Two rigidity theorems for fully nonlinear equations✩

Haizhong Li a,b, Changwei Xiong a,*

a Department of Mathematical Sciences, Tsinghua University, 100084, Beijing, PR China
b Mathematical Sciences Center, Tsinghua University, 100084, Beijing, PR China

A R T I C L E   I N F O

Article history:
Received 14 April 2014
Available online 19 September 2014
Communicated by F. Fang

MSC:
53C21
53C20

Keywords:
Liouville Theorem
Fully nonlinear equation
Locally conformally flat

A B S T R A C T

This paper is concerned with the fully nonlinear equation $\sigma_2(g) = a\sigma_1(g) + b$. The first result is to obtain the entire solutions of the equation for conformally flat metric on $\mathbb{R}^n$ under some additional assumptions, which generalizes the famous result of Chang–Gursky–Yang in [3]. The second one is to classify compact locally conformally flat manifolds with nonnegative Ricci curvature and the metric $g$ satisfying the equation.

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1. Introduction

Let $(M^n, g)$ be an $n$-dimensional compact Riemannian manifold. It is well-known that the Riemannian curvature tensor has the following decomposition:

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(A_{ik}g_{jl} + A_{jl}g_{ik} - A_{il}g_{jk} - A_{jk}g_{il}),$$

where $W$ is the Weyl curvature tensor and $A$ is the Schouten tensor defined as

$$A_g = \text{Ric}_g - \frac{R_g}{2(n-1)}g. \tag{1}$$

Since $W$ is conformally invariant, the conformal transformation of the Riemannian curvature tensor is consistent with that of the Schouten tensor $A_g$. In view of that, the Schouten tensor $A_g$ has always been playing a central role in conformal geometry.

✩ The research of the authors was supported by NSFC No. 11271214.
* Corresponding author.
E-mail addresses: hli@math.tsinghua.edu.cn (H. Li), xiongcw10@mails.tsinghua.edu.cn (C. Xiong).
Associated with Schouten tensor $A_g$, J.A. Viaclovsky in [15] introduced the so-called $\sigma_k(g)$ curvature, which is defined to be

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot A_g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $g^{-1} \cdot A_g$, i.e., $\sigma_k(g)$ is the $k$-th elementary symmetric polynomial with respect to the eigenvalues of the matrix $g^{-1} \cdot A_g$. Note that $\sigma_1(g) = tr_g(A_g)$ is just the scalar curvature $R_g$ up to a constant multiple, and $\sigma_2(g)$ has the following expression:

$$\sigma_2(g) = \frac{1}{2} (tr_g(A_g))^2 - |A_g|^2).$$

Furthermore, he found that the metric $\tilde{g}$ satisfying $\sigma_k(\tilde{g}) = \text{const}$ is in fact related to a variational problem, which can be stated precisely as follows.

Denote by $[g]$ the conformal class of the metric $g$ and by $[g]_1$ the restriction of $[g]$ to the unit volume set. Then if $k \neq n/2$ and $(M^n, g)$ is locally conformally flat, a metric $\tilde{g} \in [g]_1$ is a critical point of the functional

$$\mathcal{F}_k(\tilde{g}) = \int_M \sigma_k(\tilde{g})d\tilde{g}$$

restricted to $[g]_1$ if and only if

$$\sigma_k(\tilde{g}) = \text{const}.$$

And if $M$ is not locally conformally flat, the statement is true for $k = 1$ and $k = 2$.

From the last assertion, we can see that the cases $k = 1$ and $k = 2$ are special, and for fixed $a \in \mathbb{R}$, the metric $\tilde{g} \in [g]_1$ satisfying

$$\sigma_2(\tilde{g}) - a\sigma_1(\tilde{g}) = \text{const}$$

will correspond to the critical point of some linear combination of the functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ restricted to $[g]_1$. See also [8,9]. This is the motivation of the present paper.

The issue involved in the first part of this paper can be originated from the classical theorem of Obata [13]. There he proved that any metric conformal to the standard metric on the $n$-sphere $S^n$ with constant scalar curvature is isometric to the standard metric. Or equivalently, he got the entire solutions to the equation

$$\sigma_1(g) = \text{const}, \quad g \in [g_0] \text{ on } S^n,$$

where $g_0$ is the standard metric on $S^n$.

Via stereographic projection, equations on $S^n$ are equivalent to those on $\mathbb{R}^n$ with the assumption that the singularity at infinity is removable. So equations on $\mathbb{R}^n$ without any assumption are less restricted and more interesting. In this aspect, by using the method of moving planes, Caffarelli, Gidas and Spruck [1] classified the entire solutions of the equation

$$-\Delta u = n(n - 2)u^{\frac{n+2}{n-2}}, \quad u > 0 \text{ on } \mathbb{R}^n,$$

and prove that the solutions must be

$$u(x) = \left(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\right)^{\frac{2}{n-2}}$$

where $\lambda > 0$. This is the motivation of the present paper.
for some \(x_0 \in \mathbb{R}^n\) and \(\lambda > 0\); or equivalently the metric \(u(x)\frac{1}{\lambda}dx^2\) is isometric to the standard metric on \(\mathbb{S}^n\) via stereographic projection.

Note that in our notation, Eq. (8) is equivalent to the following equation for \(k = 1\):

\[
\sigma_k(g) = \text{const}, \quad g \in [dx^2] \text{ on } \mathbb{R}^n,
\]

(10)

where \(dx^2\) is the standard Euclidean metric. Thus after the celebrated work of Caffarelli et al. [1], it is natural to consider the equation for general \(k\). This kind of results is also referred to as Liouville-type theorems. For general \(k \geq 2\), J.A. Viaclovsky [16] got the Liouville Theorem but under a strong growth condition at infinity. For the important case \(k = 2\), in [2,3], A. Chang, M. Gursky and P. Yang proved the Liouville Theorem for \(n = 4, 5\) and for \(n \geq 6\) with a finite volume assumption. At last, A. Li and Y.Y. Li [12] dealt with general case \(k \geq 2\) without additional assumption. See also the work of Guan and Wang [7] on general locally conformally flat manifold.

