It turns out that the set equations of the Rauzy fractals are controlled by the stepped surface. An explicit expression of the sets D_1 and D_2 is given in [3], from which one immediately deduces the following facts:

Lemma 5 ([3, 18])**.** *Using the notation above:*

i) for any $(a,i) \in D_1 \cup D_2$*, there exists an element* $[z,i^*] \in S$ *such that* $\phi \circ \pi(z) = a$*;*

ii) $(0,1),(0,2) \in D_1 \cup D_2$;

iii) $(n_{ij})_{1\le i\le 2} = {}^tM_\sigma$*, where* n_{ij} *counts the number of elements* (a,j) *in the set* D_i*.*

4.3. **A tiling associated with the stepped surface.** Projecting the stepped surface $\overline{S}$ onto V', we first obtain a tiling $\mathcal{J}'$ of V':

$$\mathcal{J}' = \{\pi(\overline{[z,i^*]});\ [z,i^*] \in S\}.$$

Applying the linear transformation ϕ, we then get a tiling $\mathcal{J}$ of the real line:

$$\mathcal{J} = \{\phi \circ \pi(\overline{[z, i^*]});\ [z, i^*] \in S\}.$$

The tiling $\mathcal{J}$ is a quasi-periodic tiling with two prototiles. Indeed

$$\mathcal{J} = \{\phi \circ \pi(z) + J_i;\ [z, i^*] \in S\},$$

where

$$J_1 = \phi \circ \pi\overline{[0, 1^*]} = [-1 + \alpha, 0],\ J_2 = \phi \circ \pi\overline{[0, 2^*]} = [0, \alpha].$$

We label the tiles of $\mathcal{J}$ on the right side of the origin by the sequence $T_0, T_1, T_2, \ldots$, where T_{n+1} is the rightside neighbour of T_n. Likewise we label the tiles of $\mathcal{J}$ on the left side of the origin by $T_{-1}, T_{-2}, \ldots$. One has $\mathcal{J} = \{T_k;\ k \in \mathbb{Z}\}$. We furthermore define the two-sided sequence $(g_k)_{k\in\mathbb{Z}}$ as the sequence of left endpoints of the tiles T_k. An arithmetic description of the sequence $(g_k)_{k\in\mathbb{Z}}$ is given in Section 6.

4.4. **Set equations of connected Rauzy fractals.** According to Corollary 2, if σ is a primitive invertible substitution, then there exists a real number h such that $X_1 = [-1 + \alpha + h, h], X_2 = [h, h + \alpha]$, that is,

$$X_1 = J_1 + h, \quad X_2 = J_2 + h,$$

where $J_1 = [-1 + \alpha, 0]$ and $J_2 = [0, \alpha]$ are the two prototiles of the tiling $\mathcal{J}$.

Suppose $X_i + a$ is a piece in the subdivision of $\frac{X_1}{\beta'}$ according to the set equations, that is, $(a, i) \in D_1$; there exists an element $[z, i^*] \in S$ such that $\phi \circ \pi(z) = a$ by Lemma 5. Let $k \in \mathbb{Z}$ such that $\phi \circ \pi\overline{[z, i^*]} = T_k$; then

$$X_i + a = J_i + h + a = T_k + h.$$

We thus can introduce two subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathcal{J}$ such that

$$\frac{X_1}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_1} T\right) + h, \quad \frac{X_2}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_1} T\right) + h.$$

On the one hand, the tiles in $\mathcal{D}_1 \cup \mathcal{D}_2$ do not overlap according to [5] and [3]. On the other hand, these tiles must form a connected patch of $\mathcal{J}$ since $X_1, X_2, X_1 \cup X_2$ are intervals, according to Theorem 3. Hence we have proved that

Theorem 4. *Let $X_1 = [-1 + \alpha + h, h], X_2 = [h, h + \alpha]$ be the Rauzy fractals of the primitive invertible substitution σ. Then*

$$\frac{X_1}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_1} T\right) + h, \quad \frac{X_2}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_2} T\right) + h,$$

where $\mathcal{D}_1, \mathcal{D}_2$ *and* $\mathcal{D}_1 \cup \mathcal{D}_2$ *are connected patches of the tiling* $\mathcal{J}$.

