On locally strongly convex affine hypersurfaces with parallel cubic form. Part I

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Abstract
In this paper, we study locally strongly convex affine hypersurfaces of $\mathbb{R}^{n+1}$ that have parallel cubic form with respect to the Levi-Civita connection of the affine Berwald–Blaschke metric; it is known that they are affine spheres. In dimension $n \leq 7$ we give a complete classification of such hypersurfaces; in particular, we present new examples of affine spheres.

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1. Introduction

We denote by $\mathbb{R}^{n+1}$ the real unimodular-affine space equipped with its canonical flat connection $D$ and a parallel volume form $\omega$. Let $M := M^n$ be a differentiable, connected $C^\infty$-manifold of dimension $n \geq 2$, and let $F : M^n \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface immersion with equiaffine (unimodular) normal $\xi$. This normalization induces an equiaffinely invariant geometry on $M$ ([1]). We denote by $h$ its affine Blaschke–Berwald metric which is semi-Riemannian, by $\nabla$ its Weingarten form and by $\hat{\nabla}$ its induced affine connection. Let $\hat{\nabla}$ be the Levi-Civita connection of the affine metric $h$. The difference tensor $K$ is defined by $K(X,Y) := K_{XY} := \nabla_X Y - \nabla_Y X$; it is symmetric as both connections are torsion free.

Define the cubic form $C$ by $C := h$; it is related to the difference tensor by

$$C(X,Y,Z) = \frac{1}{2} h(X,Y,Z).$$

(1.1)

It follows from its definition that $C$ is a totally symmetric tensor of type $(0,3)$. The classical theorem of Blaschke–Pick–Berwald states that $C$ vanishes identically on $M$ if and only if $M$ is an open part of a non-degenerate quadric ([16]).
In this paper, we will consider locally strongly convex hypersurfaces satisfying the condition $\nabla C = 0$. The convexity condition is equivalent to the fact that the affine metric $h$ is definite; it is positive definite for appropriate orientation of the affine normal $\xi$. For details concerning the general affine hypersurface theory we refer to the monographs $[25]$ from 1991, $[14]$ from 1993, and $[18]$ from 1994.

Clearly, the condition $\nabla C = 0$ is equivalent to $\nabla K = 0$. For surfaces this condition was studied by M. Magid and K. Nomizu in $[15]$, and for hypersurfaces by F. Dillen et al. in $[5,8]$.

Other authors studied a related problem, namely the relation $\nabla C = 0$. Such hypersurfaces are quadrics or improper affine spheres. For details see the papers $[17]$ by K. Nomizu and U. Pinkall, $[7,26,28]$ by L. Vrancken and F. Dillen $[2]$ by N. Bokan et al., and $[9,10]$ by S. Gigena (as Gigena does not specify to which connection the cubic form is parallel the reader himself has to find out which connection is used).

From $[2]$ it has been known that the condition $\nabla C = 0$ implies that the hypersurface is an affine sphere. In contrast to the Euclidean situation, where “spheres” only differ by their radius, the situation in the unimodular geometry is quite different: It is well known (see $[3]$) that the class of affine spheres is very large and that one is far from a classification; even under strong additional assumptions there only exist partial classifications. One of the most famous examples is the local classification of affine spheres with constant sectional curvature of the Blaschke–Berwald metric; this classification was finished in dimension $n = 2$ in $[24]$, for locally strongly convex hypersurfaces in $[30]$ (see also $[14]$, Chapter 2), and for non-degenerate hypersurfaces with non-vanishing Pick invariant $J$ in $[29]$; for $J = 0$ the final classification is still an open problem, but many examples are known in this case.

This situation justifies the further study of this topic to find new subclasses and examples of affine spheres. An interesting subclass recently was treated in the papers of C. Scharlach $[21–23]$. On the other hand, the interest in affine spheres comes from the diversity of different subclasses and the many known examples, on the other hand from the diversity of methods that are applied for their investigation. Examples for different methods are:

(i) the study of nonlinear fourth order PDEs for the local and global classification of locally strongly convex affine spheres, in the global case under completeness conditions for the Blaschke metric, see e.g. $[14]$;
(ii) the application of groups acting on the tangent space and preserving certain invariants pointwise $[21–23]$;
(iii) the study of properties of the cubic forms; see e.g. $[29,30]$ and below.

Considering again the condition $\nabla C = 0$ from above, in dimensions $n = 2, 3, 4$ detailed classification theorems were established, mainly under the additional condition that the hypersurfaces are locally strongly convex. We recall the following results, using an obvious notation for coordinates in $\mathbb{R}^{n+1}$:

**Theorem A.** (See $[2,8]$.) Let $M$ be an $n$-dimensional locally strongly convex hypersurface of $\mathbb{R}^{n+1}$. If $\nabla C = 0$, then $M^n$ is a locally affine-homogeneous hyperbolic affine sphere.

**Theorem B.** (See $[15]$.) Let $M^2$ be an affine non-degenerate surface in $\mathbb{R}^3$ with $\nabla C = 0$. Then either $M^2$ is an open part of a non-degenerate quadric (i.e. $C = 0$) or $M^2$ is affinely equivalent to an open part of one of the following three surfaces:

(i) $xyz = 1$,
(ii) $x(y^2 + z^2) = 1$,
(iii) $z = xy + \frac{1}{3}y^3$ (the Cayley surface).

