PERIODS OF $\beta$-EXPANSIONS AND LINEAR RECURRENT SEQUENCES

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Abstract. Let $\beta > 1$ be a Pisot number. It is well known that a number $x$ has periodic $\beta$-expansion if and only if $x \in \mathbb{Q}(\beta)$. When $\beta$ is a quadratic Pisot unit, we show that the period of the $\beta$-expansion of $x$ is determined by a linear recurrent sequence related to $\beta$ and $x$. Particularly, if $\beta = (\sqrt{5} + 1)/2$ is the golden number, then the periods of the $\beta$-expansions are determined by the prominent Fibonacci sequence.

1. Introduction

1.1. $\beta$-numeration system. Let $\beta > 1$ be a real number. The $\beta$-transformation is a piecewise linear transformation on $[0, 1)$ defined by

$$T_{\beta} : x \mapsto \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \alpha \rfloor$ is the largest integer not exceeding $\alpha$. By iterating this map and considering its trajectory

$$x \xrightarrow{T_{\beta}} T_{\beta}(x) \xrightarrow{T_{\beta}^2} T_{\beta}^2(x) \xrightarrow{T_{\beta}^3} \cdots$$

with $x_i = \lfloor \beta T_{\beta}^{i-1}(x) \rfloor$, we obtain the $\beta$-expansion of $x$

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots = 0.x_1x_2x_3\ldots.$$

An expansion is finite if $(x_i)_{i\geq1}$ is eventually 0. A $\beta$-expansion is periodic if there exists $p \geq 1$ and $M \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq M$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is strictly periodic (or purely periodic). When the $\beta$-expansion of $x$ is periodic, we denote by $L_{\beta}(x)$ the minimal period of the expansion.

When $\beta = b \geq 2$ is an integer, then the $\beta$-expansion is the $b$-adic expansion. In this case, it is well known that:

(i) A number $x \in [0, 1)$ has periodic expansion if and only if $x$ is a rational number.

(ii) The expansion of $x = p/q$ is strictly periodic if and only if $\gcd\{q, b\} = 1$.

(iii) The minimum period of the $b$-adic expansion of $p/q$ coincides with the minimum period of sequence $\{b^n \pmod q\}_{n\geq0}$.

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It is natural to ask the same questions for $\beta$-expansions when $\beta$ is not an integer.

(Q1). For which $x \in [0, 1)$, is the $\beta$-expansion periodic?
(Q2). For which $x \in [0, 1)$, is the $\beta$-expansion strictly periodic?
(Q3). If the $\beta$-expansion of $x$ is periodic, what is the minimum period?

1.2. Periodic $\beta$-expansion and Pisot number. Question (Q1) has been settled down for all Pisot numbers (See Schmidt [Sch] 1980). An algebraic integer strictly greater than 1 is a Pisot number if all its algebraic conjugates have modulus strictly less than 1. A number is called Pisot unit if it is a Pisot number and an algebraic unit. Let $\mathbb{Q}(\beta)$ be the smallest field containing rational numbers and $\beta$. It is well known that

Theorem A (Schmidt [Sch]) Let $\beta$ be a Pisot number. Then $x \in [0, 1)$ has periodic $\beta$-expansion if and only if $x \in \mathbb{Q}(\beta) \cap [0, 1)$.

The $\beta$-expansion with a Pisot base has drawn the attention of many mathematicians. For a Pisot unit $\beta$, Rauzy ([Rau] 1982) and Thurston ([Thu] 1989) have constructed a self-similar tiling system. The tiles (usually have fractal boundaries) are called Rauzy fractals or atomic surfaces. Several interesting dynamical system are related to Rauzy fractals (cf. [AI, IR2]).

According to the behavior of the $\beta$-expansion, several algebraic properties of $\beta$ have been defined, for example, the (F)-property introduced by Frougny and Solomyak ([FS]) and the (W)-property introduced by Akiyama [Aki]. Further investigation of these properties can be found in [Hol] (for the (F)-property) and [ARS] (for the (W)-property).

Akiyama [Aki] shows that the algebraic properties of $\beta$ characterize the tiling and dynamical properties of the associated Rauzy fractals.