5. Invertible substitutions with a given incidence matrix

In this section, we give a more detailed description of the Rauzy fractals of invertible substitutions with a given incidence matrix.

5.1. A list of invertible substitutions with a given incidence matrix.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix. A very interesting result on invertible substitutions is given in [31]:

Lemma 6. [31] *Let* M *be a primitive unimodular matrix. The number of invertible substitutions with incidence matrix* M *is equal to* $a+b+c+d-1$.

Let σ be an invertible substitution with incidence matrix $M_\sigma = M$. According to Lemma 5, $(0,1),(0,2) \in D_1 \cup D_2$, so that we have

$$T_{-1}, T_0 \in \mathcal{D}_1 \cup \mathcal{D}_2. \tag{6}$$

By Lemma 5 (iii), we have that

$$\mathrm{Card}\mathcal{D}_1 = \mathrm{Card}D_1 = a+b, \quad \mathrm{Card}\mathcal{D}_2 = \mathrm{card}D_2 = c+d. \tag{7}$$

Let us assume that the determinant of M is equal to 1. We will not need to consider in the sequel the case $\det(M) = -1$, but a similar study can be conducted. In this case $1/\beta' = \beta > 0$ so that $\frac{X_1}{\beta'}$ is on the left side of $\frac{X_2}{\beta'}$. Hence by Theorem 4, the patch $\mathcal{D}_1$ is on the left side of $\mathcal{D}_2$. By Theorem 4, (6) and (7), we infer that there exists k with $1 \le k \le a+b+c+d-1$ such that

$$\begin{aligned} \mathcal{D}_1 &= \{T_{-k}, T_{-k+1}, \dots, T_{-k+a+b-1}\}, \\ \mathcal{D}_2 &= \{T_{-k+a+b}, T_{-k+a+b+1)}, \dots, T_{-k+a+c+b+d-1}\}. \end{aligned} \tag{8}$$

Hence there are at most $a+b+c+d-1$ invertible substitutions with incidence matrix M, and their set equations are deduced from (8). On the other hand, Lemma 6 asserts that there are exactly $a+b+c+d-1$ such substitutions. Since the set equations for different substitutions are distinct, we conclude that there is a one-to-one correspondence between the invertible substitutions with incidence matrix M and the set equations determined by (8). We denote these substitutions by σ_k, $1 \le k \le a+b+c+d-1$.

5.2. **Intersection point of Rauzy fractals.** For each of the substitutions σ_k defined in the previous section, there exists ρ_k such that s_{α,ρ_k} (which means indifferently either $\overline{s}_{\alpha,\rho_k}$ or $\underline{s}_{\alpha,\rho_k}$) is a periodic point of σ_k, according to Lemma 4.

Let $1 \le k \le a+b+c+d-1$. Let $X_1 = [-1+\alpha+h_k, h_k]$, $X_2 = [h_k, \alpha+h_k]$ be the Rauzy fractals of σ_k. One has $\rho_k = 1-\alpha-h_k$. We use below the connectedness and the self-similarity of the Rauzy fractals to determine h_k and thus ρ_k. Let us recall that $(g_k)_{k\in\mathbb{Z}}$ denote the sequence of left endpoints of the tile T_k in $\mathcal{J}$.

Theorem 5. *Let M be a primitive matrix with* $\det M = 1$. *Let σ_k, $1 \le k \le a+b+c+d-1$, be the invertible substitutions with incidence matrix M. Let β be the maximal eigenvalue of M. We assume* $\det M = 1$. *Then*

$$h_k = \frac{g_{-k+a+b}}{\beta - 1}.$$

Proof. On the one hand, $\frac{X_1}{\beta'} \cap \frac{X_1}{\beta'} = \{(\beta')^{-1}h_k\} = \{\beta h_k\}$. On the other hand, this intersection point is the right end-point of the interval $\cup\{T + h_k;\ T \in \mathcal{D}_1\}$, that is, the right end-point of $T_{-k+a+b-1} + h_k$. So we get $g_{-k+a+b} + h_k = \beta h_k$, and $h_k = \frac{g_{-k+a+b}}{\beta-1}$. □