**Theorem C.** (See $[5]$.) Let $M^3$ be a 3-dimensional, locally strongly convex affine hypersurface in $\mathbb{R}^4$ with $\nabla C = 0$. Then either $M^3$ is an open part of a quadric (i.e. $C = 0$) or, up to an affine equivalence and a suitable homothetic transformation, $M^3$ is an open part of one of the following two hypersurfaces:

(i) $xyzw = 1$,
(ii) $(y^2 - z^2 - w^2)^3x^2 = 1$.

**Theorem D.** (See $[8]$.) Let $M^4$ be a 4-dimensional locally strongly convex affine hypersurface in $\mathbb{R}^5$ with $\nabla C = 0$. Then either $M^4$ is an open part of a quadric (i.e. $C = 0$) or, up to an affine equivalence and a suitable homothetic transformation, $M^4$ is an open part of one of the following three hypersurfaces:

(i) $xyzwt = 1$,
(ii) $(y^2 - z^2 - w^2 - t^2)^3x = 1$,
(iii) $(z^2 - w^2 - t^2)^3(xy)^2 = 1$. 
Remark 1. Consider the following class of hypersurfaces in $\mathbb{R}^{n+1}$, represented by

$$
\prod_{i=1}^{k} \left( x^2_{i,p_i+1} - \sum_{j=1}^{p_i} x^2_{i,j} \right)^{p_i+1} (y_1 \cdots y_{q+1})^2 = 1,
$$

(1.2)

where $n = \sum_{i=1}^{k} (p_i + 1) + q$, and where

$$(x_1, \ldots, x_1; p_1+1, x_2; 2, \ldots, x_2; p_2+1, \ldots, x_k; 1, \ldots, x_k; p_k+1, y_1, \ldots, y_{q+1})$$

is affine coordinates of $\mathbb{R}^{n+1}$. Obviously, this class gives all examples of locally strongly convex hypersurfaces with $\hat{\nabla}C = 0$ for $n = 2, 3, 4$. An analysis of the examples shows that they all are Calabi-type composition products of lower dimensional hyperbolic affine spheres that have parallel cubic form, see [6,11] for more details.

In this paper, we systematically investigate the condition $\hat{\nabla}C = 0$ under the restriction that the hypersurfaces in $\mathbb{R}^{n+1}$ are locally strongly convex. In particular, if $n = 5, 6, 7$, we give a complete classification for all such hypersurfaces. Therefore, together with the results cited above, now all locally strongly convex hypersurfaces satisfying $\nabla C = 0$ and $2 \leq n \leq 7$ are classified.

Our main result is the following classification.

**Classification Theorem.** Let $M^n$ ($n \leq 7$) be a $n$-dimensional locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\hat{\nabla}C = 0$. Then either

(i) $M^n$ is an open part of a locally strongly convex quadric (i.e. $C = 0$), or

(ii) $M^n$ is obtained as the Calabi product of a lower dimensional hyperbolic affine sphere with parallel cubic form and a point, or

(iii) $M^n$ is obtained as the Calabi product of two lower dimensional hyperbolic affine spheres with parallel cubic form, or

(iv) $n = 5$ and $(M^3, h)$ is isometric with $SL(3, \mathbb{R})/SO(3)$, and the immersion is affinely equivalent to the standard embedding of $SL(3, \mathbb{R})/SO(3) \hookrightarrow \mathbb{R}^6$; see [18], pp. 106–113.

The proof of the above Classification Theorem is an immediate consequence of Theorems B, C, D, 4.1, 5.1, 6.1, 6.2, 7.1 and Corollary 7.1.

We would like to point out the following: While, up to dimension $n = 4$, all examples are of type (1.2), in dimension $n = 5$ there appears the new example (iv). Moreover, it is the only example that we know of an affine sphere with unimodular Einstein metric that is not of constant sectional curvature.

A direct calculation shows that the example (iv) can be characterized as follows:

**Corollary.** Let $M^5$ be a 5-dimensional locally strongly convex affine hypersurface in $\mathbb{R}^6$ with $\hat{\nabla}C = 0$. If $(M^5, h)$ is Einstein and possesses non-constant sectional curvature then, modulo affine equivalences and homothetic transformations, it must be an open part of $SL(3, \mathbb{R})/SO(3)$.

2. Preliminaries

For a hypersurface $F : M = M^n \hookrightarrow \mathbb{R}^{n+1}$, we recall the notation from the beginning of the introduction. We also recall some standard definitions: let $S$ be the affine shape operator (or Weingarten operator). $M^n$ is called an affine sphere if $S = \lambda\text{id}$; one easily proves $\lambda = \text{const.} = H$ where $H := \frac{1}{n}\text{trace } S$ is the affine mean curvature. $F$ is called a proper affine sphere if $H \neq 0$; if $H > 0$, the proper affine sphere is called elliptic, for $H < 0$ hyperbolic. If $H = 0$, the affine sphere is called improper or parabolic. For a proper affine sphere the affine normal satisfies $\xi(p) = H(F(p) - c)$, where $c$ is a constant vector, called the center of $F(M^n)$; for simplicity, we choose $c$ as origin. For an improper affine sphere the affine normal field is constant.