On the other hand, Ito and his coauthors ([HI, IS, IR1]) employs the Rauzy fractals to study the $\beta$-expansions. He and his coauthors answered Question (Q2) when $\beta$ is a Pisot unit (see section 1.3). The goal of this paper is to answer Question (Q3) when $\beta$ is a quadratic Pisot unit.

1.3. Strictly periodic $\beta$-expansion with Pisot unit base. Let $\beta$ be a Pisot unit of degree $d$. By using Rauzy fractals, a region $K$ is constructed to serve as a Markov partition of a group automorphism on $d$-dimensional torus ([Pra][IR2]). It is proved that the region $K$ completely characterizes the strictly periodic $\beta$-expansions in base $\beta$. The result is first obtained for quadratic Pisot unit by [HI], generalized to a family of Pisot units by [IS], and generalized to all Pisot units by [IR2].

Here we state only the result for quadratic Pisot unit, which is needed in the present paper. A number $\beta > 1$ is a quadratic Pisot unit if and only if $\beta$ satisfies $\beta^2 = n\beta + 1$ ($n \geq 1$) or $\beta^2 = n\beta - 1$ ($n \geq 3$). Since $1, \beta$ is a basis of the field $\mathbb{Q}(\beta)$, for a number $x \in \mathbb{Q}(\beta)$, $x$ can be uniquely expressed as $x = x_1 + x_2\beta$ with
The number \( x' = x_1 + x_2\beta' \) is the conjugate of \( x \) in \( \mathbb{Q}(\beta) \) by definition, where \( \beta' \) is the algebraic conjugate of \( \beta \).

**Theorem B** ([HI, IS, IR1])

(i) If \( \beta > 1 \) satisfies \( \beta^2 = n\beta + 1 \) then \( x \in [0,1] \) has strictly periodic \( \beta \)-expansion if and only if \( x \in \mathbb{Q}(\beta) \cap [0, 1] \) and \( (x, x') \) belongs to the closed set \( K \) in Figure 1.a except that \( x = 1/\beta \) and \( x = 1 \).

(ii) If \( \beta > 1 \) satisfies \( \beta^2 = n\beta - 1 \) then \( x \in [0,1] \) has strictly periodic \( \beta \)-expansion if and only if \( x \in \mathbb{Q}(\beta) \cap [0, 1] \) and \( (x, x') \) belongs to the closed set \( K \) in Figure 1.b.

If we use the concept of “weakly admissible”, then we have a uniform statement of Theorem B, which is discussed in full detail in Section 2.

As a consequence, if \( \beta \) satisfies \( \beta^2 = n\beta + 1 \), then every rational number in \( [0,1) \) has strictly periodic \( \beta \)-expansion since the segment \( \overrightarrow{oa} \) belongs to \( K \) in Figure 1.a (this result is first proved by Schmidt [Sch]); if \( \beta \) satisfies \( \beta^2 = n\beta - 1 \), then none of the \( \beta \)-expansions of rational numbers \( \neq 0 \) is strictly periodic.

1.4. **Periods of \( \beta \)-expansions with quadratic Pisot unit base.** The main purpose of this paper is to investigate the periods of \( \beta \)-expansions.

When \( \beta \) satisfies \( \beta^2 = n\beta + 1 \), the length of period of \( \beta \)-expansion of \( p/q \) has been studied by Schmidt [Sch]. He characterized the function \( L_\beta(p/q) \) by a certain dynamical system.

Using the dynamical and tiling properties of Rauzy fractal, we obtain a satisfactory answer to Question (Q3) when \( \beta \) is a quadratic Pisot unit. Precisely, we show that \( L_\beta(x) \) is determined by a linear recurrent sequence related to \( \beta \) and \( x \).

Let \( \beta \) be a quadratic Pisot unit with minimal polynomial \( P(x) = x^2 - a_1x - a_0 \) with \( a_0 = \pm 1 \). We denote by \( \mathbb{Z}[^\beta] \) the set

\[
\mathbb{Z}[^\beta] := \{ c_0 + c_1\beta; \ c_0, c_1 \in \mathbb{Z} \}.
\]

For \( x \in \mathbb{Q}(\beta) \), \( x \neq 0 \), let \( q \geq 1 \) be the smallest integer such that \( qx \in \mathbb{Z}[^\beta] \). Notice that \( 1, a_0\beta^{-1} \) is a basis of \( \mathbb{Z}[^\beta] \), hence \( x \) can be written uniquely as

\[
x = \frac{u_0(a_0\beta^{-1}) + u_1}{q}
\]
with $u_0, u_1$ integers and $\gcd\{u_0, u_1, q\} = 1$.