Theorem 6. *Let M be a primitive matrix with* $\det M = 1$. *Let $\sigma_1, \sigma_2, \ldots, \sigma_{a+b+c+d-1}$ be the invertible substitutions with incidence matrix M. Let $G := \{g_k;\ k \in \mathbb{Z}\}$. Then the Sturmian word $s_{\alpha,\rho}$ is a periodic point of the substitution σ_k if and only if*

$$0 \le \rho \le 1 \text{ and } (\rho + \alpha - 1) \in \frac{G}{1-\beta}.$$

Proof. By Theorem 5, one has $h_{a+b+c+d-1} < \cdots < h_2 < h_1$. Hence a real number h belongs to the set $\{h_1, h_2, \ldots, h_{a+b+c+d-1}\}$ if and only if

$$h \in \frac{G}{\beta-1} \text{ and } h_{a+b+c+d-1} \le h \le h_1. \tag{9}$$

It remains to determine the values h_1 and $h_{a+b+c+d-1}$. For the substitution σ_1, the set $\mathcal{D}_1$ is equal to $\{T_{-1}, T_0, \ldots, T_{a+b-2}\}$. By Lemma 5, the numbers of tiles in $\mathcal{D}_1$ of length $1-\alpha$ and α are a and b respectively. Since $|T_1| = 1-\alpha$, we have

$$g_{a+b-1} = (a-1)(1-\alpha) + b\alpha = (\beta-1)(1-\alpha).$$

Here we use the equality $a(1-\alpha)+b\alpha = \beta(1-\alpha)$, which follows from the fact that $(1-\alpha,\alpha)$ is an expanding eigenvector of M. Therefore $h_1 = 1-\alpha$. A similar argument shows that $h_{a+b+c+d-1} = -\alpha$.

Remember now that $\rho_k = 1-\alpha-h_k$. The theorem follows from (9). □

6. The stepped surface

In this section, we give an arithmetic description of the stepped surface. We first define the two-sided word $(t_n)_{n\in\mathbb{Z}}$ as:

$$\forall n \in \mathbb{Z},\ t_n = \begin{cases} 1, & \text{if } |T_n| = 1-\alpha \\ 2, & \text{if } |T_n| = \alpha. \end{cases}$$

It is well known that Sturmian words can also be described as cutting sequences, see for instance [23]. One checks acccording to [3] that $(t_n)_{n\in\mathbb{Z}}$ is the upper two-sided cutting sequence of the line $V' : y = \frac{\alpha'-1}{\alpha'}$. Hence

$$t_{-1}t_{-2}t_{-3}\cdots = 1s_{\gamma,\gamma}, \quad t_0t_1t_2\cdots = 2s_{\gamma,\gamma}, \tag{10}$$

where

$$\gamma = \frac{\alpha'-1}{2\alpha'-1}. \tag{11}$$

Let $R_\gamma : x \mapsto x+\gamma$ be the rotation of the torus $\mathbb{T}^1$ of angle γ. We deduce from (10) that for all k

$$|t_{-1}t_{-2}\dots t_{-k}|_1 \cdot \gamma + |t_{-1}t_{-2}\dots t_{-k}|_2 \cdot (\gamma-1) = R_\gamma^k(0)$$

$$|t_0t_1\dots t_{k-1}|_1 \cdot \gamma + |t_0t_1\dots t_{k-1}|_2 \cdot (\gamma-1) = R_\gamma^k(0).$$

By definition of $(g_k)_{k\in\mathbb{Z}}$, one has for every nonnegative k

$$g_{-k} = -|t_{-1}t_{-2}\dots t_{-k}|_1 \cdot (1-\alpha) - |t_{-1}t_{-2}\dots t_{-k}|_2 \cdot \alpha,$$

$$g_k = |t_0t_1\dots t_{k-1}|_1 \cdot (1-\alpha) + |t_0t_1\dots t_{k-1}|_2 \cdot \alpha.$$

Hence

$$\forall k \in \mathbb{Z},\ \frac{g'_k}{2\alpha'-1} = R_\gamma^{-k}(0), \tag{12}$$

where g'_k denotes the conjugate of g_k. This thus provides an arithmetic description of the stepped surface.

Recall that $G := \{g_k;\ k\in\mathbb{Z}\}$, where the sequence (g_k) is defined in Section 5.2. Denote $\mathbb{Z}[\alpha] := \{m\alpha+n;\ m,n\in\mathbb{Z}\}$.

