ON SUBSTITUTION INVARIANT STURMIAN WORDS: 
AN APPLICATION OF RAUZY FRACTALS

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Abstract. Sturmian words are infinite words that have exactly 
\(n + 1\) factors of length \(n\) for every positive integer \(n\). A Sturmian 
word \(s_{\alpha,\rho}\) is also defined as a coding over a two-letter alphabet of 
the orbit of the point \(\rho\) under the action of the irrational rotation 
\(R_\alpha: x \mapsto x + \alpha \pmod{1}\). Yasutomi characterized in [34] all the 
pairs \((\alpha, \rho)\) such that the Sturmian word \(s_{\alpha,\rho}\) is a fixed point of 
some non-trivial substitution. By investigating the Rauzy fractals 
associated with invertible substitutions, we give an alternative 
geometric proof of Yasutomi’s characterization.

1. Introduction

1.1. Sturmian words and substitution invariance. Sturmian words 
are infinite words over a binary alphabet, say, \(\{1, 2\}\), that have exactly 
\(n + 1\) factors of length \(n\) for every positive integer \(n\). Sturmian words 
can also be defined in a constructive way as follows. Let \(0 < \alpha < 1\). 
Let \(T^1 = \mathbb{R}/\mathbb{Z}\) denote the one-dimensional torus. The rotation of angle 
\(\alpha\) of \(T^1\) is defined by \(R_\alpha: T^1 \to T^1, x \mapsto x + \alpha\). For a given real 
number \(\alpha\), we introduce the following two partitions of \(T^1\):

\[
I_1 = [0, 1 - \alpha), \quad I_2 = [1 - \alpha, 1); \quad I_1 = (0, 1 - \alpha], \quad I_2 = (1 - \alpha, 1].
\]

Tracing the orbit of \(R^n_\alpha(\rho)\), we define two infinite words:

\[
\underline{s}_{\alpha,\rho}(n) = \begin{cases} 
1 & \text{if } R^n_\alpha(\rho) \in I_1, \\
2 & \text{if } R^n_\alpha(\rho) \in I_2,
\end{cases}
\]

\[
\overline{s}_{\alpha,\rho}(n) = \begin{cases} 
1 & \text{if } R^n_\alpha(\rho) \in \overline{I}_1, \\
2 & \text{if } R^n_\alpha(\rho) \in \overline{I}_2.
\end{cases}
\]

It is well known ([13, 25]) that an infinite word is a Sturmian word 
if and only if it is equal either to \(\overline{s}_{\alpha,\rho}\) or to \(\underline{s}_{\alpha,\rho}\) for some irrational 
number \(\alpha\). The word \(\underline{s}_{\alpha,\rho}\) is called lower Sturmian word whereas the 
word \(\overline{s}_{\alpha,\rho}\) is called upper Sturmian word. The notation \(s_{\alpha,\rho}\) stands in

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all that follow indifferently for $\overline{s}_{\alpha,\rho}$ or for $s_{\alpha,\rho}$ when there is no need to distinguish between both. A detailed description of Sturmian words can be found in [23], see also [28].

Let $\{1, 2\}^*$ be the free monoid over $\{1, 2\}$ endowed with the concatenation operation. A non-erasing homomorphism $\sigma$ of the free monoid $\{1, 2\}^*$ is called a substitution. An infinite word $s \in \{1, 2\}^\mathbb{N}$ is a fixed point of the substitution $\sigma$ if $\sigma(s) = s$; it is called a periodic point of $\sigma$ if $\sigma^k(s) = s$ for some $k \geq 1$.

It is well known that the famous Fibonacci word, that is, the fixed point of the Fibonacci substitution $1 \mapsto 12, 2 \mapsto 1$, is a Sturmian word. It is thus natural to ask when a Sturmian sequence is a fixed point of some non-trivial substitution. More precisely, we want to know

**Question 1.** For which $\alpha$ and $\rho$, is the Sturmian word $s_{\alpha,\rho}$ (resp. $\overline{s}_{\alpha,\rho}$) a fixed point of some non-trivial substitution?

By non-trivial substitution, we mean here a substitution that is distinct from the identity. In all that follows, we say that a Sturmian word is substitution invariant if it is a fixed point of a non-trivial substitution.

There is a large literature devoted to Question 1. The first step has been made in [14] (Theorem A below). When $\rho = \alpha$, we have $s_{\alpha,\alpha} = \overline{s}_{\alpha,\alpha}$ since $\alpha$ is an irrational number. We thus denote this word by $s_{\alpha,\alpha}$. It is usually called the characteristic word of $\alpha$. For a number $x$ in a quadratic field, we denote by $x'$ the conjugate of $x$ in this field.

**Theorem A.** (Crisp and al. [14]) Let $0 < \alpha < 1$ be an irrational number. Then the following two conditions are equivalent:

(i) the characteristic word $s_{\alpha,\alpha}$ is substitution invariant.
(ii) $\alpha$ is a quadratic irrational with $\alpha' \notin [0, 1]$.

A quadratic number $\alpha$ with $0 < \alpha < 1$ and $\alpha' \notin [0, 1]$ is called a Sturm number according to [2]. Let us note that the simplification of Condition (ii) in Theorem A to its present form is due to [2]. Furthermore, the expression of the substitutions which fix $s_{\alpha,\alpha}$ can be explicitly obtained from the continued fraction expansion of $\alpha$ (see [14]).

For more results on the homogeneous case (that is, the case $\rho = \{n\alpha\}$ for $n \in \mathbb{Z}$, where $\{x\}$ denotes the fractional part of $x$), see for instance [7, 8, 11, 16, 21, 23]; for results in the non-homogeneous case, see [22, 26, 6]. Some variants of Question 1 are also considered in [27, 9].

Yasutomi has given a complete answer to Question 1 in [34]. Its characterization involves the conjugates of the quadratic real number
Theorem 1 (Yasutomi [34]). Let $0 < \alpha < 1$ and $0 \leq \rho \leq 1$. Then $s_{\alpha, \rho}$ is substitution invariant if and only if the following two conditions are satisfied:

(i) $\alpha$ is an irrational quadratic number and $\rho \in \mathbb{Q}(\alpha)$;
(ii) $\alpha' > 1$, $1 - \alpha' \leq \rho' \leq \alpha'$ or $\alpha' < 0$, $\alpha' \leq \rho' \leq 1 - \alpha'$.

Remark 1. Let us notice the symmetry between both cases in Assertion (ii) of Theorem 1. Indeed let $E : 1 \mapsto 1, 2 \mapsto 2$ be the substitution exchanging letters; then $s_{\alpha, \rho}$ is substitution invariant if and only if so does $E(s_{\alpha, \rho}) = s_{1-\alpha, 1-\rho}$ (respectively $s_{1-\alpha, 1-\rho}$); furthermore, $(\alpha, \rho)$ satisfies $\alpha' > 1$, $1 - \alpha' \leq \rho' \leq \alpha'$ if and only if $(1 - \alpha, 1 - \rho)$ satisfies $\alpha' < 0$, $\alpha' \leq \rho' \leq 1 - \alpha'$.