The curvature tensors $R$ and $\mathring{R}$ of $\nabla$ and $\mathring{\nabla}$, resp., are related to $S$ and $K$ by two equations of Gauß type

$$
R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY,
$$

$$
\mathring{R}(X, Y)Z = \frac{1}{2} \left[ h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y \right] - [K_X, K_Y]Z.
$$

In particular, for affine spheres we have $S = H\text{id}$ and thus

$$
\mathring{R}(X, Y)Z = H \cdot (h(Y, Z)X - h(X, Z)Y) - [K_X, K_Y]Z.
$$

(2.1)

We also recall the relation

$$
$$

(2.2)

Finally, $K$ satisfies the apolarity condition for its trace; it reads $\text{tr } K_X = 0$ for all $X$. 

3. The construction of an appropriate orthonormal basis

In this section, we consider an \( n \)-dimensional, locally strongly convex affine hypersurface \( M^n \) in \( \mathbb{R}^{n+1} \) which has parallel cubic form, i.e. \( \nabla C = 0 \). Thus, according to Theorem A, \( M^n \) is a hyperbolic affine sphere with affine shape operator \( S = H \cdot \text{id} \), \( H < 0 \).

Since \( \nabla C = 0 \) implies that \( h(C, C) \) is constant, there are two cases. If \( h(C, C) = 0 \), then \( C = 0 \) and \( M^n \) is an open part of a quadric. Otherwise, \( C \) never vanishes, and we assume this in this section from now on.

Let \( p \in M^n \). Now we will review the construction of a typical orthonormal basis with respect to the affine metric \( h \) for \( T_p M^n \), which was introduced by Ejiri and has been widely applied, and proved to be very useful for various purposes, see e.g. [8,30]. The idea is to construct from the (1,2) tensor \( K \) a self adjoint operator at a point; then one extends the eigenbasis to a local field.

Let \( p \in M^n \) and \( UM_p = \{ u \in T_p M^n \mid h(u, u) = 1 \} \). Since \( M^n \) is locally strongly convex, \( UM_p \) is compact. We define a function \( f \) on \( UM_p \) by \( f(u) = h(K_u u, u) \). Let \( e_1 \) be an element of \( UM_p \) at which the function \( f \) attains an absolute maximum. Since \( K \neq 0 \), we easily see that \( f(e_1) > 0 \).

Let \( u \in UM_p \) such that \( h(u, e_1) = 0 \), and define a function \( g \) by \( g(t) := f(t e_1 + \sin tu) \). Then we have
\[
\begin{align*}
g'(0) &= 3h(K_{e_1} e_1, u), \\
g''(0) &= 6h(K_{e_1} u, u) - 3f(e_1), \\
g''(0) &= 6f(u) - 21h(K_{e_1} e_1, u).
\end{align*}
\]

Since \( g \) attains an absolute maximum at \( t = 0 \), we have \( g'(0) = 0 \), i.e. \( h(K_{e_1} e_1, u) = 0 \). So \( e_1 \) is an eigenvector of \( K_{e_1} \), say associated to the eigenvalue \( \lambda_1 \). Let \( e_2, e_3, \ldots, e_n \) be orthonormal vectors, orthogonal to \( e_1 \), which are the remaining eigenvectors of the operator \( K_{e_1} \), associated to the eigenvalues \( \lambda_2, \lambda_3, \ldots, \lambda_n \). Further, since \( e_1 \) is an absolute maximum of \( f \), we know that \( g''(0) \leq 0 \), and if \( g'(0) = 0 \), then \( g'''(0) = 0 \). This implies the following

**Lemma 3.1.** For every \( i \geq 2 \), we have \( \lambda_1 - 2\lambda_i \geq 0 \). If, for some \( i, \lambda_i = \frac{1}{2} \lambda_1 \) then
\[
f(e_i) = h(K_{e_i} e_i, e_i) = 0.
\]

Furthermore, from the apolarity condition, we have \( \sum_{i=1}^{n} \lambda_i = 0 \). Now \( \hat{\nabla} K = 0 \) together with (2.1) imply that, for any \( i \geq 2 \),
\[
0 = \left( \hat{R}(e_1, e_i) K(e_1, e_1) \right) = \hat{R}(e_1, e_i) \lambda_1 e_1 - 2K(\hat{R}(e_1, e_i) e_1, e_1) = (2\lambda_i - \lambda_1)(H - \lambda_1 \lambda_i + \lambda_i^2)e_i,
\]
that is
\[
(2\lambda_i - \lambda_1)(H - \lambda_1 \lambda_i + \lambda_i^2) = 0, \quad \forall i \geq 2.
\]
(3.1)

Now we can state

**Lemma 3.2.** (See [8].) Let \( M^n \) be a hyperbolic affine sphere, thus \( S = H \cdot \text{id} \) with \( H < 0 \), and with parallel cubic form. Then, for every \( p \in M^n \), there exists an orthonormal basis \( \{ e_j \}_{1 \leq j \leq n} \) (if necessary, we rearrange the order), satisfying \( K_{e_i} e_j = \lambda_j e_j \), and there exists a number \( i, \ 1 \leq i < n, \) such that
\[
\lambda_2 = \lambda_3 = \cdots = \lambda_i = \frac{1}{2} \lambda_1, \quad \lambda_{i+1} = \cdots = \lambda_n = -i_1 + \frac{1}{2(n-i)} \lambda_1 =: \mu
\]
and
\[
-H = \frac{\lambda_1^2 (i+1)^2 + 2(i+1)(n-i)}{4(n-i)^2}.
\]
(3.3)

