A NOTE ON STABILITY OF MINIMAL SURFACES
IN $n$-DIMENSIONAL HYPERBOLIC SPACE $H^n(c)$

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Abstract. We improve a result of Barbosa-do Carmo about stability of minimal surfaces in $n$-dimensional hyperbolic space $H^n(c)$.

1. Introduction. In [1] Barbosa and do Carmo obtain the following well-known result

**Theorem 1** [1]. Let $M$ be a minimal surface in an $n$-dimensional hyperbolic space $H^n(c)$. Assume that $D$ is a simply connected compact domain with piecewise smooth boundary on $M$. Let $A$ denote the second fundamental form of $M$. If

(1) \[ \int_D (|c| + \frac{|A|^2}{2})dv < \frac{4\pi}{3}, \]

then $D$ is stable.

In this note, we improve the Theorem above as follows

**Theorem 2**. Let $M$ be a minimal surface in an $n$-dimensional hyperbolic space $H^n(c)$. Assume that $D$ is a simply connected compact domain with piecewise smooth boundary on $M$. Let $A$ denote the second fundamental form of $M$. If

(2) \[ \int_D \big( \frac{|c|}{5} + \frac{|A|^2}{2} \big)dv < \frac{4\pi}{3}, \]

then $D$ is stable.

*Remark.* Obviously, our condition (2) is better than condition (1) of Barbosa-do Carmo’s.


*Key Words and Phrases:* minimal surface, stability, hyperbolic space.
2. Preliminaries. Let $H^n(c)$ be an $n$-dimensional simply connected space of constant negative curvature $c$; we also call it the hyperbolic space. Let $M$ be a minimal surface in $H^n(c)$; we denote by $K$ the Gauss curvature of $M$ with respect to the induced metric $ds^2_M$. Let $A$ be the second fundamental form of $M$.

We need the following lemmas to prove Theorem 2.

smallskip Lemma 1. If $M$ be a minimal surface in $H^n(c)$, then

$$|\nabla(|A|^2)|^2 \leq 2|A|^2|\nabla A|^2. \tag{3}$$

Proof. Let $M$ be a minimal surface $H^n(c)$. By an elementary observation one can see that at each point the dimension of the image of the second fundamental form $A$ of $M$ is at most 2. Thus we may choose $e_3, \ldots, e_n$ so that $h^\alpha_{ij} = 0$ for all $i, j$ and $\alpha \geq 5$, i.e., we may choose the basis $e_1, e_2, \ldots, e_n$ so that the component $h^\alpha_{ij}$ of $A$ satisfy

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h^5_{ij}) = \cdots = (h^n_{ij}) = O, \tag{4}$$

for some functions $\lambda$ and $\mu$. Let $|A|^2 = \sum_{\alpha,i,j} (h^\alpha_{ij})^2$ be the square length of the second fundamental form $A$ of $M$ and $K_N = \sum_{\alpha,\beta,i,j} R_{\alpha\beta ij}$ be the normal scalar curvature of $M$. By (4) and Ricci equation we easily check that $|A|^2 = 2(\lambda^2 + \mu^2)$, $K_N = 16\lambda^2\mu^2$.

Noting $\sum_k (h^\alpha_{11k})^2 = \sum_k (h^\alpha_{12k})^2$, $3 \leq \alpha \leq n$, by (4), we have

$$|\nabla(|A|^2)|^2 = 4 \sum_k \left( \sum_{i,j,\alpha} h^\alpha_{ij} h^\alpha_{ij} \right)^2$$

$$= 16 \sum_k \left( \lambda h^3_{11k} + \mu h^4_{12k} \right)^2$$

$$\leq 16 \sum_k (\lambda^2 + \mu^2)[(h^3_{11k})^2 + (h^4_{12k})^2]$$

$$= 8|A|^2 \sum_k [(h^3_{11k})^2 + (h^4_{11k})^2]. \tag{5}$$

On the other hand, we have

$$|\nabla A|^2 = 2 \sum_{i,k,\alpha} (h^\alpha_{ii1k})^2 = 4 \sum_k (h^\alpha_{11k})^2$$

$$\geq 4 \sum_k [(h^3_{11k})^2 + (h^4_{11k})^2]. \tag{6}$$

We get (3) from (5) and (6). The proof of Lemma 1 is completed.
Lemma 2. If $M$ be a minimal surface in $H^n(c)$, then

$$
\frac{1}{2} \Delta(|A|^2) = |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4 + 2(\lambda^2 - \mu^2)^2
\geq |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4.
$$