As a corollary of Theorem 1, we easily obtain:

Corollary 1. Let $\alpha$ be a Sturm number. Then

(i) for any $\rho \in \mathbb{Q} \cap (0, 1)$, $s_{\alpha, \rho} = s_{\alpha, \rho}$ is substitution invariant.
(ii) The sturmian word $s_{\alpha, \{n\alpha\}}$ is substitution invariant if and only if $n = -1, 0, 1$. In total we obtain exactly five substitution invariant Sturmian words

$$\{21s_{\alpha, \alpha}, 12s_{\alpha, \alpha}, 2s_{\alpha, \alpha}, 1s_{\alpha, \alpha}, s_{\alpha, \alpha}\}$$

in the homogeneous case.

Let us note condition (ii) is also proved in [34] and in [16].

Proof. (i) Since $\rho$ is a rational number, we have $\rho' = \rho$. Hence the condition of Condition (ii) of Theorem 1 is fulfilled if $\alpha' > 1$ or $\alpha' < 0$.

(ii) Let us first assume that $\alpha' > 1$. Let $n, p \in \mathbb{Z}$ such that $\rho = \{n\alpha\} = n\alpha - p$.

For $n = -1, 0, 1$, we have $\rho = 1 - \alpha, 0, \alpha$ respectively, so that $\rho' = 1 - \alpha', 0, \alpha'$. Hence $\rho' \in [1 - \alpha', \alpha']$. Therefore $s_{\alpha, \rho}$ and $s_{\alpha, \rho}$ are substitution invariant.

For $n \geq 2$, $\rho' = n\alpha' - p > \alpha'$ since $p = \lfloor n\alpha \rfloor \leq n - 1$; for $n \leq -2$, $\rho' = n\alpha' - p < 1 - \alpha'$ since $p = \lfloor n\alpha \rfloor \geq n\alpha - 1 \geq n$. Therefore, $s_{\alpha, \rho}$ and $s_{\alpha, \rho}$ are substitution invariant.

We deduce the case $\alpha' < 0$ by applying Remark 1.

1.2. Invertible substitutions. Let $\sigma$ be a substitution over \{1, 2\} and let $M_{\sigma} = (m_{ij})$ be its incidence matrix, where $m_{ij}$ counts the number of occurrences of the letter $i$ in $\sigma(j)$. We assume that $\det M_{\sigma} =$
\( \pm 1 \) (the substitution is said \textit{unimodular}) and \( M_\sigma \) is \textit{primitive} (\( M_\sigma^n \) has only positive entries for some non-negative integer \( n \)).

A substitution is said \textit{invertible} if it is an automorphism of the free group generated by the alphabet \( \{1, 2\} \). Note that if \( \sigma \) is an invertible substitution, then its incidence matrix is unimodular.

**Theorem B** ([33]). Every invertible substitution over \( \{1, 2\} \) is a composition of the following three invertible substitutions:

\[
1 \mapsto 2, 2 \mapsto 1; \ 1 \mapsto 12, 2 \mapsto 1; \ 1 \mapsto 21, 2 \mapsto 1.
\]

Question 1 is related to invertible substitution according to the following well-known result (see for instance [23]).

**Theorem 2.** A Sturmian word is substitution invariant if and only if it is a fixed point of some primitive and invertible substitution.

Let us illustrate the main idea of the proof of Theorem 1 in [34]. According to the three substitutions in Theorem B, S. Ito and S. Yasutomi [21] define three transformations from \([0, 1]^2\) to \([0, 1]^2\), namely:

\[
T_1(\alpha, \rho) = \left( \frac{\alpha}{1 + \rho}, \frac{\rho}{1 + \alpha} \right), \quad T_2(\alpha, \rho) = \left( \frac{1}{2 - \alpha}, \frac{\rho}{2 - \alpha} \right),
\]

\[
T_3(\alpha, \rho) = (1 - \alpha, 1 - \rho).
\]

Then it is proved that a Sturmian sequence \( s_{\alpha, \rho} \) is substitution invariant if and only if there exists a sequence \( S_1, \ldots, S_n \) with \( S_i \in \{T_1, T_2, T_3\} \) such that \( (\alpha, \rho) = S_1 \circ \cdots \circ S_n(\alpha, \rho) \). Since there are three transformations, the task of determining such \( (\alpha, \rho) \) is tedious. Yasutomi’s original proof of Theorem 1 in [34] is somewhat technique and lengthy.

1.3. \textbf{Rauzy fractals: an alternative proof of Theorem 1.} Rauzy fractals (first introduced in [30] in the Tribonacci case) are compact attractors of a graph-directed iterated function system associated with a substitution of Pisot type.

Let us first describe an intuitive approach to Rauzy fractals for two-letter substitutions. We give a more formal definition in Section 2. Let \( \sigma \) be a primitive and unimodular substitution over \( \{1, 2\} \). Let \( s = s_0s_1s_2 \ldots \) be a periodic point of \( \sigma \). Let \( (1 - \alpha, \alpha) \) be the eigenvector of \( M_\sigma \) corresponding to the Perron-Frobenius eigenvalue. We shall call \( \alpha \) the \textit{characteristic length} of the matrix \( M_\sigma \) or of the substitution \( \sigma \), according to the context.

We define an oriented walk on the real line as follows. We start from the origin; in the \( n \)-th step, if \( s_{n-1} = 1 \), we move to the right side by
\( \alpha \); if \( s_{n-1} = 2 \), we move to the left side by \( 1 - \alpha \). Taking the closure of the orbit of the origin under this transformation, we obtain

\[
X = \text{closure} \left\{ |s_0 s_1 \ldots s_{k-1}|_1 \cdot \alpha + |s_0 s_1 \ldots s_{k-1}|_2 \cdot (\alpha - 1); \quad k \geq 0 \right\},
\]

where \( |s_0 s_1 \ldots s_{n-1}|_j \) denotes the number of occurrences of the letter \( j \) in the word \( s_0 s_1 \ldots s_n \). Furthermore, we define

\[
\begin{align*}
X_1 &= \text{closure} \left\{ |s_0 s_1 \ldots s_{k-1}|_1 \cdot \alpha + |s_0 s_1 \ldots s_{k-1}|_2 \cdot (\alpha - 1); \quad k \geq 0, s_k = 1 \right\}, \\
X_2 &= \text{closure} \left\{ |s_0 s_1 \ldots s_{k-1}|_1 \cdot \alpha + |s_0 s_1 \ldots s_{k-1}|_2 \cdot (\alpha - 1); \quad k \geq 0, s_k = 2 \right\}.
\end{align*}
\]

We shall show in Section 2 that \( X = X_1 \cup X_2 \) is an affine image of the so-called Rauzy fractal. By abuse of language, we call \( X_1, X_2 \) in (2) the Rauzy fractals of \( \sigma \); in particular, (2) is well defined for any Sturmian word \( s \), and we also call \( X_1, X_2 \) the Rauzy fractals of the Sturmian word \( s \).

