

# PERIODS OF $\beta$ -EXPANSIONS AND LINEAR RECURRENT SEQUENCES

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ABSTRACT. Let  $\beta > 1$  be a Pisot number. It is well known that a number  $x$  has periodic  $\beta$ -expansion if and only if  $x \in \mathbb{Q}(\beta)$ . When  $\beta$  is a quadratic Pisot unit, we show that the period of the  $\beta$ -expansion of  $x$  is determined by a linear recurrent sequence related to  $\beta$  and  $x$ . Particularly, if  $\beta = (\sqrt{5} + 1)/2$  is the golden number, then the periods of the  $\beta$ -expansions are determined by the prominent Fibonacci sequence.

## 1. Introduction

1.1.  **$\beta$ -numeration system.** Let  $\beta > 1$  be a real number. The  $\beta$ -transformation is a piecewise linear transformation on  $[0, 1)$  defined by

$$T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor,$$

where  $\lfloor \alpha \rfloor$  is the largest integer not exceeding  $\alpha$ . By iterating this map and considering its trajectory

$$x \xrightarrow{x_1} T_\beta(x) \xrightarrow{x_2} T_\beta^2(x) \xrightarrow{x_3} \dots$$

with  $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$ , we obtain the  $\beta$ -expansion of  $x$

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots = 0.x_1x_2x_3\dots$$

An expansion is *finite* if  $(x_i)_{i \geq 1}$  is eventually 0. A  $\beta$ -expansion is *periodic* if there exists  $p \geq 1$  and  $M \geq 1$  such that  $x_k = x_{k+p}$  holds for all  $k \geq M$ ; if  $x_k = x_{k+p}$  holds for all  $k \geq 1$ , then it is *strictly periodic* (or purely periodic). When the  $\beta$ -expansion of  $x$  is periodic, we denote by  $L_\beta(x)$  the minimal period of the expansion.

When  $\beta = b \geq 2$  is an integer, then the  $\beta$ -expansion is the  $b$ -adic expansion. In this case, it is well known that:

- (i) A number  $x \in [0, 1)$  has periodic expansion if and only if  $x$  is a rational number.
- (ii) The expansion of  $x = p/q$  is strictly periodic if and only if  $\gcd\{q, b\} = 1$ .
- (iii) The minimum period of the  $b$ -adic expansion of  $p/q$  coincides with the minimum period of sequence  $\{b^n \pmod{q}\}_{n \geq 0}$ .

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It is natural to ask the same questions for  $\beta$ -expansions when  $\beta$  is not an integer.

(Q1). For which  $x \in [0, 1)$ , is the  $\beta$ -expansion periodic?

(Q2). For which  $x \in [0, 1)$ , is the  $\beta$ -expansion strictly periodic?

(Q3). If the  $\beta$ -expansion of  $x$  is periodic, what is the minimum period?

**1.2. Periodic  $\beta$ -expansion and Pisot number.** Question (Q1) has been settled down for all Pisot numbers (See Schmidt [Sch] 1980). An algebraic integer strictly greater than 1 is a *Pisot number* if all its algebraic conjugates have modulus strictly less than 1. A number is called *Pisot unit* if it is a Pisot number and an algebraic unit. Let  $\mathbb{Q}(\beta)$  be the smallest field containing rational numbers and  $\beta$ . It is well known that

**Theorem A** (Schmidt [Sch]) *Let  $\beta$  be a Pisot number. Then  $x \in [0, 1)$  has periodic  $\beta$ -expansion if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1)$ .*

The  $\beta$ -expansion with a Pisot base has drawn the attention of many mathematicians. For a Pisot unit  $\beta$ , Rauzy ([Rau] 1982) and Thurston ([Thu] 1989) have constructed a self-similar tiling system. The tiles (usually have fractal boundaries) are called *Rauzy fractals* or *atomic surfaces*. Several interesting dynamical system are related to Rauzy fractals (cf. [AI, IR2]).

According to the behavior of the  $\beta$ -expansion, several algebraic properties of  $\beta$  have been defined, for example, the (F)-property introduced by Frougny and Solomyak ([FS]) and the (W)-property introduced by Akiyama [Aki]. Further investigation of these properties can be found in [Hol] (for the (F)-property) and [ARS] (for the (W)-property).

Akiyama [Aki] shows that the algebraic properties of  $\beta$  characterize the tiling and dynamical properties of the associated Rauzy fractals.