In this paper, along the approach of Obata [13] and Chang et al. [3], we prove the following:

**Theorem 1.** Let \(g = v^{-2}dx^2\) be a conformal metric on \(\mathbb{R}^n, n \geq 4\), satisfying

\[
\sigma_2(g) = a\sigma_1(g) + b, \quad b > 0.
\]

Moreover, if \(n \geq 6\), we assume \(\text{vol}(g) = \int_{\mathbb{R}^n} v^{-n}dx < \infty\). If \(n = 5\), we assume \(a > -\sqrt{\frac{5}{7}}b\). Then \(v = p|x|^2 + q_1x_1 + r\) for constants \(p, q_1, r\). In particular, \(g\) is obtained by pulling back to \(\mathbb{R}^n\) the round metric on \(\mathbb{S}^n\).

When \(a = 0\), Theorem 1 reduces to the result in [3]. And our result is another example showing the power of Obata’s method.

In the second part of this paper we study the problem to classify compact locally conformally flat Riemannian manifolds with \(\sigma_2(g) - a\sigma_1(g) = b\) being constant. The corresponding problem for \(\sigma_1(g)\) i.e. scalar curvature being constant is handled by Tani [14], Wegner [17] and Cheng [4] (see Theorem 7 below for precise statement). And the case \(\sigma_k(g)(k \geq 2)\) being constant was considered by Z.J. Hu, H. Li and U. Simon [10]. See also [5,6,11] for relevant work in the hypersurface context. By applying the method of [10], we prove the following theorem.

**Theorem 2.** Let \((M^n, g)\) \((n \geq 3)\) be a compact locally conformally flat manifold with

\[
\sigma_2(g) = a\sigma_1(g) + b,
\]

(11)

where \(a\) and \(b\) are constants satisfying \(a^2 + 2b \geq 0\). Moreover, assume the Ricci tensor is semi-positive definite. Then \((M^n, g)\) is either a space form or a space \(\mathbb{S}^1 \times N^{n-1}\) with \(N\) a space form.

When \(a = 0\), the result reduces to that of [10].

The present paper is built up as follows. Some notations and lemmas for proving Theorem 1 are prepared in Section 2. And Section 3 is devoted to the proof of Theorem 1. Relatively independent, Section 4 includes the proof of Theorem 2.

**2. Preliminaries**

Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold. The Schouten tensor \(A_g\) is defined as

\[
A_g = \text{Ric}_g - \frac{R_g}{2(n-1)}g,
\]

where \(\text{Ric}_g\) is the Ricci tensor and \(R_g\) the scalar curvature.
Recall the $\sigma_k(g)$ scalar curvature is given by

$$\sigma_k(g) = \sigma_k(g^{-1} \cdot A_g),$$

where $\sigma_k$ is the $k$-th elementary symmetric polynomial on $\mathbb{R}^n$.

For locally conformally flat manifolds $(M^n, g)$, it is well known that $A_g$ is a Codazzi tensor. That is, it satisfies

$$A_{ij,k} = A_{ik,j}.$$  

Using this property, J.A. Viaclovsky [16] proved that the $k$-th Newtonian tensor $T_k$ is divergence-free. In this paper we are only concerned with the first two Newtonian tensors $T_1$ and $T_2$, which are defined as follows:

$$T_1 = \sigma_1(g)g - A_g,$$

$$T_2 = \sigma_2(g)g - \sigma_1(g)A_g + A_g^2.$$  

Now go back to our equation $\sigma_2(A_g) = a\sigma_1(A_g) + b$. Related to our equation is the following tensor:

$$T = T_2 - \frac{n-2}{n-1}aT_1.$$  

Denote by $L$ the trace-free part of $T$. That is,

$$L = T_2 - \frac{n-2}{n-1}aT_1 - \frac{n-2}{n}(\sigma_2 - a\sigma_1)g.$$  

It is easy to see that when $\sigma_2 - a\sigma_1 = b$ is a constant, $L$ is divergence-free. Furthermore, $L$ has two important properties as follows.

**Proposition 3.** Denote by $E$ the trace-free part of Ricci tensor. Assume that $\sigma_2(g) - a\sigma_1(g) + \frac{na^2}{2(n-1)} > 0$ and $\sigma_1(g) - \frac{n}{n-1}a > 0$. Then

1. $\langle L, E \rangle \leq 0$, with equality if and only if $E = 0$.
2. $|L|^2 \leq -\frac{2(n-2)}{n}(\sigma_1(g) - \frac{n}{n-1}a)\langle L, E \rangle$.

**Proof.** (1) Let $\tilde{A} = A_g - \frac{a}{n-1}g$. And for simplicity, we write $\sigma_i$ for $\sigma_i(g)$, $i = 1, 2$. Then

$$\sigma_1(\tilde{A}) = \sigma_1 - \frac{n}{n-1}a > 0,$$

$$\sigma_2(\tilde{A}) = \sigma_2 - a\sigma_1 + \frac{na^2}{2(n-1)} > 0.$$  

And by direct computation we have

$$T_2(\tilde{A}) = \sigma_2(\tilde{A})g - \sigma_1(\tilde{A})\tilde{A} + \tilde{A}^2$$

$$= T_2 - \frac{n-2}{n-1}aT_1 + \frac{(n-2)a^2}{2(n-1)}g.$$  

Therefore,
\[ \langle L, E \rangle = \left\langle T_2 - \frac{n-2}{n-1} aT_1 - \frac{n-2}{n} \left( \sigma_2 - a\sigma_1 \right) g, E \right\rangle \]

\[ = \left\langle T_2(\tilde{A}) - \frac{(n-2)a^2}{2(n-1)} g - \frac{n-2}{n} \left( \sigma_2 - a\sigma_1 \right) g, E \right\rangle \]

\[ = \left\langle T_2(\tilde{A}) - \frac{n-2}{n} \sigma_2(\tilde{A}) g, E \right\rangle, \]

where in the last line we have used that \( L \) is trace-free.

Using the contracted formula \( \text{tr}(T_k A) = (k + 1)\sigma_{k+1} \) and Newton–Maclaurin inequality, we obtain

\[ \langle L, E \rangle = 3\sigma_3(\tilde{A}) - \frac{n-2}{n} \sigma_2(\tilde{A}) \sigma_1(\tilde{A}) \leq 0, \]

and the equality holds if and only if \( \tilde{A} = \lambda g \), that is \( E = 0 \).