One easily checks that the Rauzy fractals of the Sturmian word \( s_{\alpha, \rho} \) are intervals equal to \( X_1 = [-\rho, 1 - \alpha - \rho] \), \( X_2 = [1 - \alpha - \rho, 1 - \rho] \). The periodic points of an invertible substitution are Sturmian (see Lemma 4), and hence the associated Rauzy fractals are intervals. Furthermore, one has

**Theorem 3 ([12]).** Let \( \sigma \) be a primitive unimodular substitution over \( \{1, 2\} \). Then the Rauzy fractals \( X_1, X_2 \) and \( X_1 \cup X_2 \) are intervals if and only if \( \sigma \) is invertible.

A simple proof of this result is given in Section 3.

Since Theorem 1 is an elementary and important result, it is worth giving a proof that is more transparent and accessible. This is the main purpose of the present paper. Let us note that a geometric proof based on the use of cut-and-project schemes has also been given in [4].

Let us give a sketch of our proof. By Theorem 2 and Theorem 3, a Sturmian word is substitution invariant if and only if it is a fixed point of some primitive substitution with connected Rauzy fractals. The principal idea used here is to study the Rauzy fractals associated with invertible primitive substitutions following the approach of [15].

Let \( \sigma \) be an invertible substitution with characteristic length \( \alpha \). Then \( \alpha \) is a Sturm number, and the Rauzy fractals \( X_1, X_2 \) are intervals with length \( 1 - \alpha \) and \( \alpha \) respectively. Suppose \( s = s_{\alpha, \rho} \) is a periodic point of \( \sigma \), then \( \rho = h + 1 - \alpha \), where \( \{h\} = X_1 \cap X_2 \).

Let \( L \) be the line \( y = \frac{1 - \alpha}{\alpha} \), where \( \alpha' \) is the algebraic conjugate of \( \alpha \). A broken line in \( \mathbb{R}^2 \), the so-called stepped surface, is associated with the line \( L \), defined as a discretization of \( L \).
The sets $X_1, X_2$ have a self-similar structure, and the set equation is controlled by the stepped surface of $L$ (see Lemma 5 and Theorem 4). Hence, by connectedness and self-similarity of the Rauzy fractals, we express the intersection $X_1 \cap X_2$ in terms of the stepped surface (see Theorem 5).

Then we show that the stepped surface is associated with the rotation $R_\gamma$ with $\gamma = \frac{\alpha' - 1}{2\alpha' - 1}$, which may be considered as the dual rotation of $R_\alpha$. An algebraic characterization of the stepped surface is obtained (see Theorem 7). This allows us to get an algebraic description of the intersection set $X_1 \cap X_2$ for an invertible substitution $\sigma$, which yields a proof of Theorem 1.

Let us note that Rauzy fractals have numerous applications in number theory, ergodic theory, dynamical systems, fractal geometry and tiling theory (see for instance [3, 18, 19, 20, 30, 32], and Chap. 7 in [28]). The present paper contains a new application of Rauzy fractals to Sturmian words.

This paper is organized as follows. We first recall in Section 2 some basic facts on Rauzy fractals. We then discuss in Section 3 the connectedness of Rauzy fractals for a two-letter alphabet. Theorem 3 is proved in this section. In Section 4, we study the set equations of Rauzy fractals, especially in the invertible case. The intersection set $X_1 \cap X_2$ for invertible substitutions is determined in Section 5. In Section 6, an algebraic characterization of the stepped surface is given. Theorem 1 of Yasutomi is proved in Section 7.

2. Rauzy fractals

In this section we recall some basic facts on Rauzy fractals. We present here all the definitions in the case of a two-letter alphabet since it is enough for our purpose. Let us note that the notation, which is adapted from [18], is slightly different from [3].

2.1. Sturm numbers. Let $\sigma$ be a primitive unimodular substitution over $\{1, 2\}$. Let $\beta$ be the maximal eigenvalue of its incidence matrix $M_\sigma$. Its algebraic conjugate $\beta'$ is also an eigenvalue of $M_\sigma$. By Perron-Frobenius' theorem, we have $\beta > 1$. Now $\beta \beta' = \det M = \pm 1$ implies $|\beta'| < 1$. Therefore $\beta$ is a Pisot number and the substitution $\sigma$ is said to be of Pisot type.

It is well-known that the densities of letters exist in fixed points of primitive substitutions (see [29]). Furthermore the vector of densities of the letters 1 and 2 that we denote $(1-\alpha, \alpha)$, with $0 \leq \alpha \leq 1$, is easily seen to be an expanding eigenvector, that is, an eigenvector associated
with the expanding eigenvalue $\beta$. The real number $\alpha$ is quadratic; the vector $(1 - \alpha', \alpha')$ is an eigenvector associated with the eigenvalue $\beta'$. Still by Perron-Frobenius' theorem, the coordinates $1 - \alpha'$, $\alpha'$ cannot be both positive, hence $\alpha'(1 - \alpha') \leq 0$, which implies that $\alpha' \not\in [0, 1[. Hence $\alpha$ is a Sturm number.

Conversely, any Sturm number is the characteristic length of a primitive unimodular matrix $M$ of size $2 \times 2$. Indeed if $\alpha$ is a Sturm number, then $s_{\alpha, \alpha}$ is a fixed point of an invertible primitive substitution $\sigma$ following Theorem A, and hence $\alpha$ is the characteristic length of $M_\sigma$. We thus have proved the lemma below. Let us note that in all that follows all primitive matrices are assumed to be square matrices with non-negative integral entries.

**Lemma 1.** A number $\alpha$ is a Sturm number if and only if there exists a primitive unimodular matrix $M$ of size $2 \times 2$ such that $(1 - \alpha, \alpha)$ is an expanding eigenvector of $M$. In consequence, if the Sturmian word $s_{\alpha, \rho}$ is substitution invariant, then this implies that $\alpha$ is a Sturm number.

**2.2. Definition of Rauzy fractals.** Let $\vec{e}_1, \vec{e}_2$ be the canonical basis of $\mathbb{R}^2$. Let $f : \{1, 2\}^* \rightarrow \mathbb{Z}^2$ be the Parikh map, also called abelianization homomorphism, defined by $f(w) = |w|_1 \vec{e}_1 + |w|_2 \vec{e}_2$, where $|w|_i$ denotes the number of occurrences of the letter $i$ in $w$.

Denote by $V$ be the expanding eigenspace of the matrix $M_\sigma$ corresponding to the eigenvalue $\beta$, and $V'$ the contracting eigenspace corresponding to $\beta'$. The expanding subspace is generated by the vector $\vec{v} = (1 - \alpha, \alpha)$, therefore the contracting subspace is generated by the vector $\vec{v}' = (1 - \alpha', \alpha')$. Then $V \oplus V' = \mathbb{R}^2$ is a direct sum decomposition of $\mathbb{R}^2$. According to this direct sum, two natural projections are defined:

$$
\pi : \mathbb{R}^2 \rightarrow V' \quad \text{and} \quad \pi' : \mathbb{R}^2 \rightarrow V.
$$

We define the Rauzy fractal associated with $\sigma$ as the closure of the projection according to $\pi$ of the vertices of the broken line (illustrated in Figure 1) obtained by applying the map $f$ to the prefixes of a given periodic point of $\sigma$.