Therefore, for a locally strongly convex affine hypersurface with parallel cubic form, we have to deal with \( n-1 \) cases \( \{ C_i \}_{1 \leq i \leq n-1} \) as follows:

**Case** \( C_1 \). \( \lambda_2 = \lambda_3 = \cdots = \lambda_n = -\frac{\lambda_1}{n-1} \).

**Case** \( C_2 \). \( \lambda_2 = \cdots = \lambda_i = \frac{1}{2} \lambda_1 \) and \( \lambda_{i+1} = \cdots = \lambda_n = -\frac{i+1}{2(n-i)} \lambda_1 \) for \( 2 \leq i \leq n-1 \).

We are going to discuss the cases step by step. Firstly, we have the important observation:

**Lemma 3.3.** If \( i > \frac{1}{2}(2n-1) \), then the case \( C_i \) does not occur.
Lemma 3.4. If $2n = 1 \mod 3$, then for the case $C_{\hat{n}}$, we have:

$$h(K_{e_j} e_k, e_l) = 0, \quad \text{for all } j, k, l \geq \hat{n} + 1. \quad (3.4)$$

Proof. From the proof of Lemma 3.3 we see that $f(\alpha e_j + \beta e_k + \gamma e_l) = 0$ for all $j, k, l \geq \hat{n} + 1$ and for arbitrary real $\alpha, \beta, \gamma$. Then the claim follows. □

To treat the general case $C_{i}$ with $i \leq \hat{n}$ we introduce the notations $D_2 := \text{span}(e_2, \ldots, e_1)$ and $D_3 := \text{span}(e_{i+1}, \ldots, e_n)$. It turns out that the subspaces $D_2$ and $D_3$ satisfy remarkable relations which we are going to state in the following Lemmas 3.5-3.9.

Lemma 3.5. For the case $C_{i}$, if $v \in D_2$ and $w \in D_3$, then $K_v w \in D_2$.

Proof. From (2.2) and $\hat{R}K = 0$, we have

$$0 = (\hat{R}(v, w)K)(e_1, e_1) = \hat{R}(v, w)K(e_1, e_1) - 2K(\hat{R}(v, w)e_1, e_1)$$

$$= \lambda_1 \hat{R}(v, w)e_1 - 2K_{e_1}(\hat{R}(v, w)e_1). \quad (3.5)$$

which shows that $\hat{R}(v, w)e_1 \in D_2$. Furthermore, (2.1) gives

$$\hat{R}(v, w)e_1 = -[K_v, K_w]e_1 = -K_v K_w e_1 + K_w K_v e_1 = \frac{n + 1}{2(n - i)} K_v w,$$

from (3.5) we then have $K_{e_1} K_v w = \frac{1}{2} \lambda_1 K_v w$, i.e., $K_v w \in D_2$. □

Lemma 3.6. If $\dim D_2 \geq 1$, then for any $v_1, v_2, v_3 \in D_2$, $h(K_{v_1} v_2, v_3) = 0$.

Proof. This is a direct corollary of Lemma 3.1. □

For the general case $C_{i}$ with $\dim D_2 \geq 1$, let us define the bilinear map $L$ on $D_2$ by

$$L(v_1, v_2) := K(v_1, v_2) - \frac{1}{2} \lambda_1 h(v_1, v_2)e_1, \quad v_1, v_2 \in D_2. \quad (3.6)$$

Lemma 3.7. For the case $C_{i}$ with $i \geq 2$, we have $L: D_2 \times D_2 \rightarrow D_3$; moreover, $L$ is isotropic in the sense that

$$h(L(v, v), L(v, v)) = \frac{n + 1}{4(n - i)} \lambda^2 \frac{1}{2} (h(v, v))^2, \quad v \in D_2. \quad (3.7)$$

Proof. From Lemma 3.6 and the definition of $L$, it can be seen that, for all $v_1, v_2 \in D_2$,

$$h(L(v_1, v_2), e_1) = h(L(v_1, v_2), v) = 0.$$

This proves the claim that $L: D_2 \times D_2 \rightarrow D_3$.