On the other hand, Ito and his coauthors ([HI, IS, IR1]) employs the Rauzy fractals to study the  $\beta$ -expansions. He and his coauthors answered Question (Q2) when  $\beta$  is a Pisot unit (see section 1.3). The goal of this paper is to answer Question (Q3) when  $\beta$  is a quadratic Pisot unit.

**1.3. Strictly periodic  $\beta$ -expansion with Pisot unit base.** Let  $\beta$  be a Pisot unit of degree  $d$ . By using *Rauzy fractals*, a region  $K$  is constructed to serve as a Markov partition of a group automorphism on  $d$ -dimensional torus ([Pra][IR2]). It is proved that the region  $K$  completely characterizes the strictly periodic  $\beta$ -expansions in base  $\beta$ . The result is first obtained for quadratic Pisot unit by [HI], generalized to a family of Pisot units by [IS], and generalized to all Pisot units by [IR2].

Here we state only the result for quadratic Pisot unit, which is needed in the present paper. A number  $\beta > 1$  is a quadratic Pisot unit if and only if  $\beta$  satisfies  $\beta^2 = n\beta + 1$  ( $n \geq 1$ ) or  $\beta^2 = n\beta - 1$  ( $n \geq 3$ ). Since  $1, \beta$  is a basis of the field  $\mathbb{Q}(\beta)$ , for a number  $x \in \mathbb{Q}(\beta)$ ,  $x$  can be uniquely expressed as  $x = x_1 + x_2\beta$  with

$x_1, x_2 \in \mathbb{Q}$ . The number  $x' = x_1 + x_2\beta'$  is the conjugate of  $x$  in  $\mathbb{Q}(\beta)$  by definition, where  $\beta'$  is the algebraic conjugate of  $\beta$ .

**Theorem B** ([HI, IS, IR1]) (i) If  $\beta > 1$  satisfies  $\beta^2 = n\beta + 1$  then  $x \in [0, 1]$  has strictly periodic  $\beta$ -expansion if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1]$  and  $(x, x')$  belongs to the closed set  $K$  in Figure 1.a except that  $x = 1/\beta$  and  $x = 1$ .

(ii) If  $\beta > 1$  satisfies  $\beta^2 = n\beta - 1$  then  $x \in [0, 1]$  has strictly periodic  $\beta$ -expansion if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1]$  and  $(x, x')$  belongs to the closed set  $K$  in Figure 1.b.

If we use the concept of “weakly admissible”, then we have a uniform statement of Theorem B, which is discussed in full detail in Section 2.

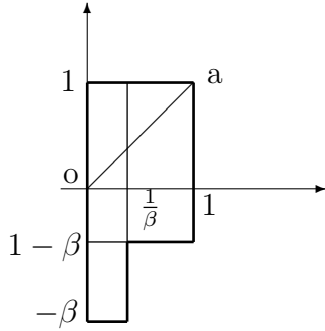


Figure 1.a

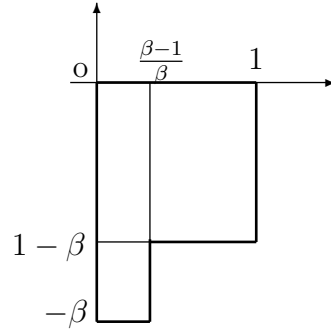


Figure 1.b

As a consequence, if  $\beta$  satisfies  $\beta^2 = n\beta + 1$ , then every rational number in  $[0, 1)$  has strictly periodic  $\beta$ -expansion since the segment  $o\vec{a}$  belongs to  $K$  in Figure 1.a (this result is first proved by Schmidt [Sch]); if  $\beta$  satisfies  $\beta^2 = n\beta - 1$ , then none of the  $\beta$ -expansions of rational numbers  $\neq 0$  is strictly periodic.

**1.4. Periods of  $\beta$ -expansions with quadratic Pisot unit base.** The main purpose of this paper is to investigate the periods of  $\beta$ -expansions.

When  $\beta$  satisfies  $\beta^2 = n\beta + 1$ , the length of period of  $\beta$ -expansion of  $p/q$  has been studied by Schmidt [Sch]. He characterized the function  $L_\beta(p/q)$  by a certain dynamical system.

Using the dynamical and tiling properties of Rauzy fractal, we obtain a satisfactory answer to Question (Q3) when  $\beta$  is a quadratic Pisot unit. Precisely, we show that  $L_\beta(x)$  is determined by a linear recurrent sequence related to  $\beta$  and  $x$ .