Let $\{u_k\}$ be the sequence of integers defined by the initial set $u_0, u_1$ and the recurrence relation

\[ u_{k+1} = a_1 u_k + a_0 u_{k-1} \quad (k \geq 1). \]

We denote this sequence by $u_k = u_k(u_0, u_1)$. It is easy to show that $\{u_k \pmod{q}\}_{k \geq 0}$ is always strictly periodic ([Eng, W]). It is interesting that this sequence characterizes the periods of $\beta$-expansion.

**Theorem 1.1.** Let $\beta$ be a quadratic Pisot unit. Suppose $x \neq 0$ and

\[ x = \frac{u_0 (a_0 \beta^{-1}) + u_1}{q} \]

with $u_0, u_1, q$ integers and $\gcd\{u_0, u_1, q\} = 1$. If the $\beta$-expansion of $x$ is strictly periodic, then the periods coincide with the periods of the sequence $\{u_k(u_0, u_1) \pmod{q}\}_{k \geq 0}$.

Theorem 1.1 is the main result of this paper. It is proved in Section 2. The fact the region $K$ translationally tiles $\mathbb{R}^2$ plays a crucial role in our proof (see Figure 2).

![Figure 2. A tiling by $K$](image)

In the following, we give a second characterization of function $L_\beta(x)$. Denote by $D = a_1^2 + 4a_0$ the discriminant of the polynomial $P(x)$. Let $d_0$ be the maximal square-free factor of $D$, clearly $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d_0})$. Let $D_0$ be the discriminant of the field $\mathbb{Q}(\beta)$, then $D_0 = d_0$ when $d_0 \equiv 1 \pmod{4}$ and $D_0 = 4d_0$ otherwise. We note that $I = \sqrt{D/D_0}$, the index of the polynomial $P(x)$ in $K$, is an integer. (See for example Hecke [Hec]).

Let $q \geq 1$ be an integer, then the sequence $\beta^k$ modulo $q$ is strictly periodic since $\beta$ is an algebraic unit. We will prove in Section 3 that this sequence also characterize the periods of the $\beta$-expansions.

**Theorem 1.2.** Let $\beta$ be a quadratic Pisot unit. Suppose the $\beta$-expansion of $x$ is strictly periodic and $q$ is the smallest integer such that $qx \in \mathbb{Z}[\beta]$. If $q$ is prime to $D/D_0$, then the periods of the $\beta$-expansion of $x$ coincide with the periods of the sequence $\{qx\beta^k \pmod{q}\}_{k \geq 0}$. 
Let us denote $N(\beta) = \beta \beta' = -a_0$ the norm of $\beta$. When $N(\beta) = -1$, every rational number $p/q$ has strictly periodic $\beta$-expansion. If $q$ is prime to $D/D_0$, then the periods coincides with the periods of the sequence $\beta^k \pmod{q}$, which is an analogue of the result of the $b$-adic expansion.

If $q$ is not prime to $D/D_0$, then $L(\beta)(x)$ is usually a multiple of the minimum period of the sequence \(\{qx\beta^k \pmod{q}\}_{k \geq 0}\).

Example 1.3. Let $\beta = (1 + \sqrt{5})/2$ (which satisfies $\beta^2 = \beta + 1$) be the golden number. Then every rational number $p/q$ in $[0, 1)$ has strictly periodic $\beta$-expansion by Theorem B. According to Theorem 1.1, the minimum period $L(\beta)(p/q)$ coincides with the minimum period of the sequence $f_k$ modulo $q$, where $f_k$ is the famous Fibonacci sequence which is defined by

\[
    f_0 = 0, \quad f_1 = 1, \quad f_{k+1} = f_k + f_{k-1} \quad (k \geq 1).
\]

The first several terms of the sequence are

\[
    0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots
\]

For example, since the sequence $f_k$ modulo 7 is

\[
    0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, \ldots
\]

and has minimum period 16, we conclude that $L(\beta)(1/7) = 16$. Actually the $\beta$-expansion of $1/7$ is $0.00001010100\ldots$. We easily show that $L(\beta)(p/7) = 16$ holds for all $1 \leq p \leq 6$.