To prove (3.7), we choose $v \in D_2$ with $h(v, v) = 1$. We calculate the left-hand side (LHS) and the right-hand side (RHS) of the equation

$$\hat{R}(e_1, v)K(v, v) = 2K(\hat{R}(e_1, v)v, v),$$

and apply Lemma 3.5 to obtain:
Lemma 3.9. For every $v \in \mathcal{D}_2$, we immediately obtain (3.12). This proves Lemma 3.7. □

Since $L : \mathcal{D}_2 \times \mathcal{D}_2 \to \mathcal{D}_3$ is isotropic, we see from (3.7) that, if $\dim \mathcal{D}_2 \geq 1$, then $\dim(\operatorname{Im} L) \geq 1$. Moreover, from [19] we have the following

Lemma 3.8. (See [19], also [27].) If $\dim \mathcal{D}_2 \geq 1$, then for orthonormal vectors $X, Y, Z$ and $W$ in $\mathcal{D}_2$, there holds

$$h(L(X, X), L(Y, Y)) = 0, \quad (3.8)$$

$$h(L(X, X), L(Y, Y)) + 2h(L(X, Y), L(X, Y)) = \frac{n + 1}{4(n - i)} \lambda_1^2, \quad (3.9)$$

$$h(L(X, X), L(Y, Z)) + 2h(L(X, Y), L(X, Z)) = 0, \quad (3.10)$$

$$h(L(X, Y), L(Z, W)) + h(L(X, Z), L(W, Y)) + h(L(X, W), L(Y, Z)) = 0. \quad (3.11)$$

If $\dim \mathcal{D}_2 \geq 1$ and $\operatorname{Im}(L) \neq \mathcal{D}_3$, we can say more about the bilinear map $L$:

Lemma 3.9. Assume in case $\mathcal{E}_i$ that $\dim \mathcal{D}_2 \geq 1$ and $\operatorname{Im}(L) \neq \mathcal{D}_3$. Then, for any $v_1, v_2 \in \mathcal{D}_2$ and $w \in \mathcal{D}_3$ such that $w \perp \operatorname{Im}(L)$, we have

$$K(L(v_1, v_2), w) = -\frac{(n + 1)(i + 1)}{4(n - i)^2} h(v_1, v_2) \lambda_1^2 w. \quad (3.12)$$

Proof. For every $v \in \mathcal{D}_2$ and $w \perp \operatorname{Im}(L)$, we apply Lemmas 3.5, 3.7 and (2.1) to obtain

$$K_v w = \sum_{j=2}^i h(K(v, w), e_j)e_j = \sum_{j=2}^i h(K(v, e_j), w)e_j = \sum_{j=2}^i h(L(v, e_j), w)e_j = 0,$$

$$\hat{R}(e_1, v)w = -K_{e_1}K_v w + K_vK_{e_1}w = -\frac{\lambda_1}{2}K_v w + K_v \left(-\frac{i + 1}{2(n - i)} \lambda_1 \right)w = 0.$$

Then, for $v_1, v_2$ and $w$ as in the assumptions, the following equation

$$\hat{R}(e_1, v_1)K(v_2, w) = K(\hat{R}(e_1, v_1)v_2, w) + K(v_2, \hat{R}(e_1, v_1)w)$$

is equivalent to $K(\hat{R}(e_1, v_1)v_2, w) = 0$. On the other hand, we have

$$\hat{R}(e_1, v_1)v_2 = Hh(v_1, v_2)e_1 - K_{e_1}K_v v_2 + K_v K_{e_1} v_2$$

$$= Hh(v_1, v_2)e_1 - K_{e_1} \left[\frac{\lambda_1}{2} h(v_1, v_2)e_1 + L(v_1, v_2)\right] + \frac{\lambda_1}{2} K_v v_2$$

$$= \left(H - \frac{\lambda_1^2}{2}\right) h(v_1, v_2)e_1 - K_{e_1}L(v_1, v_2) + \frac{\lambda_1}{2} \left[\frac{\lambda_1}{2} h(v_1, v_2)e_1 + L(v_1, v_2)\right]$$

$$= \left(H - \frac{\lambda_1^2}{4}\right) h(v_1, v_2)e_1 + \frac{n + 1}{2(n - i)} \lambda_1 L(v_1, v_2);$$

from this and (3.3), we immediately obtain (3.12). □
4. Hypersurfaces with the case $C_1$

In this section we consider the case $C_1$ locally strongly convex affine hypersurfaces with parallel cubic form and $n \geq 3$. In fact, we have the following general result in this case.

**Theorem 4.1.** Let $F : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersurface with parallel and non-vanishing cubic form. If, for a fixed $p \in M^n$, there exists an orthonormal basis $\{e_i\}_{1 \leq i \leq n}$ of $T_p M^n$ such that

\[
K_{e_1} e_1 = \lambda_1 e_1, \quad K_{e_i} e_i = \mu_i e_i, \quad 2 \leq i \leq n,
\]

\[
\lambda_1 + (n-1) \mu = 0, \quad H = -\frac{n}{(n-1)^2} \lambda^2_1,
\]

then $F$ is a Calabi product of a point with a hyperbolic affine sphere in $\mathbb{R}^n$ with parallel cubic form.

**Proof.** We extend the basis $\{e_i\}_{1 \leq i \leq n}$ by parallel translation along geodesics (with respect to $\hat{\nabla}$) through $p$ to a normal neighborhood around $p$. By the properties of parallel translation this gives a local $h$-orthonormal basis $\{E_i\}_{1 \leq i \leq n}$ on a neighborhood of $p$. Since $\hat{\nabla}K = 0$, it follows that

\[
K_{E_i} E_1 = \lambda_1 E_1, \quad K_{E_i} E_1 = \mu E_1, \quad 2 \leq i \leq n,
\]

holds at every point in a normal neighborhood. Since $\hat{\nabla}K = 0$ and $\hat{\nabla}_E E_1$ is $h$-orthogonal to $E_1$, we have

\[
0 = (\hat{\nabla}_E K)(E_1, E_1) = \lambda_1 \hat{\nabla}_E E_1 - 2K(\hat{\nabla}_E E_1, E_1) = \frac{n+1}{n-1} \hat{\nabla}_E E_1, \quad i = 1, 2, \ldots, n,
\]

i.e. $E_1$ is a parallel vector field with respect to $\hat{\nabla}$, and this gives $\nabla_X E_1 = K(X, E_1)$ for all $X$.