Let  $\beta$  be a quadratic Pisot unit with minimal polynomial  $P(x) = x^2 - a_1x - a_0$  with  $a_0 = \pm 1$ . We denote by  $\mathbb{Z}[\beta]$  the set

$$\mathbb{Z}[\beta] := \{c_0 + c_1\beta; c_0, c_1 \in \mathbb{Z}\}.$$

For  $x \in \mathbb{Q}(\beta)$ ,  $x \neq 0$ , let  $q \geq 1$  be the smallest integer such that  $qx \in \mathbb{Z}[\beta]$ . Notice that  $1, a_0\beta^{-1}$  is a basis of  $\mathbb{Z}[\beta]$ , hence  $x$  can be written uniquely as

$$x = \frac{u_0(a_0\beta^{-1}) + u_1}{q}$$

with  $u_0, u_1$  integers and  $\gcd\{u_0, u_1, q\} = 1$ .

Let  $\{u_k\}$  be the sequence of integers defined by the initial set  $u_0, u_1$  and the recurrence relation

$$u_{k+1} = a_1 u_k + a_0 u_{k-1} \quad (k \geq 1). \quad (1.1)$$

We denote this sequence by  $u_k = u_k(u_0, u_1)$ . It is easy to show that  $\{u_k \pmod{q}\}_{k \geq 0}$  is always strictly periodic ([Eng, W]). It is interesting that this sequence characterizes the periods of  $\beta$ -expansion.

**Theorem 1.1.** *Let  $\beta$  be a quadratic Pisot unit. Suppose  $x \neq 0$  and  $x = \frac{u_0(a_0\beta^{-1})+u_1}{q}$  with  $u_0, u_1, q$  integers and  $\gcd\{u_0, u_1, q\} = 1$ . If the  $\beta$ -expansion of  $x$  is strictly periodic, then the periods coincide with the periods of the sequence  $\{u_k(u_0, u_1) \pmod{q}\}_{k \geq 0}$ .*

Theorem 1.1 is the main result of this paper. It is proved in Section 2. The fact the region  $K$  translationally tiles  $\mathbb{R}^2$  plays a crucial role in our proof (see Figure 2).

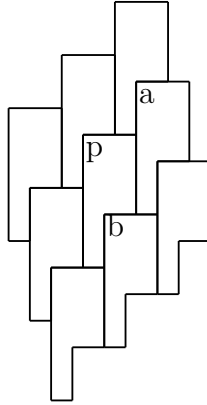


Figure 2. A tiling by  $K$

In the following, we give a second characterization of function  $L_\beta(x)$ . Denote by  $D = a_1^2 + 4a_0$  the discriminant of the polynomial  $P(x)$ . Let  $d_0$  be the maximal square-free factor of  $D$ , clearly  $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d_0})$ . Let  $D_0$  be the discriminant of the field  $\mathbb{Q}(\beta)$ , then  $D_0 = d_0$  when  $d_0 \equiv 1 \pmod{4}$  and  $D_0 = 4d_0$  otherwise. We note that  $I = \sqrt{D/D_0}$ , the index of the polynomial  $P(x)$  in  $K$ , is an integer. (See for example Hecke [Hec]).

Let  $q \geq 1$  be an integer, then the sequence  $\beta^k$  modulo  $q$  is strictly periodic since  $\beta$  is an algebraic unit. We will prove in Section 3 that this sequence also characterizes the periods of the  $\beta$ -expansions.

**Theorem 1.2.** *Let  $\beta$  be a quadratic Pisot unit. Suppose the  $\beta$ -expansion of  $x$  is strictly periodic and  $q$  is the smallest integer such that  $qx \in \mathbb{Z}[\beta]$ . If  $q$  is prime to  $D/D_0$ , then the periods of the  $\beta$ -expansion of  $x$  coincide with the periods of the sequence  $\{qx\beta^k \pmod{q}\}_{k \geq 0}$ .*

Let us denote  $N(\beta) = \beta\beta' = -a_0$  the norm of  $\beta$ . When  $N(\beta) = -1$ , every rational number  $p/q$  has strictly periodic  $\beta$ -expansion. If  $q$  is prime to  $D/D_0$ , then the periods coincides with the periods of the sequence  $\beta^k \pmod{q}$ , which is an analogue of the result of the b-adic expansion.

If  $q$  is not prime to  $D/D_0$ , then  $L_\beta(x)$  is usually a multiple of the minimum period of the sequence  $\{qx\beta^k \pmod{q}\}_{k \geq 0}$ .