Theorem 7. *One has*

$$G = \{g\in\mathbb{Z}[\alpha];\ 0 \le g' < 2\alpha'-1\} \text{ when } \alpha' > 1,$$
$$G = \{g\in\mathbb{Z}[\alpha];\ 2\alpha'-1 < g' \le 0\} \text{ when } \alpha' < 0.$$

Proof. We assume that $\alpha' > 1$. The case $\alpha' < 0$ can be handled similarly. Notice that

$$\{R_\gamma^k(0);\ k\in\mathbb{Z}\} = \{m\gamma+n;\ 0 \le m\gamma+n < 1\}.$$

This together with (12) imply that

$$\begin{aligned} G &= \{g;\ g' = m(\alpha'-1)+n(2\alpha'-1);\ m,n\in\mathbb{Z},\ 0\le g' < 2\alpha'-1\} \\ &= \{g;\ g = m(\alpha-1)+n(2\alpha-1);\ m,n\in\mathbb{Z},\ 0\le g' < 2\alpha'-1\} \\ &= \{g\in\mathbb{Z}[\alpha];\ 0\le g' < 2\alpha'-1\}. \end{aligned}$$

□

Remark 2. For a Sturm number α, it is easy to check that $\gamma = \frac{\alpha'-1}{2\alpha'-1}$ is also a Sturm number. We say that γ is the *dual* of α. One checks that γ and α are dual of each other. In some sense, the rotation R_γ is the dual rotation of R_α.

7. Proof of Theorem 1

In this section, we prove Theorem 1.

Theorem 1. (Yasutomi [34]) *Let $0<\alpha<1$ and $0\le\rho\le 1$. Then $s_{\alpha,\rho}$ is substitution invariant if and only if the following two conditions are satisfied:*

(i) α is an irrational quadratic number and $\rho\in\mathbb{Q}(\alpha)$;

(ii) $\alpha'>1,\ 1-\alpha'\le\rho'\le\alpha'$ or $\alpha'<0,\ \alpha'\le\rho'\le 1-\alpha'$.

7.1. An algebraic lemma.

We first need a preliminary lemma.

Lemma 7. *Let β be a quadratic algebraic unit, and α be an irrational number in $\mathbb{Q}(\beta)$. Then for any $\rho\in\mathbb{Q}(\beta)$, there exists an arbitrary large even number n such that $\rho(\beta^n-1)\in\mathbb{Z}[\alpha]$.*

Proof. Let $\mathcal{A}$ denote the ring of algebraic integers in $\mathbb{Q}(\beta)$. First we claim that for any $\rho\in\mathbb{Q}(\beta)$, there exists an arbitrary large even number n such that $\rho(\beta^n-1)\in\mathcal{A}$. Let $\alpha\in\mathcal{A}$ such that $\alpha\rho\in\mathcal{A}$. Then at least two terms in the sequence $(\alpha\rho\beta^n)_{n\ge 0}$ belong to the same residue class modulo the principal ideal of $\mathcal{A}$ generated by α. Hence $\alpha\rho(\beta^{n_1}-\beta^{n_2})$ is divisible by α for some $n_1>n_2$. Since β is an algebraic unit, $\alpha\rho(\beta^n-1)$ is also divisible by α for $n=n_1-n_2$. This proves our claim. Let us note furthermore that obviously we can choose n to be an even number. We thus have proved that for every $N>0$,

$$\mathbb{Q}(\beta) = \bigcup_{n\ge N}\frac{\mathcal{A}}{\beta^{2n}-1}.$$

We then prove that there exists a rational number K such that $\mathcal{A}\subset K\mathbb{Z}[\alpha]$. Indeed let d be the square-free integer such that $\mathbb{Q}(\beta)=\mathbb{Q}(\sqrt{d})$. Then there exist integers a,b and $c\ne 0$ such that $\alpha=\frac{a+b\sqrt{d}}{c}$.

Notice that $b \neq 0$ since α is irrational. It is well known that any element in $\mathcal{A}$ must have the form $(m\sqrt{d}+n)/2$. Since

$$(m\sqrt{d}+n)/2 = \frac{mc\alpha - ma + nb}{2b}$$

is an element of $\frac{\mathbb{Z}[\alpha]}{2b}$, our assertion is true by taking $K = \frac{1}{2b}$.