1.5. Linear recurrent sequence modulo $q$. The period of linear recurrent sequence modulo $q$ has been studied as early as 1920 by Carmichael [Car] and 1931 by Engstrom [Eng]. By using Dedekind’s theorem on decomposition of primes in number field, Engstrom obtained very general results. Wall [W] (1960) studied the Fibonacci sequence modulo $q$. He obtained many precise and interesting results by primitive method. A complete investigation of linear recurrent sequence of degree 2 modulo $q$ is done in [WY].

2. Proof of Theorem 1.1

In this section, first we investigate the weakly admissible $\beta$-expansions, which is closely related to the boundary of the region $K$. Then we show that $K$ can tile $\mathbb{R}^2$. Thanks to this tiling property, finally we prove Theorem 1.1.

2.1. Admissible and weakly admissible. Let

\[
    \Omega = \{x_1x_2 \ldots ; \ 0.x_1x_2 \ldots \ \text{is a } \beta\text{-expansion}\}.
\]

We may endow the space $\Omega$ with discrete metric ([Wal]). The space $\Omega$ is not complete under this metric. Let $\tilde{\Omega}$ be the completion of $\Omega$. We will see that $\tilde{\Omega}$ consists of weakly admissible sequences.
Formally we may consider the trajectory of 1:

\[ 1 \xrightarrow{b_1} T_\beta(1) \xrightarrow{b_2} T_\beta^2(1) \xrightarrow{b_3} \cdots \]

We call \( b_1 b_2 b_3 \ldots \) the expansion of 1 and denote it by \( d_\beta(1) \). Define

\[
d_\beta^*(1) = \begin{cases} 
d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite,} \\
(b_1 \ldots b_{d-1}(b_d - 1))^{\infty} & \text{if } d_\beta(1) = b_1 \ldots b_{d-1}b_d \text{ is finite with } b_d \neq 0,
\end{cases}
\]

A sequence over the alphabet \( \{0, 1, 2, \ldots, \lfloor \beta \rfloor\} \) is \emph{admissible} if and only if starting from any place of the sequence, the right side truncation is lexicographically strictly less than \( d_\beta^*(1) \). If the right side truncations are less than or equal to \( d_\beta^*(1) \), then the sequence is called \emph{weakly admissible}.

A sequence is a \( \beta \)-expansion of a certain number if and only if this sequence is admissible. If a \( \beta \)-expansion is infinite, then it is admissible and weakly admissible. If a \( \beta \)-expansion of \( x \) is finite, then \( x \) has another infinite expansion in base \( \beta \) which is weakly admissible but not admissible. One can show that \( \Omega \) is the set of all weakly admissible sequences (cf. [P, Thu]).

Each weakly admissible sequence \((x_i)_{i \geq 1}\) determines a real number \( x \), and we call \((x_i)\) the weakly admissible \( \beta \)-expansion of \( x \). Let \( \text{Pur}(\beta) \) be the set of the numbers in [0, 1) with strictly periodic \( \beta \)-expansions, and let \( \text{Pur}'(\beta) \) be the set of the numbers in [0, 1] with strictly periodic, weakly admissible \( \beta \)-expansions. The difference between \( \text{Pur}'(\beta) \) and \( \text{Pur}(\beta) \) is very small.

\textbf{Theorem 2.1.} Let \( \beta > 1 \) be a real number. If \( d_\beta(1) \) is infinite, then \( \text{Pur}(\beta) = \text{Pur}'(\beta) \); if \( d_\beta(1) = b_1b_2 \ldots b_d \) is finite, then \( \text{Pur}'(\beta) = \text{Pur}(\beta) \cup \{r_1, r_2, \ldots, r_d\} \), where \( r_i = 0.b_i \ldots b_d \).