(2) In terms of the trace-free Ricci tensor, we have

\[ A_g = E + \frac{1}{n} \sigma_1 g, \]

\[ A_g^2 = E^2 + \frac{2}{n} \sigma_1 E + \frac{1}{n^2} \sigma_1^2 g, \]

which implies

\[ T_1 = \sigma_1 g - A_g = \frac{n-1}{n} \sigma_1 g - E, \]

\[ T_2 = \sigma_2 g - \sigma_1 A_g + A_g^2 \]

\[ = E^2 - \frac{n-2}{n} \sigma_1 E + \left( \sigma_2 - \frac{n-1}{n^2} \sigma_1^2 \right) g. \]

Consequently,

\[ |L|^2 = \left| T_2 - \frac{n-2}{n-1} aT_1 - \frac{n-2}{n} \left( \sigma_2 - a\sigma_1 \right) g \right|^2 \]

\[ = \left| E^2 + \left( \frac{n-2}{n-1} a - \frac{n-2}{n} \sigma_1 \right) E - \frac{\text{tr} E^2}{n} g \right|^2 \]

\[ = \text{tr} E^4 + 2\left( \frac{n-2}{n-1} a - \frac{n-2}{n} \sigma_1 \right) \text{tr} E^3 + \left( \frac{n-2}{n-1} a - \frac{n-2}{n} \sigma_1 \right)^2 \text{tr} E^2 - \frac{1}{n} \left( \text{tr} E^2 \right)^2. \]

Meanwhile, we have

\[ \langle L, E \rangle = \text{tr} E^3 + \left( \frac{n-2}{n-1} a - \frac{n-2}{n} \sigma_1 \right) \text{tr} E^2. \]

Therefore,

\[ |L|^2 = \text{tr} E^4 + 2\left( \frac{n-2}{n-1} a - \frac{n-2}{n} \sigma_1 \right) \langle L, E \rangle - \left( \frac{n-2}{n-1} a - \frac{n-2}{n} \sigma_1 \right)^2 \text{tr} E^2 - \frac{1}{n} \left( \text{tr} E^2 \right)^2. \]

Now by Lemma 2.3 in [3], there holds

\[ \text{tr} E^4 \leq \frac{n^2 - 3n + 3}{n(n-1)} \left( \text{tr} E^2 \right)^2. \]
Hence, we obtain

\[
\text{tr } E^4 - \left( \frac{n - 2}{n - 1} a - \frac{n - 2}{n} \sigma_1 \right)^2 \text{tr } E^2 - \frac{1}{n} (\text{tr } E^2)^2 \\
\leq \frac{(n - 2)^2}{n(n - 1)} (\text{tr } E^2)^2 - \left( \frac{n - 2}{n - 1} a - \frac{n - 2}{n} \sigma_1 \right)^2 \text{tr } E^2 \\
= \frac{(n - 2)^2}{n(n - 1)} \text{tr } E^2 \left( \text{tr } E^2 - n(n - 1) \left( \frac{a}{n - 1} - \frac{\sigma_1}{n} \right)^2 \right) \\
= \frac{(n - 2)^2}{n(n - 1)} \text{tr } E^2 \left( -2\sigma_2 + 2a\sigma_1 - \frac{n}{n - 1} a^2 \right) \leq 0.
\]

Then the conclusion follows immediately. \(\Box\)

Next we need a proposition to determine the range which our \(\sigma_1(g)\) belongs to.

**Proposition 4.** (See [2] for \(n = 4\).) There exists no conformal metric \(g = u^{\frac{4}{n-2}} dx^2\) on \(\mathbb{R}^n\) such that its scalar curvature \(R_g \leq -c\) for fixed positive constant \(c\).

**Proof.** Assume that there exists such \(g\). Under the transformation \(g = u^{\frac{4}{n-2}} dx^2\), we have

\[
-\Delta u = \frac{n - 2}{4(n - 1)} R_g u^{\frac{n+2}{n-2}} \leq -u^{\frac{n+2}{n-2}},
\]

where and thereafter the notation “\(\lesssim (\gtrsim \text{ resp.})\)” means “\(\leq (\geq \text{ resp.})\) up to a harmless constant multiple”.

For \(\rho > 0\), let \(\eta\) be a cut off function supported in the ball \(B(2\rho)\) with \(\eta \equiv 1\) on \(B(\rho)\). And assume \(||\nabla \eta|| \lesssim \frac{1}{\rho} \). Then multiplying the above inequality by \(u\eta^n\) and integrating it on \(\mathbb{R}^n\), there holds

\[
\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} \eta^n dx \lesssim \int_{\mathbb{R}^n} \Delta u \cdot u\eta^n dx = -\int_{\mathbb{R}^n} |\nabla u|^2 \eta^n dx - \int_{\mathbb{R}^n} n\eta^{n-1} u \langle \nabla u, \nabla \eta \rangle dx \\
\leq -\int_{\mathbb{R}^n} |\nabla u|^2 \eta^n dx + \int_{\mathbb{R}^n} (|\nabla u|^2 \eta^n + C_1 \eta^{n-2} u^2 |\nabla \eta|^2) dx \\
= C_1 \int_{\mathbb{R}^n} \eta^{n-2} u^2 |\nabla \eta|^2 dx \\
\lesssim \frac{1}{\rho^2} \int_{\rho \leq |x| \leq 2\rho} \eta^{n-2} u^2 dx.
\]

Next using the Hölder inequality, we get

\[
\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} \eta^n dx \lesssim \frac{1}{\rho^2} \left( \int_{\rho \leq |x| \leq 2\rho} \eta^n u^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \left( \int_{\rho \leq |x| \leq 2\rho} 1 dx \right)^{\frac{2}{n}} \\
\lesssim \left( \int_{\rho \leq |x| \leq 2\rho} \eta^n u^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}},
\]

from which it is easy to see that \(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx < \infty\). Then letting \(\rho \to \infty\) in the above inequality, we get \(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx = 0\). So \(u \equiv 0\), which is a contradiction. \(\Box\)
Now with this proposition at hand, we can determine the range of \( \sigma_1(g) \). Note that

\[
a \sigma_1(g) + b = \sigma_2(g) \leq \frac{n-1}{2n} \sigma_1^2(g).
\]