Proof. Denote the matrix $(h^\alpha_{ij})$ by $H^\alpha$, $3 \leq \alpha \leq n$. By Gauss-Codazzi-Ricci equations it was shown in [4] that

$$
\frac{1}{2} \Delta(|A|^2) = \sum_{\alpha, i, j, k} (h^\alpha_{ij})^2 + \sum_{\alpha, i, j, k, l} h^\alpha_{ij} (h^\alpha_{kl} R_{l i j k} + h^\alpha_{k j} R_{i l j k})
+ \sum_{\alpha, \beta, i, j, k} h^\alpha_{ij} h^\beta_{jk} R^\beta_{i j k} = |\nabla A|^2 + \sum_{\alpha, \beta} \text{tr}(H\alpha H\beta - H\beta H\alpha)^2 - \sum_{\alpha, \beta} (\text{tr}(H\alpha H\beta))^2 + 2c|A|^2.
$$

By (4), it is easy to check the following formulas

$$
\sum_{\alpha, \beta} \text{tr}(H\alpha H\beta - H\beta H\alpha)^2 = -16\lambda^2 \mu^2, \quad \sum_{\alpha, \beta} (\text{tr}(H\alpha H\beta))^2 = 4(\lambda^4 + \mu^4).
$$

Substituting (8) into (7), we get

$$
\frac{1}{2} \Delta(|A|^2) = |\nabla A|^2 + 2c|A|^2 - 8(\lambda^2 + \mu^2)^2 + 4(\lambda^4 + \mu^4)
= |\nabla A|^2 + 2c|A|^2 - \frac{3}{2}|A|^4 + 2(\lambda^2 - \mu^2)^2.
$$

We completed the proof of Lemma 2.

The following proposition is crucial to prove our Theorem 2.

Proposition 1. Let $M$ be a minimal surface in $H^n(c)$, $ds_M^2$ be the induced metric. Then the Gauss curvature $K$ of the conformal metric $ds^2 = \sigma ds_M^2$ satisfies $K \leq 2$, where

$$
\sigma = \frac{|c|}{5} + \frac{|A|^2}{2} > 0.
$$

Proof. By Gauss equation $2K = 2c - |A|^2$,

$$
\sigma = \frac{|c|}{5} + \frac{|A|^2}{2} = \frac{4c}{5} - K.
$$

Thus we can define a conformal metric $\overline{ds}^2 = \sigma ds_M^2$ on $M$. As it is wellknown, the Gauss curvature $\overline{K}$ of $\overline{ds}^2$ satisfies (for example, see [2])

$$
-\sigma \overline{K} = -K + \frac{1}{2} \frac{\Delta \sigma}{\sigma} - \frac{|\nabla \sigma|^2}{2\sigma^2}.
$$
where $\Delta$ is the Laplacian operator of the metric $ds^2_M$.

By (9) and (10), we get

$$-\sigma K = \sigma - \frac{4c}{5} + \frac{1}{2} \Delta \sigma = -\frac{|\nabla \sigma|^2}{2\sigma^2}. \tag{11}$$

By use of Lemma 1, Lemma 2 and (9),

$$\frac{1}{2} \Delta \sigma = \frac{1}{4} \Delta (|A|^2) \geq \frac{1}{2} |\nabla A|^2 + c|A|^2 - \frac{3}{4} |A|^4 \tag{12}$$

Combining (11) with (12), we obtain

$$K \leq 2.$$

We completed the proof of Proposition 1.

3. The Proof of Theorem 2. By use of our Proposition 1, we can prove Theorem 2 in the same way as Barbosa and do Carmo did in [1] for Theorem 1. So we omit the proof of Theorem 2 here.

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