We define two local distributions $T_0$ and $T_1$ on $M^n$ by

\[
T_0 : q \mapsto T_0|q = \text{span}\{E_1(q)\},
\]

\[
T_1 : q \mapsto T_1|q = \{u \in T_p M^n \mid h(u, E_1(q)) = 0\}.
\]

Since $\hat{\nabla}_X E_1 = 0$, we have $\hat{\nabla}_{T_0} T_0 \subset T_0$ and $\hat{\nabla}_{T_1} T_0 \subset T_0$. Since $T_0$ and $T_1$ are $h$-orthogonal this then implies that also $\hat{\nabla}_X T_1 \subset T_1$ for any vector field $X$. Therefore it follows from the de Rham decomposition theorem ([12], pp. 187) that $(M^n, h)$ is locally isometric to a Riemannian product $\mathbb{R} \times M_1$, where $M_1$ is an $(n-1)$-dimensional manifold. Moreover, since $E_1 \in T_0$, after identification $E_1$ is tangent to the $\mathbb{R}$-component, and we write $E_1 = \frac{\partial}{\partial r}$, whereas $\{E_2, \ldots, E_n\}$ is a basis of $M_1$.

From the above we can solve the following system of equations

\[
\begin{align*}
E_1(\eta_1) &= (\mu - \lambda_1) \eta_1, \quad U(\eta_1) = 0; \\
E_1(\eta_2) &= -\mu \eta_2, \quad U(\eta_2) = 0, \quad U \in T_p M_1.
\end{align*}
\]

(4.2)

to obtain

\[
\eta_1 = c_1 e^{(\mu - \lambda_1) \xi}, \quad \eta_2 = c_2 e^{-\mu \xi}
\]

(4.3)

for some positive constants $c_1$ and $c_2$. We define two maps from $M^n$ to $\mathbb{R}^{n+1}$ as follows:

\[
G_1 := e^{(\mu - \lambda_1) \xi}(-\mu F + E_1), \quad G_2 := e^{-\mu \xi}(-HF + \mu E_1).
\]

(4.4)

**Claim 1.** $G_1$ is a constant vector field.

**Proof.** In fact, for the original immersion $F : M^n \hookrightarrow \mathbb{R}^{n+1}$ we have $dF = \sum_{i=1}^n E_i \omega_i$; the affine normal satisfies $\xi = -HF$, and $\nabla_{E_i} E_1 = K_{E_i} E_1 = \lambda_i E_1$, $\lambda_2 = \cdots = \lambda_n = \mu$; then we get:

\[
dG_1 = (\mu - \lambda_1) e^{(\mu - \lambda_1) \xi}(-\mu F + E_1) \omega_1 + e^{(\mu - \lambda_1) \xi} \left( -\mu \sum_{i=1}^n E_i \omega_i + \sum_{i=1}^n \nabla_{E_i} E_1 \omega_i - HF \omega_1 \right)
\]

\[
= -[\mu(\mu - \lambda_1) + H] F e^{(\mu - \lambda_1) \xi} dt = 0,
\]

here we used the relation $\mu(\mu - \lambda_1) = -H$. This proves Claim 1. \qed

**Claim 2.** $G_2$ defines an $(n - 1)$-dimensional hyperbolic affine sphere (contained in some $\mathbb{R}^n$ to which $G_1$ is a transversal vector field in $\mathbb{R}^{n+1}$).
Proof. From the definition we see that
\[
dG_2 = -\mu e^{-\mu t} (-HF + \mu E_1)\omega_1 + e^{-\mu t} \left(-H \sum_{i=1}^{n} E_i \omega_i + \mu \sum_{i=1}^{n} \nabla_{E_i} E_1 \omega_i - \mu HF \omega_1 \right)
\]
\[= e^{-\mu t} (\mu^2 - H) \sum_{i=2}^{n} E_i \omega_i.
\]
Since \(G_1\) is a constant vector field, it follows that \(G_2\) is contained in some \(\mathbb{R}^n\) with \(G_1\) a transversal vector field to this \(\mathbb{R}^n\).