Solving this inequality, it follows that

\[
\sigma_1(g) \leq \frac{n}{n-1} a - \sqrt{\left(\frac{n}{n-1} a\right)^2 + \frac{2n}{n-1} b},
\]

or

\[
\sigma_1(g) \geq \frac{n}{n-1} a + \sqrt{\left(\frac{n}{n-1} a\right)^2 + \frac{2n}{n-1} b}.
\]

Since \( b > 0 \) and \( \sigma_1(g) = \frac{n-2}{2(n-1)} R_g \), we see that \( R_g \) satisfies either \( R_g \leq -c \) or \( R_g \geq c \). But we can exclude the former case by Proposition 4. Therefore we obtain

\[
\sigma_1(g) = \frac{n-2}{2(n-1)} R_g \geq 0,
\]

and

\[
\sigma_1(g) - \frac{n}{n-1} a \geq \sqrt{\left(\frac{n}{n-1} a\right)^2 + \frac{2n}{n-1} b} > 0.
\]

The last ingredient we need is a “tail” estimate, whose proof relies on the following well-known technical lemma.

**Lemma 5.** (See [3].) Let \( g = v^{-2}dx^2 \) be a conformal metric on \( \mathbb{R}^n \) with \( R_g \geq 0 \). Then there exists a constant \( c \) such that \( v(x) \leq c|x|^2 \) for all \( |x| \) sufficiently large.

Now we can state our “tail” estimate and prove it.

**Proposition 6.** Let \( g = v^{-2}dx^2 \) be a conformal metric on \( \mathbb{R}^n, n \geq 4 \). Suppose that \( \sigma_2(g) - a \sigma_1(g) = b > 0 \). Moreover, if \( n \geq 6 \), assume \( \int_{\mathbb{R}^n} v^{-n}dx \lesssim 1 \). If \( n = 5 \), assume \( a > -\sqrt{\frac{2}{7}} b \). Then there holds

\[
\int_{A_{\rho}} \left( \sigma_1(g) - \frac{n}{n-1} a \right) |\nabla v|^2 v^{1-n} dx \lesssim \rho^2, \quad \text{for } \rho \gg 1,
\]

where \( A_{\rho} = B(2\rho) \setminus B(\rho) \).

**Proof.** Denote \( g_0 = dx^2 \). Then we have

\[
\frac{1}{n-2} A_g = \frac{\nabla^2 v}{v} - \frac{1}{2} \frac{|\nabla v|^2}{v^2} g_0 + \frac{1}{n-2} A_{g_0} = \frac{\nabla^2 v}{v} - \frac{1}{2} \frac{|\nabla v|^2}{v^2} g_0.
\]

Therefore,
\[
\sigma_2\left(\frac{1}{n-2} A_g\right) = v^4 \sigma_2\left(g_0^{-1} \cdot \frac{1}{n-2} A_g\right) = \frac{v^4}{2} \left(v^2\left(\frac{1}{n-2} A_g\right) - \left|\frac{1}{n-2} A_g\right|^2\right) = \frac{v^4}{2} \left(-v^{-2}|\nabla^2 v|^2 + v^{-2}(\Delta v)^2 - (n-1)v^{-3} \Delta v|\nabla v|^2 + \frac{n(n-1)}{4} v^{-4} |\nabla v|^4\right).
\]

Multiplying both sides by \(2v^{-n}\) and using the Bochner formula

\[|\nabla^2 v|^2 = \frac{1}{2} \Delta |\nabla v|^2 - \langle \nabla v, \nabla \Delta v\rangle,\]

we arrive at

\[
\frac{2}{(n-2)^2} \sigma_2(A_g) v^{-n} = \frac{1}{2} v^{2-n} \Delta |\nabla v|^2 + v^{2-n} \langle \nabla v, \nabla \Delta v\rangle + v^{2-n} (\Delta v)^2 - (n-1)v^{1-n} \Delta v|\nabla v|^2 + \frac{1}{4} n(n-1)v^{-n} |\nabla v|^4.
\]

Take a new cut-off function \(\eta\) supported on \(B(\frac{\rho}{2}) \setminus B(\frac{1}{2}\rho)\) with \(\eta \equiv 1\) on \(A_\rho\) and \(|\nabla^k \eta| \lesssim \rho^{-k}, k = 1, 2\). Multiply the above equality by \(\eta^4 v^n\) and integrate over \(\mathbb{R}^n\). Next do the integration by parts to eliminate the derivative terms of \(v\) of degree bigger than 2. After some tedious computation, we obtain the following equality. For more details, we refer to [3].

\[
\frac{2}{(n-2)^2} \int_{\mathbb{R}^n} \sigma_2(A_g) v^{\alpha-n} \eta^4 dx = \frac{n-4-3\alpha}{2} \int_{\mathbb{R}^n} v^{1+\alpha-n} |\nabla v|^2 \Delta v \cdot \eta^4 dx + \int_{\mathbb{R}^n} v^{2+\alpha-n} \Delta v \cdot \nabla \eta^4 dx
\]

\[
+ \left(\frac{1}{4} n(n-1) - \frac{1}{2} (2+\alpha-n)(1+\alpha-n)\right) \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^4 \eta^4 dx
\]

\[
- \frac{1}{2} (2+\alpha-n) \int_{\mathbb{R}^n} v^{1+\alpha-n} |\nabla v|^2 \nabla v \cdot \nabla \eta^4 dx + \int_{\mathbb{R}^n} v^{2+\alpha-n} \nabla^2 \eta^4 (\nabla v, \nabla v) dx
\]

\[
- \int_{\mathbb{R}^n} v^{2+\alpha-n} |\nabla v|^2 \Delta \eta^4 dx.
\]

Substituting

\[
\Delta v = \frac{n}{2} |\nabla v|^2 v^{-1} + \frac{1}{n-2} \sigma_1(g)v^{-1}
\]

\[
= \frac{n}{2} |\nabla v|^2 v^{-1} + \frac{1}{n-2} \left(\sigma_1(g) - \frac{n}{n-1} \sigma_1\right) v^{-1} + \frac{1}{n-2} \frac{n}{n-1} av^{-1}
\]

into the first line, we get

\[
\frac{2}{(n-2)^2} \int_{\mathbb{R}^n} \sigma_2(g) v^{\alpha-n} \eta^4 dx
\]

\[
= a_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \left(\sigma_1(g) - \frac{na}{n-1}\right) \eta^4 dx + b_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^4 \eta^4 dx + M_1 + M_2 + \frac{na(n-4-3\alpha)}{2(n-1)(n-2)} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx,
\]