More precisely, let $s = (s_k)_{k \geq 0}$ be a periodic point of $\sigma$. We first define

$$
Y = \{ f(s_0 \ldots s_{k-1}); \ k \geq 0 \},
$$

where the notation $s_0 \ldots s_{k-1}$ stands for the empty word when $k = 0$. We then divide $Y$ into two parts:

$$
Y_1 = \{ f(s_0 \ldots s_{k-1}); \ s_k = 1 \}, \ Y_2 = \{ f(s_0 \ldots s_{k-1}); \ s_k = 2 \}.
$$
Projecting $Y_1, Y_2$ onto the contracting eigenspace $P$ and taking its closure, we get

$$\vec{X}_1 = \pi(Y_1), \quad \vec{X}_2 = \pi(Y_2).$$

We call $\vec{X}_1$ and $\vec{X}_2$ the Rauzy fractals of the substitution $\sigma$. It is shown that the Rauzy fractals are independent of the choice of the periodic point in the definition, according to [18].

Clearly the Rauzy fractals $\vec{X}_1$ and $\vec{X}_2$ are one-dimensional objects. An easy computation shows that

$$X_1 = \phi(\vec{X}_1), \quad X_2 = \phi(\vec{X}_2),$$

where $X_1, X_2$ are defined in (2) and $\phi$ is the linear map defined by

$$\phi : V' \to \mathbb{R}, \quad \phi\left(\frac{x \vec{v}}{\alpha - \alpha'}\right) = x.$$

By abuse of language, we also call $X$, $X_1$ and $X_2$ the Rauzy fractals of the substitution $\sigma$.

Barge and Diamond showed in [5] that every Pisot substitution over a two-letter alphabet satisfies a certain combinatorial condition, called the strong coincidence condition. Thanks to this, one can show that

**Lemma 2** ([18]). Let $\sigma$ be a primitive Pisot substitution over two letters. Then

$$\mu(X_1) = 1 - \alpha, \quad \mu(X_2) = \alpha,$$

where $\mu$ is the Lebesgue measure and $\alpha$ is the characteristic length of $\sigma$.

### 3. Connectedness of Rauzy fractals

#### 3.1. Proof of Theorem 3.

It is in general hard to decide whether the Rauzy fractals are connected (see for instance [1, 12]). However, in the two-letter case we have a complete characterization given by Theorem 3. We provide an elementary proof of this folklore result. We first need the following lemmas.
Lemma 3 ([24, 33, 7]). Let $\sigma$ be a non-trivial substitution over $\{1, 2\}$. The following three conditions are equivalent:

(i) $\sigma$ is primitive invertible;
(ii) for any Sturmian word $s$, $\sigma(s)$ is still a Sturmian word;
(iii) there exists a Sturmian word $s$ such that $\sigma(s)$ is a Sturmian word.

The equivalence between (i) and (ii) is due to [24] and [33], the equivalence with (iii) is proved in [7]. For more details, see [23].

Lemma 4. Let $\sigma$ be a non-trivial substitution over $\{1, 2\}$. The following statements are equivalent:

(i) $\sigma$ is a primitive invertible substitution;
(ii) all periodic points of $\sigma$ are Sturmian words;
(iii) there exists a periodic point of $\sigma$ which is a Sturmian word.

Proof. (i) $\Rightarrow$ (ii). Suppose $\sigma$ is a primitive invertible substitution. Let $s$ be a periodic point, say, $\sigma^k(s) = s$, with $k \geq 1$. Let $s'$ be any Sturmian word with the same initial letter as $s$. Then by Lemma 3, $(\sigma^k)^n(s')$ is a Sturmian word for any $n \geq 1$. But the sequence of words $(\sigma^k)^n(s')_{n \geq 1}$ converges to $s$. Hence $s$ has at most $n + 1$ factors of length $n$. Since $\sigma$ is both unimodular and primitive, we infer that the density of letter 1 in $s$ is irrational. Hence $s$ is aperiodic, and it is a Sturmian word.

(ii) $\Rightarrow$ (iii) is immediate.

(iii) $\Rightarrow$ (i). Let $s$ be a Sturmian periodic point of $\sigma$, say, $\sigma^k(s) = s$, with $k \geq 1$. The condition (iii) of Lemma 3 is fulfilled and $\sigma^k$ is invertible. So $\sigma$ is also invertible. \hfill \Box

Proof of Theorem 3. Let $\sigma$ be a primitive unimodular substitution over $\{1, 2\}$. We first assume that the Rauzy fractals of $\sigma$, namely $X_1$, $X_2$, and $X = X_1 \cup X_2$, are intervals. Let $s = (s_k)_{k \geq 0}$ be a periodic point of $\sigma$ which defines $X_1$ and $X_2$. Then according to Lemma 2, $|X_1| = 1 - \alpha$ and $|X_2| = \alpha$. Hence $s$ is the coding of the orbit of an irrational rotation so that it is a Sturmian word. We thus deduce that $\sigma$ is invertible from Lemma 4.

Conversely, if $\sigma$ is primitive invertible, then it has at least one periodic point $s$ and it is Sturmian. Hence the Rauzy fractals are intervals, according to Section 1. \hfill \Box

Corollary 2. Let $\sigma$ be a primitive invertible substitution. Then there exists $h \in \mathbb{Z}$ such that the Rauzy fractals satisfy

$$X_1 = [-1 + \alpha + h, h], \quad X_2 = [h, \alpha + h],$$

where $\alpha$ is the characteristic length of $\sigma$. 
3.2. Upper and lower Sturmian sequences. In this subsection we show that $s_{\alpha,\rho}$ is substitution invariant if and only if $s_{\alpha,\rho}$ is also substitution invariant.

**Proposition 1.** Let $0 < \alpha < 1$ be an irrational number and $0 < \rho < 1$. Let $\sigma$ be a non-trivial substitution. Then $s_{\alpha,\rho}$ is a fixed point of $\sigma$ if and only $s_{\alpha,\rho}$ is also a fixed point of $\sigma$.

**Proof.** Suppose $s_{\alpha,\rho} = s_0s_1s_2 \ldots$ is a fixed point of a non-trivial substitution $\sigma$. According to Lemma 3, $\sigma$ is primitive invertible. Let $X_1 = [-\rho, 1 - \alpha - \rho]$ and $X_2 = [1 - \alpha - \rho, 1 - \rho]$ be the associated Rauzy fractals of $s = s_{\alpha,\rho}$.

If the orbit of the oriented walk of $s$ does not contain $X_1 \cap X_2$, then $s_{\alpha,\rho} = s_{\alpha,\rho}$. In this case, there is nothing to prove.