\textbf{Proof.} Suppose \( x \in \text{Pur}(\beta) \), then the \( \beta \)-expansion of \( x \) is infinite and thus it is also admissible. This proves that \( \text{Pur}(\beta) \subset \text{Pur}'(\beta) \).

Suppose \( \text{Pur}(\beta) \neq \text{Pur}'(\beta) \). Take any \( x \in \text{Pur}'(\beta) \setminus \text{Pur}(\beta) \), then the \( \beta \)-expansion of \( x \) is finite, say \( x = 0.x_1x_2 \ldots x_k \). The weakly admissible \( \beta \)-expansion of \( x \) is

\[
x = 0.x_1x_2 \ldots (x_k - 1)x_{k+1}x_{k+2} \ldots
\]

where \( x_{k+1}x_{k+2} \ldots = d_\beta(1) \). Since the above expansion is strictly periodic, we have that \( d_\beta(1) \) is strictly periodic. It follows that \( d_\beta(1) \) is finite, and \( x_1x_2 \ldots (x_k - 1)x_{k+1}x_{k+2} \ldots \) coincides with a right side truncation of \( d_\beta^*(1) \). These together imply that \( x = r_i \) for some \( i \).

The following theorem is a special case of a result of [IR1].

\textbf{Theorem B'} Let \( \beta \) be a quadratic Pisot unit. Then \( x \in \text{Pur}'(\beta) \) if and only if \( x \in \mathbb{Q}(\beta) \cap [0,1] \) and \((x, x') \in K\).
When $\beta$ satisfies $\beta^2 = n\beta + 1$, $d_\beta(1) = n1$; when $\beta$ satisfies $\beta^2 = n\beta - 1$, $d_\beta(1) = -(n-1)(n-2)^\infty$. It is seen that Theorem B in Section 1 follows directly form Theorem B' and Theorem 2.1.

2.2. A tiling by $K$. Let $\beta$ be a quadratic Pisot unit with minimal polynomial $P(x) = x^2 - a_1 x - a_0$ where $a_0 = \pm 1$. Set

$$J = \{(x, x'); \ x \in \mathbb{Z}[\beta]\}.$$

Then $J$ is a lattice of $\mathbb{R}^2$ with a basis $(1,1), (\beta, \beta')$. Let us denote by $E^\circ$ the interior of a set $E$.

**Lemma 2.2.** Let $\beta$ be a quadratic Pisot unit, let $K$ be the region in Figure 1 associated with $\beta$. Then the collection $\{K + v; \ v \in J\}$ is a translation tiling of $\mathbb{R}^2$. That is, $\mathbb{R}^2 = \bigcup_{v \in J} K + v$ and $(K + v_1)\cap (K + v_2) = \emptyset$ whenever $v_1 \neq v_2, v_1, v_2 \in J$.

**Proof.** First we note that $1, 1/\beta$ is a basis of $\mathbb{Z}[\beta]$ and thus $(1, 1), (1/\beta, 1/\beta')$ is a basis of $J$ when $N(\beta) = -1$. Likewise $1, 1 - \beta^{-1}$ is a basis of $\mathbb{Z}[\beta]$ and thus $(1, 1), (1 - \beta^{-1}, 1 - (\beta')^{-1})$ is a basis of $J$ when $N(\beta) = 1$.

From Figure 2 we see clearly that $K$ tiles the plane by translation, and the translation set is a lattice with a basis $\vec{p}a, \vec{p}b$. Since $\vec{p}a = (1, 1), \vec{p}b = (1/\beta, 1/\beta')$ when $N(\beta) = -1$, and $\vec{p}a = (1, 1), \vec{p}b = (1 - \beta^{-1}, 1 - (\beta')^{-1})$ when $N(\beta) = 1$, we conclude that the translation set is $J$ and the lemma is proved. $\square$

**Lemma 2.3.** Let $\beta$ be a quadratic Pisot unit. Let $x, y \in \mathbb{Q}(\beta) \cap [0, 1)$, $x, y \neq 0$ and $x \neq y$. If $x - y \in \mathbb{Z}[\beta]$, then at most one of $x, y$ has strictly periodic $\beta$-expansion.