**Example 1.3.** Let  $\beta = (1 + \sqrt{5})/2$  (which satisfies  $\beta^2 = \beta + 1$ ) be the golden number. Then every rational number  $p/q$  in  $[0, 1)$  has strictly periodic  $\beta$ -expansion by Theorem B. According to Theorem 1.1, the minimum period  $L_\beta(p/q)$  coincides with the minimum period of the sequence  $f_k$  modulo  $q$ , where  $f_k$  is the famous *Fibonacci sequence* which is defined by

$$f_0 = 0, f_1 = 1, f_{k+1} = f_k + f_{k-1} \quad (k \geq 1).$$

The first several terms of the sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \dots$$

For example, since the sequence  $f_k$  modulo 7 is

$$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, \dots$$

and has minimum period 16, we conclude that  $L_\beta(1/7) = 16$ . Actually the  $\beta$ -expansion of  $1/7$  is  $1/7 = 0.\dot{0}000101010101000\dot{0}$ . It is easy to show that  $L_\beta(p/7) = 16$  holds for all  $1 \leq p \leq 6$ .

**1.5. Linear recurrent sequence modulo  $q$ .** The period of linear recurrent sequence modulo  $q$  has been studied as early as 1920 by Carmichael [Car] and 1931 by Engstrom [Eng]. By using Dedekind's theorem on decomposition of primes in number field, Engstrom obtained very general results. Wall [W] (1960) studied the Fibonacci sequence modulo  $q$ . He obtained many precise and interesting results by primitive method. A complete investigation of linear recurrent sequence of degree 2 modulo  $q$  is done in [WY].

## 2. Proof of Theorem 1.1

In this section, first we investigate the weakly admissible  $\beta$ -expansions, which is closely related to the boundary of the region  $K$ . Then we show that  $K$  can tile  $\mathbb{R}^2$ . Thanks to this tiling property, finally we prove Theorem 1.1.

**2.1. Admissible and weakly admissible.** Let

$$\Omega = \{x_1x_2\dots; 0.x_1x_2\dots \text{ is a } \beta\text{-expansion}\}.$$

We may endow the space  $\Omega$  with discrete metric ([Wal]). The space  $\Omega$  is not complete under this metric. Let  $\bar{\Omega}$  be the completion of  $\Omega$ . We will see that  $\bar{\Omega}$  consists of weakly admissible sequences.

Formally we may consider the trajectory of 1:

$$1 \xrightarrow{b_1} T_\beta(1) \xrightarrow{b_2} T_\beta^2(1) \xrightarrow{b_3} \dots$$

We call  $b_1b_2b_3\dots$  the expansion of 1 and denote it by  $d_\beta(1)$ . Define

$$d_\beta^*(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite,} \\ (b_1 \dots b_{d-1}(b_d - 1))^\infty & \text{if } d_\beta(1) = b_1 \dots b_{d-1}b_d \text{ is finite with } b_d \neq 0, \end{cases}$$

A sequence over the alphabet  $\{0, 1, 2, \dots, \lfloor \beta \rfloor\}$  is *admissible* if and only if starting from any place of the sequence, the right side truncation is lexicographically strictly less than  $d_\beta^*(1)$ . If the right side truncations are less than or equal to  $d_\beta^*(1)$ , then the sequence is called *weakly admissible*.

A sequence is a  $\beta$ -expansion of a certain number if and only if this sequence is admissible. If a  $\beta$ -expansion is infinite, then it is admissible and weakly admissible. If a  $\beta$ -expansion of  $x$  is finite, then  $x$  has another infinite expansion in base  $\beta$  which is weakly admissible but not admissible. One can show that  $\bar{\Omega}$  is the set of all weakly admissible sequences (cf. [P, Thu]).

Each weakly admissible sequence  $(x_i)_{i \geq 1}$  determines a real number  $x$ , and we call  $(x_i)$  the weakly admissible  $\beta$ -expansion of  $x$ . Let  $\text{Pur}(\beta)$  be the set of the numbers in  $[0, 1)$  with strictly periodic  $\beta$ -expansions, and let  $\text{Pur}'(\beta)$  be the set of the numbers in  $[0, 1]$  with strictly periodic, weakly admissible  $\beta$ -expansions. The difference between  $\text{Pur}'(\beta)$  and  $\text{Pur}(\beta)$  is very small.