(12)
where
\[
a_{\alpha,n} = \frac{n - 4 - 3\alpha}{2(n - 2)}, \quad b_{\alpha,n} = \frac{1}{4}(\alpha + 1)(n - 2\alpha - 4),
\]
\[
M_1 = \frac{1}{2}(2 + \alpha - n) \int_{\mathbb{R}^n} v^{1+\alpha-n} |\nabla v|^2 \nabla v \cdot \nabla \eta^4 dx,
\]
\[
M_2 = \int_{\mathbb{R}^n} v^{2+\alpha-n} \nabla^2 \eta^4 (\nabla v, \nabla v) dx - \int_{\mathbb{R}^n} v^{2+\alpha-n} \Delta \eta^4 |\nabla v|^2 dx.
\]
On the other hand, we have
\[
\frac{1}{n - 2} \sigma_1(g) = v \Delta v - \frac{n}{2} |\nabla v|^2,
\]
which implies
\[
\frac{1}{n - 2} \int_{\mathbb{R}^n} \sigma_1(g) v^{\alpha-n} \eta^4 dx = \int_{\mathbb{R}^n} v^{1+\alpha-n} \eta^4 \Delta v dx - \frac{n}{2} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx
\]
\[
= - \left(1 + \alpha - \frac{n}{2}\right) \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx - \int_{\mathbb{R}^n} v^{1+\alpha-n} \nabla v \cdot \nabla \eta^4 dx
\]
\[
= - \left(1 + \alpha - \frac{n}{2}\right) \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx + \frac{1}{2 + \alpha - n} \int_{\mathbb{R}^n} v^{2+\alpha-n} \cdot \Delta \eta^4 dx.
\]
Multiplying the above equality by \(-\frac{2a}{n-2}\) and adding it to (12) gives rise to
\[
\frac{2}{(n-2)^2} \int_{\mathbb{R}^n} (\sigma_2(g) - a\sigma_1(g)) v^{\alpha-n} \eta^4 dx
\]
\[
= a_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \left(\sigma_1(g) - \frac{na}{n-1}\right) \eta^4 dx
\]
\[
+ b_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^4 \eta^4 dx + M_1 + M_2 + \frac{na(n - 4 - 3\alpha)}{2(n - 1)(n - 2)} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx
\]
\[
+ \frac{2a(1 + \alpha - \frac{n}{2})}{n - 2} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx - \frac{2a}{(n-2)(2 + \alpha - n)} \int_{\mathbb{R}^n} v^{2+\alpha-n} \cdot \Delta \eta^4 dx,
\]
or simplified to
\[
\frac{2}{(n-2)^2} \int_{\mathbb{R}^n} (\sigma_2(g) - a\sigma_1(g)) v^{\alpha-n} \eta^4 dx
\]
\[
= a_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \left(\sigma_1(g) - \frac{na}{n-1}\right) \eta^4 dx
\]
\[
+ b_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^4 \eta^4 dx + M_1 + M_2
\]
\[
+ c_{\alpha,n} \int_{\mathbb{R}^n} v^{\alpha-n} |\nabla v|^2 \eta^4 dx + d_{\alpha,n} \int_{\mathbb{R}^n} v^{2+\alpha-n} \cdot \Delta \eta^4 dx,
\]
(13)
where
\[ c_{\alpha,n} = \frac{a(-n^2 + n\alpha - 4\alpha + 2n - 4)}{2(n-1)(n-2)}, \]
\[ d_{\alpha,n} = -\frac{2a}{(n-2)(2+\alpha-n)}. \]

Note that \( M_1 \) and \( M_2 \) can be estimated as follows.

\[ |M_1| \lesssim \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla v|^3 \eta^3 v^{1+n - \alpha} \lesssim \frac{1}{\rho} \left( \int_{\text{spt}\eta} |\nabla v|^4 v^{\alpha-n} \eta^4 \right)^{\frac{3}{4}} \left( \int_{\text{spt}\eta} v^{4+\alpha-n} \right)^{\frac{1}{4}}, \]
\[ |M_2| \lesssim \frac{1}{\rho^2} \int_{\mathbb{R}^n} |\nabla v|^2 \eta^2 v^{2+n - \alpha} \lesssim \frac{1}{\rho^2} \left( \int_{\text{spt}\eta} |\nabla v|^4 v^{\alpha-n} \eta^4 \right)^{\frac{1}{4}} \left( \int_{\text{spt}\eta} v^{4+\alpha-n} \right)^{\frac{1}{4}}, \]

where and thereafter \( \text{spt}\eta \) denotes the support set of \( \eta \) and the Euclidean volume element \( dx \) is omitted.

Next we have to do the analysis case by case.

**Case 1** \((n \geq 6)\). We choose \( \alpha = 0 \). Then \( a_{0,n} > 0 \) and \( b_{0,n} > 0 \). It follows that
\[
\int_{\mathbb{R}^n} v^{-n} \eta^4 \gtrsim a_{0,n} \int_{\mathbb{R}^n} \left( \sigma_1(g) - \frac{na}{n-1} \right) |\nabla v|^2 v^{-n} \eta^4 + b_{0,n} \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4 - |M_1| - |M_2| 
\]
\[
- |c_{0,n}| \int_{\mathbb{R}^n} \eta^4 v^{-n} |\nabla v|^2 - |d_{0,n}| \int_{\mathbb{R}^n} v^{2-n} |\Delta\eta|^4. 
\]

Under the assumption that \( \int_{\mathbb{R}^n} v^{-n} \lesssim 1 \), we get
\[
|M_1| \lesssim \frac{1}{\rho} \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4 \right)^{\frac{3}{4}} \left( \int_{\text{spt}\eta} v^{4-n} \right)^{\frac{1}{4}}, 
\]
\[
\lesssim \frac{1}{\rho} \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4 \right)^{\frac{3}{4}} \left( \int_{\text{spt}\eta} v^{-n} \right)^{\frac{n-4}{n}} \left( \int_{\text{spt}\eta} 1 \right)^{\frac{1}{4}}, 
\]
\[
\lesssim \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4 \right)^{\frac{3}{4}}. 
\]