**Proof.** Suppose one of $(x, x'), (y, y')$, say $(x, x')$, is an inner point of the region $K$. From $K + J$ is a tiling of $\mathbb{R}^2$, we infer that $(y, y')$ is an inner point of the tile $K + (y - x, (y - x') \cdot K$. Hence $(y, y')$ is not in $K$ and thus the $\beta$-expansion of $y$ is not strictly periodic. The lemma holds in this case.

Now we consider the case that both $(x, x'), (y, y')$ are on the boundary of $K$. It is easy to check that

$$\{x \in \mathbb{Q}(\beta); \ (x, x') \in \partial K\} = \begin{cases} \{0, \beta^{-1}, 1\}, & \text{when } N(\beta) = -1 \\ \{0, 1 - \beta^{-1}\}, & \text{when } N(\beta) = 1. \end{cases}$$

When $N(\beta) = -1$, only 0 belongs to $\text{Pur}(\beta)$, $\beta^{-1}$ and 1 belong to $\text{Pur}'(\beta)$ but not belong to $\text{Pur}(\beta)$. When $N(\beta) = 1$, 0, $1 - \beta^{-1} \in \text{Pur}(\beta)$. So there is at most one number satisfying $x \neq 0$; $(x, x')$ is on the boundary of $K$ and $x \in \text{Pur}(\beta)$. The lemma is hence proved in this case. $\square$

**Remark 2.4.** By the above discussion, we see that $\mathbb{Z}[\beta] \cap \text{Pur}(\beta) = \{0\}$ when $N(\beta) = -1; \mathbb{Z}[\beta] \cap \text{Pur}(\beta) = \{0, 1 - \beta^{-1}\}$ when $N(\beta) = 1$. Since $1 - \beta^{-1} = 0.(n-2)^\infty$ when $N(\beta) = 1$, one check directly that Theorem 1.1 holds when $q = 1$.
2.3. Carry sequence. Let
\begin{equation}
P(x) = x^2 - a_1 x - a_0 \tag{2.1}
\end{equation}
be the minimal polynomial of \( \beta \). Let \( q \geq 1 \) be the smallest integer such that \( qx \in \mathbb{Z}[\beta] \). Then \( x \) can be written uniquely as
\begin{equation}
x = \frac{u_0(a_0 \beta^{-1}) + u_1}{q} \tag{2.2}
\end{equation}
with \( u_0, u_1 \) integers and \( \gcd\{u_0, u_1, q\} = 1 \).

We define a sequence \( \tilde{u}_k \) as following. Set \( \tilde{u}_0 = u_0, \tilde{u}_1 = u_1 \). Suppose \( \tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_k \) is defined, we define \( \tilde{u}_{k+1} \) to be the unique integer such that
\begin{equation}
\tilde{u}_{k+1} \equiv a_1 \tilde{u}_k + a_0 \tilde{u}_{k-1} \pmod{q} \quad \text{and} \quad 0 \leq \tilde{u}_k(a_0 \beta^{-1}) + \tilde{u}_{k+1} < q. \tag{2.3}
\end{equation}
Let us call \( \{\tilde{u}_k\}_{k \geq 0} \) the carry sequence of \( x \).

Carry sequence is first introduced by Hollander [Hol] in the case \( q = 1 \), i.e., for \( x \in \mathbb{Z}[\beta] \); it has been used in [AR, ARS]. Here we generalize it to all \( x \in \mathbb{Q}(\beta) \). The carry sequence is closely related to the \( \beta \)-expansion.

Let \( x = 0.x_1 x_2 \ldots \) be the \( \beta \)-expansion of \( x \). Then by (2.1), (2.2) and (2.3), one has
\begin{equation}
T_\beta(x) = \beta x - x_1 = \frac{\tilde{u}_1(a_0 \beta^{-1}) + (a_1 u_1 + a_0 u_0 - x_1 q)}{q} = \frac{\tilde{u}_1(a_0 \beta^{-1}) + \tilde{u}_2}{q}.
\end{equation}
In general, it is easy to show by induction that
\begin{equation}
T_\beta^k(x) = \frac{\tilde{u}_k(a_0 \beta^{-1}) + \tilde{u}_{k+1}}{q}.
\end{equation}
Therefore \( x \) has strictly periodic \( \beta \)-expansion if and only if its carry sequence \( \tilde{u}_k \) is strictly periodic. Moreover, \( L_\beta(x) \) equals to the minimal period of the sequence \( \tilde{u}_k \). 