**Theorem 2.1.** *Let  $\beta > 1$  be a real number. If  $d_\beta(1)$  is infinite, then  $\text{Pur}(\beta) = \text{Pur}'(\beta)$ ; if  $d_\beta(1) = b_1b_2\dots b_d$  is finite, then  $\text{Pur}'(\beta) = \text{Pur}(\beta) \cup \{r_1, r_2, \dots, r_d\}$ , where  $r_i = 0.b_i\dots b_d$ .*

**Proof.** Suppose  $x \in \text{Pur}(\beta)$ , then the  $\beta$ -expansion of  $x$  is infinite and thus it is also admissible. This proves that  $\text{Pur}(\beta) \subset \text{Pur}'(\beta)$ .

Suppose  $\text{Pur}(\beta) \neq \text{Pur}'(\beta)$ . Take any  $x \in \text{Pur}'(\beta) \setminus \text{Pur}(\beta)$ , then the  $\beta$ -expansion of  $x$  is finite, say  $x = 0.x_1x_2\dots x_k$ . The weakly admissible  $\beta$ -expansion of  $x$  is

$$x = 0.x_1x_2\dots(x_k - 1)x_{k+1}x_{k+2}\dots$$

where  $x_{k+1}x_{k+2}\dots = d_\beta^*(1)$ . Since the above expansion is strictly periodic, we have that  $d_\beta^*(1)$  is strictly periodic. It follows that  $d_\beta(1)$  is finite, and  $x_1x_2\dots(x_k - 1)x_{k+1}x_{k+2}\dots$  coincides with a right side truncation of  $d_\beta^*(1)$ . These together imply that  $x = r_i$  for some  $i$ .  $\square$

The following theorem is a special case of a result of [IR1].

**Theorem B'** *Let  $\beta$  be a quadratic Pisot unit. Then  $x \in \text{Pur}'(\beta)$  if and only if  $x \in \mathbb{Q}(\beta) \cap [0, 1]$  and  $(x, x') \in K$ .*

When  $\beta$  satisfies  $\beta^2 = n\beta + 1$ ,  $d_\beta(1) = n1$ ; when  $\beta$  satisfies  $\beta^2 = n\beta - 1$ ,  $d_\beta(1) = (n-1)(n-2)^\infty$ . It is seen that Theorem B in Section 1 follows directly from Theorem B' and Theorem 2.1.

**2.2. A tiling by  $K$ .** Let  $\beta$  be a quadratic Pisot unit with minimal polynomial  $P(x) = x^2 - a_1x - a_0$  where  $a_0 = \pm 1$ . Set

$$\mathcal{J} = \{(x, x'); x \in \mathbb{Z}[\beta]\}.$$

Then  $\mathcal{J}$  is a lattice of  $\mathbb{R}^2$  with a basis  $(1, 1), (\beta, \beta')$ . Let us denote by  $E^\circ$  the interior of a set  $E$ .

**Lemma 2.2.** *Let  $\beta$  be a quadratic Pisot unit, let  $K$  be the region in Figure 1 associated with  $\beta$ . Then the collection  $\{K + v; v \in \mathcal{J}\}$  is a translation tiling of  $\mathbb{R}^2$ . That is,  $\mathbb{R}^2 = \bigcup_{v \in \mathcal{J}} K + v$  and  $(K + v_1)^\circ \cap (K + v_2)^\circ = \emptyset$  whenever  $v_1 \neq v_2$ ,  $v_1, v_2 \in \mathcal{J}$ .*

**Proof.** First we note that  $1, 1/\beta$  is a basis of  $\mathbb{Z}[\beta]$  and thus  $(1, 1), (1/\beta, 1/\beta')$  is a basis of  $\mathcal{J}$  when  $N(\beta) = -1$ . Likewise  $1, 1 - \beta^{-1}$  is a basis of  $\mathbb{Z}[\beta]$  and thus  $(1, 1), (1 - \beta^{-1}, 1 - (\beta')^{-1})$  is a basis of  $\mathcal{J}$  when  $N(\beta) = 1$ .