Similarly, there hold
\[
|M_2| \lesssim \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4 \right)^{\frac{1}{2}}, 
\]
\[
\int_{\mathbb{R}^n} \eta^4 v^{-n} |\nabla v|^2 \lesssim \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4 \right)^{\frac{1}{2}}, 
\]
\[
\int_{\mathbb{R}^n} v^{2-n} |\Delta\eta|^4 \lesssim \rho^2 \left( \int_{\text{spt}\eta} v^{-n} \right)^{\frac{n-2}{n}} \left( \int_{\text{spt}\eta} \eta^4 \right)^{\frac{2}{n}} \lesssim 1. 
\]
Then by the Young inequality

\[ ab \leq \varepsilon a^p + C \varepsilon b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \]

it is easy to see that \(|M_1|, |M_2|\) and the term \(\int_{\mathbb{R}^n} \eta^4 v^{-n} |\nabla v|^2\) can be absorbed into \(b_0 \int_{\mathbb{R}^n} |\nabla v|^4 v^{-n} \eta^4\). In conclusion, we arrive at

\[ \int_{\mathbb{R}^n} \left( \sigma_1(g) - \frac{na}{n-1} \right) |\nabla v|^2 v^{-n} \eta^4 \leq 1. \]

Then using Lemma 5 and \(\sigma_1(g) - \frac{n}{n-1} a > 0\), we obtain

\[ \int_{\mathcal{A}_v} \left( \sigma_1(g) - \frac{na}{n-1} \right) |\nabla v|^2 v^{-n} \eta^4 \lesssim \rho^2 \int_{\mathbb{R}^n} \left( \sigma_1(g) - \frac{na}{n-1} \right) |\nabla v|^2 v^{-n} \eta^4 \lesssim \rho^2. \]

**Case 2** \((n = 5)\). We choose \(\alpha = 1\). Then Eq. (13) yields

\[
\frac{2}{5} \int_{\mathbb{R}^n} \left( \sigma_2(g) - a \sigma_1(g) \right) v^{-4} \eta^4 + \frac{1}{3} \int_{\mathbb{R}^n} v^{-4} |\nabla v|^2 \left( \sigma_1(g) - \frac{5}{4} a \right) \eta^4 + \frac{1}{2} \int_{\mathbb{R}^n} v^{-4} |\nabla v|^4 \eta^4
= M_1 + M_2 - \frac{3}{4} a \int_{\mathbb{R}^n} v^{-4} |\nabla v|^2 \eta^4 + \frac{a}{3} \int_{\mathbb{R}^n} v^{-2} \Delta \eta^4.
\]

First since

\[ |M_1| \lesssim \frac{1}{\rho} \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-4} \eta^4 \right)^{\frac{1}{2}} \left( \int_{\text{spt} \eta} 1 \right)^{\frac{1}{2}} \lesssim \varepsilon \int_{\mathbb{R}^n} |\nabla v|^4 v^{-4} \eta^4 + C \varepsilon \rho, \]

\[ |M_2| \lesssim \frac{1}{\rho^2} \left( \int_{\mathbb{R}^n} |\nabla v|^4 v^{-4} \eta^4 \right)^{\frac{1}{2}} \left( \int_{\text{spt} \eta} 1 \right)^{\frac{1}{2}} \lesssim \varepsilon \int_{\mathbb{R}^n} |\nabla v|^4 v^{-4} \eta^4 + C \varepsilon \rho, \]

\(M_1\) and \(M_2\) can be absorbed into the term \(\frac{1}{2} \int_{\mathbb{R}^n} |\nabla v|^4 v^{-4} \eta^4\).

Second we have

\[ \int_{\mathbb{R}^n} v^{-2} \cdot |\Delta \eta^4| \lesssim \varepsilon \int_{\mathbb{R}^n} v^{-4} \eta^4 + C \varepsilon \rho. \]

Thus this term can be absorbed into \(\frac{2}{5} \int_{\mathbb{R}^n} \left( \sigma_2(g) - a \sigma_1(g) \right) v^{-4} \eta^4\).

Last if \(a \geq 0\), we can drop out the term \(-\frac{3}{4} a \int_{\mathbb{R}^n} v^{-4} |\nabla v|^2 \eta^4\) and go through. If \(a < 0\), the assumption \(a > -\sqrt{\frac{2}{7}} b\) implies \(b > \frac{7}{5} a^2\). Then by the previous analysis we get

\[ \sigma_1(g) - \frac{5}{4} a \geq \sqrt{\left( \frac{5}{4} a \right)^2 + \frac{5}{2} b} \]

\[ =: (1 + \delta) \frac{9}{4} (-a), \]

where \(\delta\) is a positive number. Next direct computation shows that
Thus \(-\frac{3}{a}\int R^n v^{-4} |\nabla v|^2 \eta^4\) can be absorbed into \(\frac{1}{3} \int R^n v^{-4} |\nabla v|^2 (\sigma_1(g) - \frac{5}{4} a) \eta^4\).

In conclusion, we derive
\[
\int R^n v^{-4} |\nabla v|^2 \left(\sigma_1(g) - \frac{5}{4} a\right) \eta^4 \lesssim \rho \lesssim \rho^2.
\]

**Case 3** \((n = 4)\). We choose \(\alpha = 1\). Then Eq. (13) yields
\[
\int R^n (\sigma_2(g) - a\sigma_1(g)) v^{-3} \eta^4 + \frac{3}{4} \int R^n v^{-3} |\nabla v|^2 \left(\sigma_1(g) - \frac{4}{3} a\right) \eta^4 + \int R^n v^{-3} |\nabla v|^4 \eta^4
\]
\[= M_1 + M_2 - a \int R^n v^{-3} |\nabla v|^2 \eta^4 + a \int R^n v^{-1} \Delta \eta^4.\]

First since
\[
|M_1| \lesssim \frac{1}{\rho} \left(\int R^n |\nabla v|^4 v^{-3} \eta^4\right)^{\frac{1}{2}} \left(\int R^n v^{\frac{1}{2}}\right)^{\frac{1}{2}} \lesssim \epsilon \int R^n |\nabla v|^4 v^{-3} \eta^4 + C_\varepsilon \rho^2,
\]
\[
|M_2| \lesssim \frac{1}{\rho^2} \left(\int R^n |\nabla v|^4 v^{-3} \eta^4\right)^{\frac{1}{2}} \left(\int R^n v^{\frac{1}{2}}\right)^{\frac{1}{2}} \lesssim \epsilon \int R^n |\nabla v|^4 v^{-3} \eta^4 + C_\varepsilon \rho^2,
\]

\(M_1\) and \(M_2\) can be absorbed into the term \(\int R^n |\nabla v|^4 v^{-3} \eta^4\).