**Proof of Theorem 1.1.** Let \( \{u_k(u_0, u_1)\}_{k \geq 0} \) be the linear recurrent sequence defined by (1.1). We claim that
\begin{equation}
u_k \equiv \tilde{u}_k \pmod{q} \tag{2.4}
\end{equation}
holds for all \( k \geq 0 \). For suppose (2.4) holds for \( 1, \ldots, k \), then
\begin{equation}
\tilde{u}_{k+1} \equiv a_1 \tilde{u}_k + a_0 \tilde{u}_{k-1} \equiv a_1 u_k + a_0 u_{k-1} = u_{k+1} \pmod{q}.
\end{equation}
We remain to show that the sequences \( \tilde{u}_k \) and \( u_k \pmod{q} \) have the same periods, which completes the proof of the theorem.

Let \( h \) be a period of the sequence \( \tilde{u}_k \), then \( \tilde{u}_0 = \tilde{u}_h, \tilde{u}_1 = \tilde{u}_{h+1} \). Thus by (2.4), we have \( u_0 \equiv u_h, u_1 \equiv u_{h+1} \pmod{q} \). Therefore \( h \) is a period of the sequence \( u_k \) modulo \( q \) since \( u_{k+1} \) is completely determined by \( u_k \) and \( u_{k-1} \).
On the other hand, suppose \( h \) is a period of \( u_k \pmod{q} \). Again by (2.4), we have \( \tilde{u}_0 \equiv \tilde{u}_h, \tilde{u}_1 \equiv \tilde{u}_{h+1} \pmod{q} \). Let \( y = T^h_\beta(x) \), then
\[
x - y = \frac{(\tilde{u}_0 - \tilde{u}_h)(a_0\beta^{-1}) + (\tilde{u}_1 - \tilde{u}_{h+1})}{q} \in \mathbb{Z}[\beta].
\]
Since both \( x \) and \( y \) have strictly periodic \( \beta \)-expansion, we have \( x = y \) by Lemma 2.3. Hence \( h \) is a period of the \( \beta \)-expansion of \( x \) as well as a period of \( \tilde{u}_k \).

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need three easy lemmas. Lemma 3.1 gives the formula of the general term of the sequence \( u_k \) (see for example [Eng], [W]).

**Lemma 3.1.** The general term of \( u_k \) is given by
\[
u_k = c_1\beta^k + c_2(\beta')^k
\] (3.1)
where
\[
c_1 = \frac{u_1 + a_0\beta^{-1}u_0}{\beta - \beta'}, \quad c_2 = (c_1)' = \frac{u_1 + a_0(\beta')^{-1}u_0}{\beta' - \beta}.
\]

**Proof.** One can check the initial value of (3.1) is \( u_0, u_1 \), and \( u_k \) in (3.1) satisfies the recurrence relation (1.1). \( \square \)

Let \( H(q) \) be the minimal period of the sequence \( u_k \) modulo \( q \). Then

**Lemma 3.2.** If \( q = p_1^{e_1} \cdots p_k^{e_k} \), then
\[
H(q) = \text{lcm}\{H(p_1^{e_1}), \ldots, H(p_k^{e_k})\}.
\] (3.2)
where \( \text{lcm} \) denote the least common multiple.

Lemma 3.2 can be found in [Eng] [W]. We leave the easy proof to the reader.