From Figure 2 we see clearly that  $K$  tiles the plane by translation, and the translation set is a lattice with a basis  $\vec{p}\vec{a}, \vec{p}\vec{b}$ . Since  $\vec{p}\vec{a} = (1, 1)$ ,  $\vec{p}\vec{b} = (1/\beta, 1/\beta')$  when  $N(\beta) = -1$ , and  $\vec{p}\vec{a} = (1, 1)$ ,  $\vec{p}\vec{b} = (1 - \beta^{-1}, 1 - (\beta')^{-1})$  when  $N(\beta) = 1$ , we conclude that the translation set is  $\mathcal{J}$  and the lemma is proved.  $\square$

**Lemma 2.3.** *Let  $\beta$  be a quadratic Pisot unit. Let  $x, y \in \mathbb{Q}(\beta) \cap [0, 1)$ ,  $x, y \neq 0$  and  $x \neq y$ . If  $x - y \in \mathbb{Z}[\beta]$ , then at most one of  $x, y$  has strictly periodic  $\beta$ -expansion.*

**Proof.** Suppose one of  $(x, x'), (y, y')$ , say  $(x, x')$ , is an inner point of the region  $K$ . From  $K + \mathcal{J}$  is a tiling of  $\mathbb{R}^2$ , we infer that  $(y, y')$  is an inner point of the tile  $K + (y - x, (y - x)')$ . Hence  $(y, y')$  is not in  $K$  and thus the  $\beta$ -expansion of  $y$  is not strictly periodic. The lemma holds in this case.

Now we consider the case that both  $(x, x'), (y, y')$  are on the boundary of  $K$ . It is easy to check that

$$\{x \in \mathbb{Q}(\beta); (x, x') \in \partial K\} = \begin{cases} \{0, \beta^{-1}, 1\}, & \text{when } N(\beta) = -1 \\ \{0, 1 - \beta^{-1}\}, & \text{when } N(\beta) = 1. \end{cases}$$

When  $N(\beta) = -1$ , only 0 belongs to  $\text{Pur}(\beta)$ ,  $\beta^{-1}$  and 1 belong to  $\text{Pur}'(\beta)$  but not belong to  $\text{Pur}(\beta)$ . When  $N(\beta) = 1$ ,  $0, 1 - \beta^{-1} \in \text{Pur}(\beta)$ . So there is at most one number satisfying  $x \neq 0$ ,  $(x, x')$  is on the boundary of  $K$  and  $x \in \text{Pur}(\beta)$ . The lemma is hence proved in this case.  $\square$

**Remark 2.4.** By the above discussion, we see that  $\mathbb{Z}[\beta] \cap \text{Pur}(\beta) = \{0\}$  when  $N(\beta) = -1$ ;  $\mathbb{Z}[\beta] \cap \text{Pur}(\beta) = \{0, 1 - \beta^{-1}\}$  when  $N(\beta) = 1$ . Since  $1 - \beta^{-1} = 0.(n-2)^\infty$  when  $N(\beta) = 1$ , one check directly that Theorem 1.1 holds when  $q = 1$ .

2.3. **Carry sequence.** Let

$$P(x) = x^2 - a_1x - a_0 \quad (2.1)$$

be the minimal polynomial of  $\beta$ . Let  $q \geq 1$  be the smallest integer such that  $qx \in \mathbb{Z}[\beta]$ . Then  $x$  can be written uniquely as

$$x = \frac{u_0(a_0\beta^{-1}) + u_1}{q} \quad (2.2)$$

with  $u_0, u_1$  integers and  $\gcd\{u_0, u_1, q\} = 1$ .

We define a sequence  $\tilde{u}_k$  as following. Set  $\tilde{u}_0 = u_0, \tilde{u}_1 = u_1$ . Suppose  $\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_k$  is defined, we define  $\tilde{u}_{k+1}$  to be the unique integer such that

$$\tilde{u}_{k+1} \equiv a_1\tilde{u}_k + a_0\tilde{u}_{k-1} \pmod{q} \text{ and } 0 \leq \tilde{u}_k(a_0\beta^{-1}) + \tilde{u}_{k+1} < q. \quad (2.3)$$

Let us call  $\{\tilde{u}_k\}_{k \geq 0}$  the *carry sequence* of  $x$ .

Carry sequence is first introduced by Hollander [Hol] in the case  $q = 1$ , i.e., for  $x \in \mathbb{Z}[\beta]$ ; it has been used in [AR, ARS]. Here we generalize it to all  $x \in \mathbb{Q}(\beta)$ . The carry sequence is closely related to the  $\beta$  expansion.