Second we have
\[
\int R^n v^{-1} \cdot |\Delta \eta^4| \lesssim \epsilon \int R^n v^{-3} \eta^4 + C_\varepsilon \rho.
\]

Thus this term can be absorbed into \(\frac{1}{2} \int R^n (\sigma_2(g) - a\sigma_1(g)) v^{-3} \eta^4\).

Last recall \(\sigma_2(g) - a\sigma_1(g) = b\). Again if \(a \geq 0\) we go through. If \(a < 0\), by the previous analysis, we get
\[
\sigma_1(g) - \frac{4}{3} a \geq \sqrt{\left(\frac{4}{3} a\right)^2 + \frac{8}{3} b}
\]
\[=: (1 + \delta)\frac{4}{3}(-a),\]

where \(\delta\) is a positive number. Then direct computation leads to
\[
\frac{3}{4} \left(\sigma_1(g) - \frac{4}{3} a\right) + a \geq \frac{3}{41 + \delta} \left(\sigma_1(g) - \frac{4}{3} a\right).
\]

Thus \(-a \int R^n v^{-3} |\nabla v|^2 \eta^4\) can be absorbed into \(\frac{3}{4} \int R^n v^{-3} |\nabla v|^2 (\sigma_1(g) - \frac{4}{3} a) \eta^4\).

In summary, we obtain
\[
\int R^n v^{-3} |\nabla v|^2 \left(\sigma_1(g) - \frac{4}{3} a\right) \eta^4 \lesssim \rho^2.
\]

So far we have completed the proof of this proposition. □
3. Proof of Theorem 1

For $\rho > 0$ let $\eta$ be a cut off function supported in the ball $B(2\rho)$ with $\eta \equiv 1$ on $B(\rho)$. And assume $|\nabla \eta| \lesssim \frac{1}{\rho}$. Since $g = v^{-2}dx^2$, we get

$$E_g = -(n - 2)v\nabla_g^2(v^{-1}) + \frac{n - 2}{n}v\Delta_g(v^{-1})g.$$  

Pairing both sides with $v^{-1}\eta^2L$ and integrating the equality on $\mathbb{R}^n$, there holds

$$\int_{\mathbb{R}^n} -\langle L, E \rangle v^{-1}\eta^2 dv_g = (n - 2) \int_{\mathbb{R}^n} \langle \eta^2L, \nabla_g^2(v^{-1}) \rangle dv_g,$$

where we have used that $L$ is trace-free.

Using the divergence theorem and noting that $L$ is also divergence-free, we derive

$$\int_{\mathbb{R}^n} -\langle L, E \rangle v^{-1}\eta^2 dv_g = -(n - 2) \int_{\mathbb{R}^n} L(\nabla_g(v^{-1}), \nabla_g(\eta^2)) dv_g$$

$$\lesssim \int_{\mathbb{R}^n} |L||\nabla_g(v^{-1})||\nabla_g(\eta^2)| dv_g$$

$$\lesssim \int_{\mathbb{R}^n} |L||\nabla_g v||\nabla_g \eta|v^{-2}\eta dv_g.$$  

Taking into account Proposition 3 and the fact $\text{spt}|\nabla \eta| \subset A_\rho$, it follows that

$$\int_{\mathbb{R}^n} -\langle L, E \rangle v^{-1}\eta^2 dv_g \lesssim \int_{A_\rho} \left( \sigma_1(g) - \frac{n}{n - 1}a \right)^{\frac{1}{2}} |\langle L, E \rangle|^{\frac{1}{2}} |\nabla_g v| |\nabla_g \eta| v^{-2}\eta dv_g$$

$$\lesssim \left( \int_{A_\rho} |\langle L, E \rangle| v^{-1}\eta^2 dv_g \right)^{\frac{1}{2}} \left( \int_{A_\rho} \left( \sigma_1(g) - \frac{n}{n - 1}a \right) |\nabla_g v| |\nabla_g \eta| v^{-3} dv_g \right)^{\frac{1}{2}}$$

$$= \left( \int_{A_\rho} |\langle L, E \rangle| v^{-1}\eta^2 dv_g \right)^{\frac{1}{2}} \left( \int_{A_\rho} \left( \sigma_1(g) - \frac{n}{n - 1}a \right) |\nabla v|^2 |\nabla \eta|^2 v^{1-n}dx \right)^{\frac{1}{2}},$$

where we have used that $|\nabla_g v|^2 = v^2|\nabla v|^2$, $|\nabla_g \eta|^2 = v^2|\nabla \eta|^2$ and $dv_g = v^{-n}dx$.

Thanks to $|\nabla \eta| \lesssim \frac{1}{\rho}$, we arrive at

$$\int_{\mathbb{R}^n} -\langle L, E \rangle v^{-1}\eta^2 dv_g \lesssim \left( \int_{A_\rho} |\langle L, E \rangle| v^{-1}\eta^2 dv_g \right)^{\frac{1}{2}}$$

$$\times \left( \rho^{-2} \int_{A_\rho} \left( \sigma_1(g) - \frac{n}{n - 1}a \right) |\nabla v|^2 v^{1-n}dx \right)^{\frac{1}{2}}. \quad (14)$$

Now by Proposition 6, we know that the integral $\rho^{-2} \int_{A_\rho} \left( \sigma_1(g) - \frac{n}{n - 1}a \right) |\nabla v|^2 v^{1-n}dx$ is uniformly bounded with respect to $\rho \gg 1$. Then it is easy to see that
\[ \int_{\mathbb{R}^n} -\langle L, E \rangle v^{-1} dv_g < \infty, \]

which implies that
\[ \int_{A_\rho} |\langle L, E \rangle v^{-1} \eta^2| dv_g \to 0, \quad \text{as} \ \rho \to \infty. \]

This together with (14) immediately yields
\[ \int_{\mathbb{R}^n} -\langle L, E \rangle v^{-1} dv_g = 0. \]

As a consequence, we get \( E = 0 \). Then the conclusion of Theorem 1 follows from a standard argument [13].