As we have
\[
dG_2(V) = D_V (G_2) = (\mu^2 - H)V, \quad V \in T_1,
\]
the map \(G_2\) is an immersion. Moreover, denoting by \(\nabla^1\) the \(T_1\) component of \(\nabla\), we find for \(V, \tilde{V} \in T_1:\)
\[
D_V \ dG_2(\tilde{V}) = e^{-\mu t} (\mu^2 - H)D_V \tilde{V}
\]
\[= e^{-\mu t} (\mu^2 - H)\nabla^1 V \tilde{V} + e^{-\mu t} (\mu^2 - H)h(V, \tilde{V})(-HF)
\]
\[= e^{-\mu t} (\mu^2 - H)\nabla^1 V \tilde{V} + e^{-\mu t} (\mu^2 - H)\left[h(K(V, \tilde{V}), E_1)E_1 - Hh(V, \tilde{V})F \right]
\]
\[= dG_2(\nabla^1 V \tilde{V}) + (\mu^2 - H)h(V, \tilde{V})G_2.
\]
The above formulas imply that \(G_2\) can be interpreted as a centro-affine immersion contained in an \(n\)-dimensional vector subspace \(\mathbb{R}^n\) of \(\mathbb{R}^{n+1}\) with induced connection \(\nabla^1\) and affine metric \(h_1 = (\mu^2 - H)h\). We also note that the constant vector field \(G_1\) is transversal to the immersion \(G_2\). As before we get that \(G_2 : M_1 \rightarrow \mathbb{R}^n\) satisfies the apolarity condition and that it is a hyperbolic affine hypersphere. This proves Claim 2. \(\Box\)

From (4.4) we obtain
\[
F = -\frac{\mu}{H^2 + \mu^2} e^{(\lambda_1 - \mu)t} G_1 + \frac{1}{H^2 + \mu^2} e^{\mu t} G_2.
\]
Since \(\lambda_1 - \mu = -\eta \mu\), a reparametrization \(s = \eta \mu t\) gives
\[
F = \left(-\frac{\mu}{H^2 + \mu^2} e^{-s} G_1, \frac{1}{H^2 + \mu^2} e^{\frac{s}{\eta}} G_2\right)
\]
which shows that \(F\) is a Calabi product of a point \(G_1\) with a hyperbolic affine sphere \(G_2\). Because \(F : M^n \rightarrow \mathbb{R}^{n+1}\) is a locally strongly convex affine hypersphere with parallel cubic form, then \(G_2 : M_1 \rightarrow \mathbb{R}^n\) is a locally strongly convex hyperbolic affine sphere with parallel cubic form. This proves Theorem 4.1. \(\Box\)

5. Hypersurfaces with \(\dim(\text{Im } L) = 1\)

From now on we consider the case \(C_i\) for \(i \geq 2\). Notice that for \(C_2\) we have \(\dim(\text{Im } L) = 1\). In this section, we study the general case \(\dim(\text{Im } L) = 1\); we prove the following theorem.

Theorem 5.1. Let \(M^n\) be a locally strongly convex affine hypersurface which has parallel and non-vanishing cubic form. If \(\dim(\text{Im } L) = 1\) then \(M^n\) can be decomposed as the Calabi product of two hyperbolic affine spheres both with parallel cubic form.

First of all, we prove

Lemma 5.1. If \(\dim(\text{Im } L) = 1\) then we can choose a unit vector \(w_1 \in \text{Im}(L) \subset D_3\) such that
\[
L(v_1, v_2) = \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-1}} h(v_1, v_2) w_1, \quad \forall v_1, v_2 \in D_2.
\]

Proof. Since \(\dim(\text{Im } L) = 1\), we can choose \(\bar{w} \in \text{Im } L \subset D_3\) such that
\[
L(v_1, v_2) = \alpha(v_1, v_2) \bar{w},
\]
where \(\alpha\) is a symmetric bilinear form over \(D_2\).

Define \(Q : D_2 \rightarrow D_2\) by \(h(Q v_1, v_2) := \alpha(v_1, v_2)\). From (3.8) we have

\[Q = \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-1}} h(\alpha) w_1.
\]
Comparing the above, then Lemma 5.3 follows.

From Lemma 3.5 we see that,

\[ h(v_1, v_2) = 0\]

if \( h(v_1, v_2) = 0 \).

It follows from (3.7), (5.2) and (5.3) that, if \( h(v_1, v_2) = 0 \) then \( L(v_1, v_2) = 0 \). Now we see that

\[ h(Q v_1, v_2) = 0\]

and \( Q = \frac{\lambda_j}{2} \sqrt{\frac{n+1}{n-i}} \) for all \( v \in D_2 \), where \( \epsilon(v) = \pm 1 \). Therefore,

\[ L(v_1, v_2) = \alpha(v_1, v_2)w = \frac{\lambda_j}{2} \sqrt{\frac{n+1}{n-i}} \epsilon(v_1)h(v_1, v_2)w. \]

Since \( L \) and \( h \) are both symmetric, (5.4) implies that \( \epsilon(v_1) = \epsilon(v_2) \) for any \( v_1, v_2 \in D_2 \), i.e., \( \epsilon(v) \) is independent of \( v \). Set \( w_1 := \epsilon(v_1)w \), then we obtain (5.1). \( \square \)

In the sequel of this section, we fix the unit vector \( w_1 \in D_3 \) as in Lemma 5.1; then \( K_{e_1}w_1 = \frac{n+1}{2(n-i)} \lambda_1 w_1 \). Moreover, we have