Therefore we assume that $s_{\alpha,\rho} \neq s_{\alpha,\rho}$. Then the orbit of the oriented walk of $s$ must contain $h$, i.e., there exists a nonnegative integer $n$ such that $f(s_0s_1 \ldots s_{n-1}) = h$. One has either

\begin{align}
(3) & \quad s_{\alpha,\rho} = s_0 \ldots s_{n-1}21s_{n+1} \ldots = s_0 \ldots s_{n-1}21s_{\alpha,\rho}, \\
(4) & \quad \sigma(s_{\alpha,\rho}) = s_0 \ldots s_{n-1}21s_{n+1} \ldots = s_0 \ldots s_{n-1}21s_{\alpha,\rho}
\end{align}

or

\begin{align}
(4) & \quad \sigma(s_{\alpha,\rho}) = 1s_{\alpha,\rho}, \\
& \quad \sigma(s_{\alpha,\rho}) = 2s_{\alpha,\rho}.
\end{align}

Let us denote $s' = s_{\alpha,\rho}$. We assume that we are in case (3), the case (4) can be handled in the same way.

We claim that the Rauzy fractals of $\sigma(s')$ are also the intervals $X_1, X_2$. Indeed, by Lemma 3, $\sigma(s')$ is Sturmian so that the Rauzy fractals of $\sigma(s')$ are intervals. It is shown in [15] (as a consequence of Theorem B) that if $\sigma$ is invertible, then there exist two words $w$ and $u$ such that either $\sigma(12) = w12u$, $\sigma(21) = w21u$, or $\sigma(12) = w21u$, $\sigma(21) = w12u$. Hence there exist a finite word $w$ and an infinite word $t$ such that

\begin{align}
(5) & \quad \sigma(s) = w12t, \sigma(s') = w21t \quad \text{or} \quad \sigma(s) = w21t, \sigma(s') = w12t.
\end{align}

Hence $X(\sigma(s))$ (which is equal to $X(s)$) and $X(\sigma(s'))$ differ at one point at most. Therefore $X(s)$ and $X(\sigma(s'))$ coincide since they are intervals. Our claim is proved.

By (3) and (5), one has $X_1 \cap X_2 = \{f(s_0s_1 \ldots s_{n-1})\} = \{f(w)\}$. Since $s_0s_1 \ldots s_{n-1}$ is a prefix of $w$ and $\alpha$ is irrational, we deduce from $f(s_0s_1 \ldots s_{n-1}) = f(w)$ that $w = s_0s_1 \ldots s_{n-1}$. This is possible only if $s_0s_1 \ldots s_{n-1}$ is the empty word. Again by (3) and (5), we either have

$s = \sigma(s) = 12t, \quad \sigma(s') = 21t = s'$

or

$s = \sigma(s) = 21t, \quad \sigma(s') = 12t = s'$
Hence $s'$ is a fixed point of $\sigma$. □

**Corollary 3.** Let $0 < \alpha < 1$ be an irrational number and $0 \leq \rho \leq 1$. Then $s_{\alpha,\rho}$ is a periodic point of a non-trivial substitution $\sigma$ if and only if $\overline{s}_{\alpha,\rho}$ is also a periodic point of $\sigma$.

*Proof.* Let $s_{\alpha,\rho}$ be a periodic point that is not a fixed point of $\sigma$. Since we are on a two-letter alphabet, then there exist two words $w, u$ such that $\sigma$ satisfies $\sigma(1) = 2w$ and $\sigma(2) = 1u$. Hence $s_{\alpha,\rho}$ is a fixed point of $\sigma^2$, so that $s_{\alpha,\rho}$ is also a fixed point of $\sigma^2$ by Proposition 1, meaning that $\overline{s}_{\alpha,\rho}$ is a periodic point of $\sigma$. □

### 4. Self-similarity of Rauzy fractals

In this section, we discuss the self-similar structure of the Rauzy fractals $X_1$ and $X_2$; special attention is paid to the case $\sigma$ invertible. The stepped surface is shown to play an important role.

#### 4.1. Set equations of Rauzy fractals.

Let $\sigma$ be a primitive substitution over $\{1, 2\}$ and let $\beta$ be the Perron-Frobenius eigenvalue of $M_\sigma$.

It is well-known ([3],[32],[18]) that $X_1$ and $X_2$, and thus $X_1$ and $X_2$, have a self-similar structure, that is, both $\frac{1}{\beta}X_1$ and $\frac{1}{\beta}X_2$ are unions of translated copies of $X_1$ and $X_2$. In order to describe the corresponding set equations, we introduce the following notation: let $D_1$ (resp. $D_2$) be the set of those $(a, i) \in \mathbb{R}^2 \times \{1, 2\}$ such that $X_i + a \subset \frac{1}{\beta}X_1$ (resp. $X_i + a \subset \frac{1}{\beta}X_2$), that is,

$$\frac{1}{\beta}X_1 = \bigcup_{(a, i) \in D_1} X_i + a, \quad \frac{1}{\beta}X_2 = \bigcup_{(b, i) \in D_2} X_i + b.$$

For the explicit form of $D_1, D_2$, we refer to [3].

#### 4.2. The stepped surface.

Recall that $V'$ is the contracting eigenline of $M_\sigma$. We denote the upper closed half-plane delimited by $V'$ as $(V')^+$, and the lower open half-plane delimited by $V'$ as $(V')^-$. We define

$$S = \{[z, i^*]; \quad z \in \mathbb{Z}^2, z \in V^+ \text{ and } z - e_i \in V^-\},$$

where the notation $[z, i^*]$, for $z \in \mathbb{Z}^n$ and $i^* \in \{1^*, 2^*\}$ endows the point $z$ in $\mathbb{Z}^n$ with color $i^* = 1^*, 2^*$. Intuitively $S$ consists of the the collection of those colored points $[z,i^*]$ which are close to the contracting eigenline $V'$. 
We now define \([z, 1^*]\) (resp. \([z, 2^*]\)) as the closed line segment from \(z\) to \(z + \vec{e}_2\) (resp. to \(z + \vec{e}_1\)) (see Figure 2). Then the stepped surface \(\overline{S}\) of \(V'\) is defined as the broken line consisting of the following segments
\[
\overline{S} = \bigcup_{[z, i^*] \in S} [z, i^*].
\]

A piece of a stepped surface is depicted in Figure 3. By abuse of language, the formal set \(S\) will also be called the stepped surface of \(V'\).

\[\text{Figure 2. The segments } [0, 1^*] \text{ and } [0, 2^*].\]

\[\text{Figure 3. A piece of the stepped surface for } 1 \mapsto 12112, \quad 2 \mapsto 121.\]

It turns out that the set equations of the Rauzy fractals are controlled by the stepped surface. An explicit expression of the sets \(D_1\) and \(D_2\) is given in [3], from which one immediately deduces the following facts:

**Lemma 5 ([3, 18]).** Using the notation above:

i) for any \((a, i) \in D_1 \cup D_2\), there exists an element \([z, i^*] \in S\) such that \(\phi \circ \pi(z) = a\);

ii) \((0, 1), (0, 2) \in D_1 \cup D_2\);

iii) \((n_{ij})_{1 \leq i \leq 2} = tM_{\sigma}\), where \(n_{ij}\) counts the number of elements \((a, j)\) in the set \(D_i\).