4. Proof of Theorem 2

First recall the following well known theorem.

**Theorem 7.** (See Tani [14], Wegner [17] and Cheng [4].) Let \((M^n, g)\) \((n \geq 3)\) be a compact locally conformally flat Riemannian manifold with constant scalar curvature. If the Ricci tensor is semi-positive definite, then \((M^n, g)\) is either a space form or a space \( \mathbb{S}^1 \times N^{n-1} \) with \( N \) a space form.

Therefore, to prove Theorem 2, it suffices to show that \( M \) has constant scalar curvature. For that purpose, first we introduce some notations.

For any local orthonormal basis \( \{e_1, \cdots, e_n\} \), denote \( R_{ijkl} \) the components of the Riemannian curvature tensor. And at a point, we choose \( \{e_i\} \) as eigenvectors of Schouten tensor such that \( A_{ij} = \lambda_i \delta_{ij} \). Now we can state the lemmas which will be used in the proof.

The first lemma is due to Tani [14] (see also [4,10,17]).

**Lemma 8.** (See Tani [14].) Let \((M^n, g)\) be a locally conformally flat manifold with \( \text{Ric} \geq 0 \), then
\[ \frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 R_{ijij} \geq 0. \] (15)

**Remark 9.** From the proof of Lemma 8 (see [10]), we can check that, if the equality in (15) holds, then the eigenvalues \( \{\rho_i\} \) of Ricci tensor must be either \( \{\rho_1 = \rho_2 = \cdots = \rho_n = \rho(x) \geq 0\} \) or \( \{\rho_1 = 0, \rho_2 = \cdots = \rho_n = \rho(x) > 0\} \).

In addition, we need the following inequality of Kato type.

**Lemma 10.** (See [11] for \( a = 0 \).) Assume \( \sigma_2(g) = a \sigma_1(g) + b \) with \( a^2 + 2b \geq 0 \). Then we have
\[ \sum_{i,j,k} (A_{ij,k})^2 \geq \sum_k \left( \sum_j A_{jj,k} \right)^2. \] (16)

**Proof.** First we have
\[ \sum_{i,j} A_{ij}^2 = \sigma_1^2(g) - 2\sigma_2(g) = \sigma_1^2(g) - 2a \sigma_1(g) - 2b. \]
Taking covariant derivative on the above equality with respect to $e_k$ yields
\[
\sum_{i,j} A_{ij} A_{ij,k} = (\sigma_1(g) - a) \nabla_k \sigma_1(g).
\]
Then using the Cauchy–Schwarz inequality we obtain
\[
(\sigma_1(g) - a)^2 |\nabla \sigma_1(g)|^2 = \sum_k \left( \sum_{i,j} A_{ij} A_{ij,k} \right)^2 \\
\leq \sum_k \left( \sum_{i,j} A_{ij}^2 \sum_{i,j} (A_{ij,k})^2 \right) \\
= \sum_{i,j} A_{ij}^2 \sum_{i,j,k} (A_{ij,k})^2.
\]

At last note that
\[
\sum_{i,j} A_{ij}^2 = \sigma_1^2(g) - 2a \sigma_1(g) - 2b \\
\leq \sigma_1^2(g) - 2a \sigma_1(g) + a^2 \\
= \left( \sigma_1(g) - a \right)^2.
\]
So if $\sigma_1(g) \neq a$ at the point, the conclusion follows. Otherwise an approximation argument will let us through. $\square$

Now we are in a position to prove Theorem 2.

**Proof of Theorem 2.** By direct computation, we can obtain (see the formula (3.8) in [10])
\[
\Delta \sigma_2(g) = \text{tr} (T_1 \circ \text{Hess}(\sigma_1(g))) - \frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 R_{ijij} - \sum_{i,j,k} (A_{ij,k})^2 + \sum_{i,j} \left( \sum_{j} A_{jj,k} \right)^2,
\]
where $T_1$ is the first Newton tensor associated to $A_g$, “$\circ$” denotes the composition of two $(1,1)$-tensors, and so $\text{tr}(T_1 \circ \text{Hess}(\sigma_1(g))) = \sum_{i,j} (T_1)_{ij}(\sigma_1(g))_{ij}$.

Noting that $T_1$ is divergence-free and $M$ is compact without boundary, after integrating the above formula we get
\[
\int_M \left( -\frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 R_{ijij} - \sum_{i,j,k} (A_{ij,k})^2 + \sum_{j} \left( \sum_{j} A_{jj,k} \right)^2 \right) dv_g = 0.
\]
On the other hand, Lemma 8 and Lemma 10 show that
\[
-\frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 R_{ijij} - \sum_{i,j,k} (A_{ij,k})^2 + \sum_{j} \left( \sum_{j} A_{jj,k} \right)^2 \leq 0.
\]
Thus we must have
\[
\frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 R_{ijij} \equiv 0.
\]
Then Remark 9 shows that, the eigenvalues \( \{ \rho_i \} \) of Ricci tensor must be either \( \{ \rho_1 = \rho_2 = \cdots = \rho_n = \rho(x) \geq 0 \} \) or \( \{ \rho_1 = 0, \rho_2 = \cdots = \rho_n = \rho(x) > 0 \} \). Since \( M \) is connected and the eigenvalues of Ricci tensor are continuous, only one of the two cases can occur on \( M \). Furthermore, in view of \( \sigma_2(g) = a\sigma_1(g) + b \), we know \( \rho(x) \) is constant and its value is determined by the equation. Then the scalar curvature is constant. Now as mentioned before, this fact together with Theorem 7 gives rise to the desired result. \( \square \)

Acknowledgement

The authors would like to thank the referee for some helpful comments which made this paper more readable.

References