Let  $x = 0.x_1x_2\dots$  be the  $\beta$ -expansion of  $x$ . Then by (2.1), (2.2) and (2.3), one has

$$T_\beta(x) = \beta x - x_1 = \frac{\tilde{u}_1(a_0\beta^{-1}) + (a_1u_1 + a_0u_0 - x_1q)}{q} = \frac{\tilde{u}_1(a_0\beta^{-1}) + \tilde{u}_2}{q}.$$

In general, it is easy to show by induction that

$$T_\beta^k(x) = \frac{\tilde{u}_k(a_0\beta^{-1}) + \tilde{u}_{k+1}}{q}.$$

Therefore  $x$  has strictly periodic  $\beta$ -expansion if and only if its carry sequence  $\tilde{u}_k$  is strictly periodic. Moreover,  $L_\beta(x)$  equals to the minimal period of the sequence  $\tilde{u}_k$ .

**Proof of Theorem 1.1.** Let  $\{u_k(u_0, u_1)\}_{k \geq 0}$  be the linear recurrent sequence defined by (1.1). We claim that

$$u_k \equiv \tilde{u}_k \pmod{q} \quad (2.4)$$

holds for all  $k \geq 0$ . For suppose (2.4) holds for  $1, \dots, k$ , then

$$\tilde{u}_{k+1} \equiv a_1\tilde{u}_k + a_0\tilde{u}_{k-1} \equiv a_1u_k + a_0u_{k-1} = u_{k+1} \pmod{q}.$$

We remain to show that the sequences  $\tilde{u}_k$  and  $u_k \pmod{q}$  have the same periods, which completes the proof of the theorem.

Let  $h$  be a period of the sequence  $\tilde{u}_k$ , then  $\tilde{u}_0 = \tilde{u}_h, \tilde{u}_1 = \tilde{u}_{h+1}$ . Thus by (2.4), we have  $u_0 \equiv u_h, u_1 \equiv u_{h+1} \pmod{q}$ . Therefore  $h$  is a period of the sequence  $u_k$  modulo  $q$  since  $u_{k+1}$  is completely determined by  $u_k$  and  $u_{k-1}$ .

On the other hand, suppose  $h$  is a period of  $u_k \pmod{q}$ . Again by (2.4), we have  $\tilde{u}_0 \equiv \tilde{u}_h, \tilde{u}_1 \equiv \tilde{u}_{h+1} \pmod{q}$ . Let  $y = T_\beta^h(x)$ , then

$$x - y = \frac{(\tilde{u}_0 - \tilde{u}_h)(a_0\beta^{-1}) + (\tilde{u}_1 - \tilde{u}_{h+1})}{q} \in \mathbb{Z}[\beta].$$

Since both  $x$  and  $y$  have strictly periodic  $\beta$ -expansion, we have  $x = y$  by Lemma 2.3. Hence  $h$  is a period of the  $\beta$ -expansion of  $x$  as well as a period of  $\tilde{u}_k$ .  $\square$

### 3. Proof of Theorem 1.2

To prove Theorem 1.2, we need three easy lemmas. Lemma 3.1 gives the formula of the general term of the sequence  $u_k$  (see for example [Eng],[W]).

**Lemma 3.1.** *The general term of  $u_k$  is given by*

$$u_k = c_1\beta^k + c_2(\beta')^k \quad (3.1)$$

where

$$c_1 = \frac{u_1 + a_0\beta^{-1}u_0}{\beta - \beta'}, \quad c_2 = (c_1)' = \frac{u_1 + a_0(\beta')^{-1}u_0}{\beta' - \beta}.$$

**Proof.** One can check the initial value of (3.1) is  $u_0, u_1$ , and  $u_k$  in (3.1) satisfies the recurrence relation (1.1).  $\square$

Let  $H(q)$  be the minimal period of the sequence  $u_k$  modulo  $q$ . Then

**Lemma 3.2.** *If  $q = p_1^{e_1} \cdots p_k^{e_k}$ , then*

$$H(q) = \text{lcm}\{H(p_1^{e_1}), \dots, H(p_k^{e_k})\}. \quad (3.2)$$

where  $\text{lcm}$  denote the least common multiple.

Lemma 3.2 can be found in [Eng] [W]. We leave the easy proof to the reader.