4.3. **A tiling associated with the stepped surface.** Projecting the stepped surface \(\overline{S}\) onto \(V'\), we first obtain a tiling \(\mathcal{J}'\) of \(V'\):

\[
\mathcal{J}' = \{\pi([z, i^*]); \ [z, i^*] \in S\}.
\]
Applying the linear transformation $\phi$, we then get a tiling $\mathcal{J}$ of the real line:

$$\mathcal{J} = \{ \phi \circ \pi([z, i^*]); [z, i^*] \in S \}.$$ 

The tiling $\mathcal{J}$ is a quasi-periodic tiling with two prototiles. Indeed

$$\mathcal{J} = \{ \phi \circ \pi(z) + J_i; [z, i^*] \in S \},$$

where

$$J_1 = \phi \circ \pi[0, 1^*] = [-1 + \alpha, 0], \quad J_2 = \phi \circ \pi[0, 2^*] = [0, \alpha].$$

We label the tiles of $\mathcal{J}$ on the right side of the origin by the sequence $T_0, T_1, T_2, \ldots$, where $T_n$ is the rightside neighbour of $T_n$. Likewise we label the tiles of $\mathcal{J}$ on the left side of the origin by $T_{-1}, T_{-2}, \ldots$. One has $\mathcal{J} = \{ T_k; k \in \mathbb{Z} \}$. We furthermore define the two-sided sequence $(g_k)_{k \in \mathbb{Z}}$ as the sequence of left endpoints of the tiles $T_k$. An arithmetic description of the sequence $(g_k)_{k \in \mathbb{Z}}$ is given in Section 6.

4.4. Set equations of connected Rauzy fractals. According to Corollary 2, if $\sigma$ is a primitive invertible substitution, then there exists a real number $h$ such that $X_1 = [-1 + \alpha + h, h], X_2 = [h, h + \alpha]$, that is,

$$X_1 = J_1 + h, \quad X_2 = J_2 + h,$$

where $J_1 = [-1 + \alpha, 0]$ and $J_2 = [0, \alpha]$ are the two prototiles of the tiling $\mathcal{J}$.

Suppose $X_i + a$ is a piece in the subdivision of $\frac{X_i}{\beta'}$ according to the set equations, that is, $(a, i) \in D_1$; there exists an element $[z, i^*] \in S$ such that $\phi \circ \pi(z) = a$ by Lemma 5. Let $k \in \mathbb{Z}$ such that $\phi \circ \pi[z, i^*] = T_k$; then

$$X_i + a = J_i + h + a = T_k + h.$$ 

We thus can introduce two subsets $D_1$ and $D_2$ of $\mathcal{J}$ such that

$$\frac{X_1}{\beta'} = \left( \bigcup_{T \in D_1} T \right) + h, \quad \frac{X_2}{\beta'} = \left( \bigcup_{T \in D_2} T \right) + h.$$ 

On the one hand, the tiles in $D_1 \cup D_2$ do not overlap according to [5] and [3]. On the other hand, these tiles must form a connected patch of $\mathcal{J}$ since $X_1, X_2, X_1 \cup X_2$ are intervals, according to Theorem 3. Hence we have proved that

**Theorem 4.** Let $X_1 = [-1 + \alpha + h, h], X_2 = [h, h + \alpha]$ be the Rauzy fractals of the primitive invertible substitution $\sigma$. Then

$$\frac{X_1}{\beta'} = \left( \bigcup_{T \in D_1} T \right) + h, \quad \frac{X_2}{\beta'} = \left( \bigcup_{T \in D_2} T \right) + h,$$
where $D_1, D_2$ and $D_1 \cup D_2$ are connected patches of the tiling $J$.

5. Invertible substitutions with a given incidence matrix

In this section, we give a more detailed description of the Rauzy fractals of invertible substitutions with a given incidence matrix.

5.1. A list of invertible substitutions with a given incidence matrix.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix. A very interesting result on invertible substitutions is given in [31]:

**Lemma 6.** [31] Let $M$ be a primitive unimodular matrix. The number of invertible substitutions with incidence matrix $M$ is equal to $a + b + c + d - 1$.

Let $\sigma$ be an invertible substitution with incidence matrix $M_\sigma = M$. According to Lemma 5, $(0, 1), (0, 2) \in D_1 \cup D_2$, so that we have

$$T_{-1}, T_0 \in D_1 \cup D_2.$$ 

By Lemma 5 (iii), we have that

$$\text{Card}D_1 = \text{Card}D_1 = a + b, \quad \text{Card}D_2 = \text{card}D_2 = c + d.$$ 

Let us assume that the determinant of $M$ is equal to 1. We will not need to consider in the sequel the case $\det(M) = -1$, but a similar study can be conducted. In this case $1/\beta' = \beta > 0$ so that $\frac{X_1}{\beta'}$ is on the left side of $\frac{X_2}{\beta'}$. Hence by Theorem 4, the patch $D_1$ is on the left side of $D_2$. By Theorem 4, (6) and (7), we infer that there exists $k$ with $1 \leq k \leq a + b + c + d - 1$ such that

$$D_1 = \{T_{-k}, T_{-k+1}, \ldots, T_{-k+a+b-1}\},$$
$$D_2 = \{T_{-k+a+b}, T_{-k+a+b+1}, \ldots, T_{-k+a+c+b+d-1}\}.$$ 

Hence there are at most $a + b + c + d - 1$ invertible substitutions with incidence matrix $M$, and their set equations are deduced from (8). On the other hand, Lemma 6 asserts that there are exactly $a + b + c + d - 1$ such substitutions. Since the set equations for different substitutions are distinct, we conclude that there is a one-to-one correspondence between the invertible substitutions with incidence matrix $M$ and the set equations determined by (8). We denote these substitutions by $\sigma_k$, $1 \leq k \leq a + b + c + d - 1$. 
5.2. Intersection point of Rauzy fractals. For each of the substitutions $\sigma_k$ defined in the previous section, there exists $\rho_k$ such that $s_{\alpha,\rho_k}$ (which means indifferently either $\sigma_{\alpha,\rho_k}$ or $s_{\alpha,\rho_k}$) is a periodic point of $\sigma_k$, according to Lemma 4.

Let $1 \leq k \leq a+b+c+d-1$. Let $X_1 = [-1 + \alpha + h_k, h_k]$, $X_2 = [h_k, \alpha + h_k]$ be the Rauzy fractals of $\sigma_k$. One has $\rho_k = 1 - \alpha - h_k$. We use below the connectedness and the self-similarity of the Rauzy fractals to determine $h_k$ and thus $\rho_k$. Let us recall that $(g_k)_{k \in \mathbb{Z}}$ denote the sequence of left endpoints of the tile $T_k$ in $\mathcal{J}$.

**Theorem 5.** Let $M$ be a primitive matrix with $\det M = 1$. Let $\sigma_k$, $1 \leq k \leq a+b+c+d-1$, be the invertible substitutions with incidence matrix $M$. Let $\beta$ be the maximal eigenvalue of $M$. We assume $\det M = 1$. Then

$$h_k = \frac{g_{-k+a+b}}{\beta - 1}.$$ 

**Proof.** On the one hand, $\frac{X_1}{\beta} \cap \frac{X_2}{\beta} = \{(\beta')^{-1}h_k\} = \{\beta h_k\}$. On the other hand, this intersection point is the right end-point of the interval $\cup\{T+h_k; T \in \mathcal{D}_1\}$, that is, the right end-point of $T_{-k+a+b-1} + h_k$. So we get $g_{-k+a+b} + h_k = \beta h_k$, and $h_k = \frac{g_{-k+a+b}}{\beta - 1}$.