Therefore for any $N > 0$, we have

$$\mathbb{Q}(\beta) = \bigcup_{n \geq N} \frac{\mathcal{A}}{\beta^{2n}-1} \subset K \bigcup_{n \geq N} \frac{\mathbb{Z}[\alpha]}{\beta^{2n}-1} \subseteq \mathbb{Q}(\beta).$$

Multiplying every term of the above formula by K^{-1}, we obtain

$$\mathbb{Q}(\beta) = \bigcup_{n \geq N} \frac{\mathbb{Z}[\alpha]}{\beta^{2n}-1}. \quad \square$$

7.2. **Proof of Theorem 1.** Now we are in position to prove Theorem 1.

Necessity. Let us suppose that $s_{\alpha,\rho}$ is a fixed point of the non-trivial primitive invertible substitution σ. Let β be the maximal eigenvalue of M_σ. We may assume that $\det M = 1$, for otherwise we consider σ^2 instead of σ since $s_{\alpha,\rho}$ is also a fixed point of σ^2.

By Lemma 1, α must be a Sturm number. From Theorem 6, we deduce

$$1 - \alpha - \rho = h \in \frac{G}{\beta - 1} \subseteq \frac{\mathbb{Z}[\alpha]}{\beta - 1} \subseteq \mathbb{Q}(\beta).$$

Hence $\rho \in \mathbb{Q}(\beta) = \mathbb{Q}(\alpha)$, so that condition (i) is necessary.

Concerning (ii), we need only to consider the case $\alpha' > 1$ according to Remark 1. Notice that $s_{\alpha,\rho}$ is also a fixed point of σ^n, for any $n \geq 1$, and in particular for any even number n; furthermore, the substitutions σ^n share the same stepped surface. Hence

$$\rho + \alpha - 1 \in \frac{G}{1 - \beta^n},$$

$$\rho' + \alpha' - 1 \in \frac{\{g';\ g \in G\}}{1 - (\beta')^n}.$$

By Theorem 7, we have

$$0 \leq \rho' + \alpha' - 1 < \frac{2\alpha' - 1}{1 - (\beta')^n}.$$

Notice that the above formula holds for every even number n. Letting n tend to infinity, $(\beta')^n$ vanishes, and we conclude that $1 - \alpha' \leq \rho' \leq \alpha'$.

Sufficiency. Suppose that (α, ρ) satisfies (i) and (ii). According to Remark 1, we may assume that $\alpha' > 1$ so that $\rho' + \alpha' - 1 \in [0, 2\alpha' - 1]$. Since α is a Sturm number, there exists a primitive substitution σ such that $s_{\alpha,\alpha}$ is fixed point of σ (Theorem A). Let β the maximal eigenvalue of the incidence matrix M_σ. We may assume that $\det M_\sigma = 1$, otherwise we consider σ^2 instead of σ.

Obviously $\alpha \in \mathbb{Q}(\beta)$. Condition (i) implies that $\rho \in \mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. Hence $\rho + \alpha - 1 \in \mathbb{Q}(\beta)$ so that by Lemma 7, there exist an even number n and $g \in \mathbb{Z}[\alpha]$ such that

$$\rho + \alpha - 1 = \frac{g}{1 - \beta^n}.$$

Let us prove that g is actually an element of G. The assumptions $\alpha' > 1$ and $1 - \alpha' \leq \rho' \leq \alpha'$ imply that $0 \leq \rho' + \alpha' - 1 \leq 2\alpha' - 1$. Now $0 < 1 - (\beta')^n < 1$ since n is even. Hence

$$g' = (\rho' + \alpha' - 1)(1 - (\beta')^n) \in [0, 2\alpha' - 1)$$

so that $g \in G$ by Theorem 7. We thus have proved that $\rho + \alpha - 1 \in \frac{G}{1-\beta^n}$. This together with $0 \leq \rho \leq 1$ imply that $s_{\alpha,\rho}$ is substitution invariant (by Theorem 6). □

Acknowledgement The authors thank P. Arnoux, J. Berstel, A. Rémondière, J.-i. Tamura and Z.-Y. Wen for many valuable discussions.

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