Let \( \mathfrak{R} \) be an ideal in \( \mathbb{Q}(\beta) \). Let us denote by \( G(x, \mathfrak{R}) \) the minimal period of the sequence \( x, x\beta, x\beta^2, \ldots \), modulo \( \mathfrak{R} \). Then similar to Lemma 3.2, we have

**Lemma 3.3.** If \( q = p_1^{e_1} \cdots p_k^{e_k} \), then
\[
G(x, q) = \text{lcm}\{G(x, p_1^{e_1}), G(x, p_2^{e_1}), \ldots, G(x, p_k^{e_k})\}.
\]

**Theorem 3.4.** Let \( x_0 = u_0(a_0\beta^{-1}) + u_1 \). If \( q \) is prime to \( D/D_0 \), then the periods of the sequence \( \{u_k \pmod{q}\} \) coincide with the periods of the sequence
\[
x_0, x_0\beta, x_0\beta^2, \ldots \pmod{q}.
\]

**Proof.** Since \( a_0 = \pm 1 \), it is easy to see that both \( \{x_0\beta^k \pmod{q}\} \) and \( \{u_k \pmod{q}\} \) are strictly periodic.

According to Lemma 3.2 and Lemma 3.3, we need only show that the theorem is valid for \( p^m \) where \( p \) is prime to \( D/D_0 \).
Set $x_k = u_k(a_0 \beta^{-1}) + u_{k+1}$, $k \geq 0$. Then $x_{k+1} = \beta x_k$ and so that $x_k = x_0 \beta^k$. Hence $x_k \in \mathbb{Z}(\beta)$ are algebraic integers in $\mathbb{Q}(\beta)$.

Clearly if $h$ is a period of $\{u_k \mod q\}$, then it is a period of $\{x_k \mod q\}$.

On the other hand, suppose $h$ is a period of $\{x_k \mod q\}$. Then $x_0(\beta^h - 1)$ is divisible by $q = p^m$ and so that the algebraic conjugate $(x_0(\beta^h - 1))'$ is also divisible by $p^m$. By Lemma 3.1, we have

$$u_k = \frac{(u_1 + a_0 \beta^{-1} u_0) \beta^k - (u_1 + a_0 \beta^{-1} u_0)'(\beta')^k}{\beta - \beta'}.$$  

Therefore $u_{k+h} - u_k = \frac{x_0(\beta^h - 1) \beta^k - x_0'((\beta')^h - 1)(\beta')^k}{\beta - \beta'}$. By the assumption of $h$, we have that the numerator of the right side of (3.3) is divisible by $p^m$. Notice that $\beta - \beta' = \sqrt{D} = I \sqrt{d_0}$. Let

If $p$ is prime to $D$, then the denominator is prime to $p^m$. Hence $u_{k+h} - u_k$ is divisible by $p^m$ for all $k \geq 0$, and $h$ is a period of $\{u_k \mod p^m\}$.

If $p$ is a factor of $D_0$, then $p = \mathfrak{R}^2$ where $\mathfrak{R}$ is a prime ideal in $\mathbb{Q}(\beta)$. We divide the discussion into two cases $p \neq 2$ and $p = 2$. Recall that $D_0 = d_0$ or $4d_0$ where $d_0$ is square-free.

If $p \neq 2$, then $p^2 \nmid D_0$ and so that the power of $\mathfrak{R}$ contained in the denominator of (3.3) is 1. Since the power of $\mathfrak{R}$ contained in the numerator of (3.3) is at least $2m$, we have that $(u_{k+h} - u_k)/p^{m-1}$ is an integer and it is divisible by $p$, the norm of $\mathfrak{R}$. Therefore $u_{k+h} - u_k$ is divisible by $q = p^m$ for all $k \geq 0$.

If $p = 2$, then $D_0$ is an even number and hence $D_0 = 4d_0$. Moreover $1, \sqrt{d_0}$ is a basis of $\mathbb{Q}(\beta)$. Let $x_0(\beta^h - 1)\beta^k = X + Y \sqrt{d_0}$, then $q = 2^m$ divides $X + Y \sqrt{d_0}$ implies that $2^m | Y$. Formula (3.3) becomes

$$u_{k+h} - u_k = \frac{2Y \sqrt{d_0}}{\sqrt{D}} = \frac{2Y \sqrt{d_0}}{I \sqrt{D_0}} = \frac{2Y \sqrt{d_0}}{2I \sqrt{d_0}} = Y.$$  

Since $p = 2$ is coprime to $I$, we conclude that $u_{k+h} - u_k$ is divisible by $2^m$ for all $k \geq 0$. This completes the proof the theorem. □

Theorem 1.2 follows immediately from Theorem 1.1 and Theorem 3.4.

References


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