**Theorem 6.** Let $M$ be a primitive matrix with $\det M = 1$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{a+b+c+d-1}$ be the invertible substitutions with incidence matrix $M$. Let $G := \{g_k; k \in \mathbb{Z}\}$. Then the Sturmian word $s_{\alpha,\rho}$ is a periodic point of the substitution $\sigma_k$ if and only if

$$0 \leq \rho \leq 1 \text{ and } (\rho + \alpha - 1) \in \frac{G}{\beta - 1}.$$ 

**Proof.** By Theorem 5, one has $h_{a+b+c+d-1} < \cdots < h_2 < h_1$. Hence a real number $h$ belongs to the set $\{h_1, h_2, \ldots, h_{a+b+c+d-1}\}$ if and only if

$$h \in \frac{G}{\beta - 1} \text{ and } h_{a+b+c+d-1} \leq h \leq h_1.$$ 

It remains to determine the values $h_1$ and $h_{a+b+c+d-1}$. For the substitution $\sigma_1$, the set $\mathcal{D}_1$ is equal to $\{T_1, T_0, \ldots, T_{a+b-2}\}$. By Lemma 5, the numbers of tiles in $\mathcal{D}_1$ of length $1 - \alpha$ and $\alpha$ are $a$ and $b$ respectively. Since $|T_1| = 1 - \alpha$, we have

$$g_{a+b-1} = (a - 1)(1 - \alpha) + b\alpha = (\beta - 1)(1 - \alpha).$$ 

Here we use the equality $a(1 - \alpha) + b\alpha = \beta(1 - \alpha)$, which follows from the fact that $(1 - \alpha, \alpha)$ is an expanding eigenvector of $M$. Therefore $h_1 = 1 - \alpha$. A similar argument shows that $h_{a+b+c+d-1} = -\alpha$.

Remember now that $\rho_k = 1 - \alpha - h_k$. The theorem follows from (9).
6. The stepped surface

In this section, we give an arithmetic description of the stepped surface. We first define the two-sided word \((t_n)_{n \in \mathbb{Z}}\) as:

\[
\forall n \in \mathbb{Z}, \quad t_n = \begin{cases} 
1, & \text{if } |T_n| = 1 - \alpha \\
2, & \text{if } |T_n| = \alpha.
\end{cases}
\]

It is well known that Sturmian words can also be described as cutting sequences, see for instance [23]. One checks according to [3] that \((t_n)_{n \in \mathbb{Z}}\) is the upper two-sided cutting sequence of the line \(V' : y = \frac{\alpha' - 1}{\alpha'}\). Hence

\[
t_{-1}t_{-2}t_{-3} \cdots = 1s_{\gamma, \gamma}, \quad t_0t_1t_2 \cdots = 2s_{\gamma, \gamma},
\]

where

\[
\gamma = \frac{\alpha' - 1}{2\alpha' - 1}.
\]

Let \(R_\gamma : x \mapsto x + \gamma\) be the rotation of the torus \(T^1\) of angle \(\gamma\). We deduce from (10) that for all \(k\)

\[
|t_{-1}t_{-2} \cdots t_{-k}|_1 \cdot \gamma + |t_{-1}t_{-2} \cdots t_{-k}|_2 \cdot (\gamma - 1) = R_\gamma^k(0)
\]

\[
|t_0t_1 \cdots t_{k-1}|_1 \cdot \gamma + |t_0t_1 \cdots t_{k-1}|_2 \cdot (\gamma - 1) = R_\gamma^k(0).
\]

By definition of \((g_k)_{k \in \mathbb{Z}}\), one has for every nonnegative \(k\)

\[
g_{-k} = -|t_{-1}t_{-2} \cdots t_{-k}|_1 \cdot (1 - \alpha) - |t_{-1}t_{-2} \cdots t_{-k}|_2 \cdot \alpha,
\]

\[
g_k = |t_0t_1 \cdots t_{k-1}|_1 \cdot (1 - \alpha) + |t_0t_1 \cdots t_{k-1}|_2 \cdot \alpha.
\]

Hence

\[
\forall k \in \mathbb{Z}, \quad \frac{g_k}{2\alpha' - 1} = R_\gamma^{-k}(0),
\]

where \(g_k'\) denotes the conjugate of \(g_k\). This thus provides an arithmetic description of the stepped surface.

Recall that \(G := \{g_k; \ k \in \mathbb{Z}\}\), where the sequence \((g_k)\) is defined in Section 5.2. Denote \(\mathbb{Z}[\alpha] := \{m\alpha + n; \ m, n \in \mathbb{Z}\}\).

**Theorem 7.** One has

\[
G = \{g \in \mathbb{Z}[\alpha]; \ 0 \leq g' < 2\alpha' - 1\} \text{ when } \alpha' > 1,
\]

\[
G = \{g \in \mathbb{Z}[\alpha]; \ 2\alpha' - 1 < g' \leq 0\} \text{ when } \alpha' < 0.
\]

**Proof.** We assume that \(\alpha' > 1\). The case \(\alpha' < 0\) can be handled similarly. Notice that

\[
\{R_\gamma^k(0); \ k \in \mathbb{Z}\} = \{m\gamma + n; \ 0 \leq m\gamma + n < 1\}.
\]
This together with (12) imply that
\[ G = \{ g; \; g' = m(\alpha' - 1) + n(2\alpha' - 1); \; m, n \in \mathbb{Z}, \; 0 \leq g' < 2\alpha' - 1 \} \]
\[ = \{ g; \; g = m(\alpha - 1) + n(2\alpha - 1); \; m, n \in \mathbb{Z}, \; 0 \leq g' < 2\alpha' - 1 \} \]
\[ = \{ g \in \mathbb{Z}[\alpha]; \; 0 \leq g' < 2\alpha' - 1 \}. \]

\[ \square \]

**Remark 2.** For a Sturm number \( \alpha \), it is easy to check that \( \gamma = \frac{\alpha' - 1}{2\alpha' - 1} \) is also a Sturm number. We say that \( \gamma \) is the dual of \( \alpha \). One checks that \( \gamma \) and \( \alpha \) are dual of each other. In some sense, the rotation \( R_\gamma \) is the dual rotation of \( R_\alpha \).

### 7. Proof of Theorem 1

In this section, we prove Theorem 1.

**Theorem 1.** (Yasutomi [34]) Let \( 0 < \alpha < 1 \) and \( 0 \leq \rho \leq 1 \). Then \( s_{\alpha,\rho} \) is substitution invariant if and only if the following two conditions are satisfied:

(i) \( \alpha \) is an irrational quadratic number and \( \rho \in \mathbb{Q}(\alpha) \);
(ii) \( \alpha' > 1, \; 1 - \alpha' \leq \rho' \leq \alpha' \) or \( \alpha' < 0, \; \alpha' \leq \rho' \leq 1 - \alpha' \).