Let  $\mathfrak{R}$  be an ideal in  $\mathbb{Q}(\beta)$ . Let us denote by  $G(x, \mathfrak{R})$  the minimal period of the sequence  $x, x\beta, x\beta^2, \dots$ , modulo  $\mathfrak{R}$ . Then similar to Lemma 3.2, we have

**Lemma 3.3.** *If  $q = p_1^{e_1} \cdots p_k^{e_k}$ , then*

$$G(x, q) = \text{lcm}\{G(x, p_1^{e_1}), G(x, p_2^{e_2}), \dots, G(x, p_k^{e_k})\}.$$

**Theorem 3.4.** *Let  $x_0 = u_0(a_0\beta^{-1}) + u_1$ . If  $q$  is prime to  $D/D_0$ , then the periods of the sequence  $\{u_k \pmod{q}\}$  coincide with the periods of the sequence*

$$x_0, x_0\beta, x_0\beta^2, \dots \pmod{q}.$$

**Proof.** Since  $a_0 = \pm 1$ , it is easy to see that both  $\{x_0\beta^k \pmod{q}\}$  and  $\{u_k \pmod{q}\}$  are strictly periodic.

According to Lemma 3.2 and Lemma 3.3, we need only show that the theorem is valid for  $p^m$  where  $p$  is prime to  $D/D_0$ .

Set  $x_k = u_k(a_0\beta^{-1}) + u_{k+1}$ ,  $k \geq 0$ . Then  $x_{k+1} = \beta x_k$  and so that  $x_k = x_0\beta^k$ . Hence  $x_k \in \mathbb{Z}(\beta)$  are algebraic integers in  $\mathbb{Q}(\beta)$ .

Clearly if  $h$  is a period of  $\{u_k \pmod{q}\}$ , then it is a period of  $\{x_k \pmod{q}\}$ .

On the other hand, suppose  $h$  is a period of  $\{x_k \pmod{q}\}$ . Then  $x_0(\beta^h - 1)$  is divisible by  $q = p^m$  and so that the algebraic conjugate  $(x_0(\beta^h - 1))'$  is also divisible by  $p^m$ . By Lemma 3.1, we have

$$u_k = \frac{(u_1 + a_0\beta^{-1}u_0)\beta^k - (u_1 + a_0\beta^{-1}u_0)'(\beta')^k}{\beta - \beta'}.$$

Hence

$$u_{k+h} - u_k = \frac{x_0(\beta^h - 1)\beta^k - x_0'((\beta')^h - 1)(\beta')^k}{\beta - \beta'}. \quad (3.3)$$

By the assumption of  $h$ , we have that the numerator of the right side of (3.3) is divisible by  $p^m$ . Notice that  $\beta - \beta' = \sqrt{D} = I\sqrt{D_0}$ . Let

If  $p$  is prime to  $D$ , then the denominator is prime to  $p^m$ . Hence  $u_{k+h} - u_k$  is divisible by  $p^m$  for all  $k \geq 0$ , and  $h$  is a period of  $\{u_k \pmod{p^m}\}$ .

If  $p$  is a factor of  $D_0$ , then  $p = \mathfrak{R}^2$  where  $\mathfrak{R}$  is a prime ideal in  $\mathbb{Q}(\beta)$ . We divide the discussion into two cases  $p \neq 2$  and  $p = 2$ . Recall that  $D_0 = d_0$  or  $4d_0$  where  $d_0$  is square-free.

If  $p \neq 2$ , then  $p^2 \nmid D_0$  and so that the power of  $\mathfrak{R}$  contained in the denominator of (3.3) is 1. Since the power of  $\mathfrak{R}$  contained in the numerator of (3.3) is at least  $2m$ , we have that  $(u_{k+h} - u_k)/p^{m-1}$  is an integer and it is divisible by  $p$ , the norm of  $\mathfrak{R}$ . Therefore  $u_{k+h} - u_k$  is divisible by  $q = p^m$  for all  $k \geq 0$ .

If  $p = 2$ , then  $D_0$  is an even number and hence  $D_0 = 4d_0$ . Moreover  $1, \sqrt{d_0}$  is a basis of  $\mathbb{Q}(\beta)$ . Let  $x_0(\beta^h - 1)\beta^k = X + Y\sqrt{d_0}$ , then  $q = 2^m$  divides  $X + Y\sqrt{d_0}$  implies that  $2^m | Y$ . Formula (3.3) becomes

$$u_{k+h} - u_k = \frac{2Y\sqrt{d_0}}{\sqrt{D}} = \frac{2Y\sqrt{d_0}}{I\sqrt{D_0}} = \frac{2Y\sqrt{d_0}}{2I\sqrt{d_0}} = \frac{Y}{I}.$$

Since  $p = 2$  is coprime to  $I$ , we conclude that  $u_{k+h} - u_k$  is divisible by  $2^m$  for all  $k \geq 0$ . This completes the proof the theorem.  $\square$

Theorem 1.2 follows immediately from Theorem 1.1 and Theorem 3.4.

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