#### 7.1. An algebraic lemma

We first need a preliminary lemma.

**Lemma 7.** Let \( \beta \) be a quadratic algebraic unit, and \( \alpha \) be an irrational number in \( \mathbb{Q}(\beta) \). Then for any \( \rho \in \mathbb{Q}(\beta) \), there exists an arbitrary large even number \( n \) such that \( \rho(\beta^n - 1) \in \mathbb{Z}[\alpha] \).

**Proof.** Let \( \mathcal{A} \) denote the ring of algebraic integers in \( \mathbb{Q}(\beta) \). First we claim that for any \( \rho \in \mathbb{Q}(\beta) \), there exists an arbitrary large even number \( n \) such that \( \rho(\beta^n - 1) \in \mathcal{A} \). Let \( \alpha \in \mathcal{A} \) such that \( \alpha \rho \in \mathcal{A} \). Then at least two terms in the sequence \( (\alpha \rho \beta^n)_{n \geq 0} \) belong to the same residue class modulo the principal ideal of \( \mathcal{A} \) generated by \( \alpha \). Hence \( \alpha \rho(\beta^{n_1} - \beta^{n_2}) \) is divisible by \( \alpha \) for some \( n_1 > n_2 \). Since \( \beta \) is an algebraic unit, \( \alpha \rho(\beta^n - 1) \) is also divisible by \( \alpha \) for \( n = n_1 - n_2 \). This proves our claim. Let us note furthermore that obviously we can choose \( n \) to be an even number. We thus have proved that for every \( N > 0 \),

\[ \mathbb{Q}(\beta) = \bigcup_{n \geq N} \frac{\mathcal{A}}{\beta^{2n} - 1}. \]

We then prove that there exists a rational number \( K \) such that \( \mathcal{A} \subset K\mathbb{Z}[\alpha] \). Indeed let \( d \) be the square-free integer such that \( \mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d}). \) Then there exist integers \( a, b \) and \( c \neq 0 \) such that \( \alpha = \frac{a + b\sqrt{d}}{c}. \)
Notice that $b \neq 0$ since $\alpha$ is irrational. It is well known that any element in $A$ must have the form $(m\sqrt{d} + n)/2$. Since

$$(m\sqrt{d} + n)/2 = \frac{m\alpha - ma + nb}{2b}$$

is an element of $\mathbb{Z}[\alpha]_{2b}$, our assertion is true by taking $K = \frac{1}{2b}$.

Therefore for any $N > 0$, we have

$$Q(\beta) = \bigcup_{n \geq N} \mathcal{A} \beta^{2n-1} \subset K \bigcup_{n \geq N} \mathbb{Z}[\alpha]_{\beta^{2n-1}} \subseteq Q(\beta).$$

Multiplying every term of the above formula by $K^{-1}$, we obtain

$$Q(\beta) = \bigcup_{n \geq N} \mathbb{Z}[\alpha]_{\beta^{2n-1}}. \quad \square$$

7.2. Proof of Theorem 1. Now we are in position to prove Theorem 1.

Necessity. Let us suppose that $s_{\alpha,\rho}$ is a fixed point of the non-trivial primitive invertible substitution $\sigma$. Let $\beta$ be the maximal eigenvalue of $M_\sigma$. We may assume that $\det M = 1$, for otherwise we consider $\sigma^2$ instead of $\sigma$ since $s_{\alpha,\rho}$ is also a fixed point of $\sigma^2$.

By Lemma 1, $\alpha$ must be a Sturm number. From Theorem 6, we deduce

$$1 - \alpha - \rho = h \in \frac{G}{\beta - 1} \subseteq \frac{\mathbb{Z}[\alpha]}{\beta - 1} \subseteq Q(\beta).$$

Hence $\rho \in Q(\beta) = Q(\alpha)$, so that condition (i) is necessary.

Concerning (ii), we need only to consider the case $\alpha' > 1$ according to Remark 1. Notice that $s_{\alpha,\rho}$ is also a fixed point of $\sigma^n$, for any $n \geq 1$, and in particular for any even number $n$; furthermore, the substitutions $\sigma^n$ share the same stepped surface. Hence

$$\rho + \alpha - 1 \in \frac{G}{1 - \beta^n};$$

$$\rho' + \alpha' - 1 \in \frac{\{g'; \, g \in G\}}{1 - (\beta')n}.$$

By Theorem 7, we have

$$0 \leq \rho' + \alpha' - 1 < \frac{2\alpha' - 1}{1 - (\beta')n}.$$

Notice that the above formula holds for every even number $n$. Letting $n$ tend to infinity, $(\beta')^n$ vanishes, and we conclude that $1 - \alpha' \leq \rho' \leq \alpha'$. 
Sufficiency. Suppose that \((\alpha, \rho)\) satisfies (i) and (ii). According to Remark 1, we may assume that \(\alpha' > 1\) so that \(\rho' + \alpha' - 1 \in [0, 2\alpha' - 1]\). Since \(\alpha\) is a Sturm number, there exists a primitive substitution \(\sigma\) such that \(s_{\alpha, \alpha}\) is fixed point of \(\sigma\) (Theorem A). Let \(\beta\) the maximal eigenvalue of the incidence matrix \(M_\sigma\). We may assume that \(\det M_\sigma = 1\), otherwise we consider \(\sigma^2\) instead of \(\sigma\).

Obviously \(\alpha \in \mathbb{Q}(\beta)\). Condition (i) implies that \(\rho \in \mathbb{Q}(\alpha) = \mathbb{Q}(\beta)\). Hence \(\rho + \alpha - 1 \in \mathbb{Q}(\beta)\) so that by Lemma 7, there exist an even number \(n\) and \(g \in \mathbb{Z}[\alpha]\) such that
\[
\rho + \alpha - 1 = \frac{g}{1 - \beta^n}.
\]

Let us prove that \(g\) is actually an element of \(G\). The assumptions \(\alpha' > 1\) and \(1 - \alpha' \leq \rho' \leq \alpha'\) imply that \(0 \leq \rho' + \alpha' - 1 \leq 2\alpha' - 1\). Now \(0 < 1 - (\beta')^n < 1\) since \(n\) is even. Hence
\[
g' = (\rho' + \alpha' - 1)(1 - (\beta')^n) \in [0, 2\alpha' - 1]
\]
so that \(g \in G\) by Theorem 7. We thus have proved that \(\rho + \alpha - 1 \in \frac{G}{1 - \beta^n}\). This together with \(0 \leq \rho \leq 1\) imply that \(s_{\alpha, \rho}\) is substitution invariant (by Theorem 6). □

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References


[21] Ito S. and Yasutomi S., On continued fractions, substitutions and characteristic sequences \([nx + y] - \lfloor(n - 1)x + y\rfloor\), *Japan J. Math.* 16 (1990), 287–306.


[33] Wen Z.-X. and Wen Z.-Y., Local isomorphisms of invertible substitutions, 

[34] Yasutomi S.-I., On Sturmian sequences which are invariant under some 
         substitutions, in *Number theory and its applications (Kyoto, 1997)*, pp. 

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