**Lemma 5.2.** There exists an orthonormal basis \( \{v_1, \ldots, v_{i-1}\} \) of \( D_2 \) such that

\[ K_{e_1}v_j = \frac{\lambda_j}{2} v_j, \quad K_{w_1}v_j = \frac{\lambda_j}{2} \sqrt{\frac{n+1}{n-i}} v_j, \quad 1 \leq j \leq i-1; \]

\[ K_{e_1}v_k = \frac{\lambda_k}{2} \left( e_1 + \sqrt{\frac{n+1}{n-i}} w_1 \right) \delta_{jk}, \quad 1 \leq j, k \leq i-1. \]

**Proof.** From Lemma 3.5 we see \( K_{w_1} : D_2 \to D_2 \), and \( K_{w_1} \) is self-adjoint. Then there exists an orthonormal basis \( \{v_1, \ldots, v_{i-1}\} \) of \( D_2 \) such that \( K_{w_1}v_j = \alpha_j v_j \) with eigenvalues \( \alpha_j \). The fact that \( v_j \in D_2 \) implies that \( K_{e_1}v_j = \frac{\lambda_j}{2} v_j \), and with Lemma 5.1 we get

\[ \alpha_j = h(K_{w_1}v_j, v_j) = h(L(v_j, v_j), w_1) = \frac{\lambda_j}{2} \sqrt{\frac{n+1}{n-i}}. \]

Since \( L(v_j, v_k) = \frac{\lambda_j}{2} \sqrt{\frac{n+1}{n-i}} \delta_{jk} w_1 \), we get

\[ K_{e_1}v_k = K(v_j, v_k) = \frac{1}{2} \lambda_j \delta_{jk} e_1 + L(v_j, v_k) = \frac{\lambda_j}{2} \left( e_1 + \sqrt{\frac{n+1}{n-i}} w_1 \right) \delta_{jk}. \]

Now we are ready to prove

**Lemma 5.3.** \( K_{w_1}w_1 = -\frac{n+1}{2(n-i)} \lambda_1 e_1 + \frac{2n-3i-1}{2(n-\lambda_1)} \sqrt{\frac{n+1}{n-i}} \lambda_1 w_1 \).

**Proof.** We will use the following equations, for \( 1 \leq j \leq i-1 \):

\[ \hat{R}(e_1, v_j)K(v_j, w_1) = K(\hat{R}(e_1, v_j)v_j, w_1) + K(v_j, \hat{R}(e_1, v_j)w_1). \]

From the calculations

\[ \hat{R}(e_1, v_j)v_j = He_1 - K_{e_1}K_{e_1}v_j + K_{e_1}K_{e_1}v_j = -\frac{(n+1)^2 \lambda_j^2}{4(n-i)^2} e_1 + \frac{\lambda_j^2}{4} \left( \frac{n+1}{n-i} \right)^{3/2} w_1, \]

\[ \hat{R}(e_1, v_j)w_1 = -K_{e_1}K_{e_1}w_1 + K_{e_1}K_{e_1}w_1 = -\frac{\lambda_j^2}{4} \left( \frac{n+1}{n-i} \right)^{3/2} v_j, \]

we obtain the left-hand side (LHS) and the right-hand side (RHS) of (5.6) as follows:

\[ \text{LHS} = \frac{\lambda_j^3}{8} \left( \frac{n+1}{n-i} \right)^2 \left( -\sqrt{\frac{n+1}{n-i}} e_1 + w_1 \right). \]

\[ \text{RHS} = K \left( \frac{(n+1)^2 \lambda_j^2}{4(n-i)^2} e_1 + \frac{\lambda_j^2}{4} \left( \frac{n+1}{n-i} \right)^{3/2} w_1 \right) + K \left( v_j, -\frac{\lambda_j^2}{4} \left( \frac{n+1}{n-i} \right)^{3/2} v_j \right) = -\frac{\lambda_j^3}{8} \left( \frac{n+1}{n-i} \right)^{3/2} e_1 + \frac{\lambda_j^2}{4} \left( \frac{n+1}{n-i} \right)^{3/2} K_{w_1}w_1 + \frac{\lambda_j^3}{8} \left( \frac{n+1}{n-i} \right)^{2} \left( \frac{i+1}{n-i} - 1 \right) w_1. \]

Comparing the above, then Lemma 5.3 follows. \( \square \)
Now we have proved that, for any \( w \in D_3 \) with \( h(w_1, w) = 0 \), the following holds:

\[
h(K_{w_j}, e_1) = h(K_{w_j}, w_1) = h(K_{w_j}, v_j) = 0, \quad 1 \leq j \leq i - 1.
\]

(5.7)

This implies that \( K_{w_j} : D_3 \rightarrow D_3/\mathbb{R}w_1 \), and this map is self-adjoint; therefore there exists an orthonormal basis \( \{w_1, w_2, \ldots, w_{n-1}\} \) of \( D_3 \) satisfying

\[
K_{w_j} w_j = \mu_j w_j, \quad 2 \leq j \leq n - i.
\]

(5.8)

Lemma 5.4. We have \( \mu_2 = \cdots = \mu_{n-i} = -\frac{i+1}{2(n-i)} \sqrt{\frac{n+1}{n-i}} \lambda_1. \)

Proof. From the equation \( \tilde{R}(w_1, w_j)K(v_1, v_1) = 2K(\tilde{R}(w_1, w_j) v_1, v_1) \) we get

\[
\frac{\lambda_1}{2} \tilde{R}(w_1, w_j) \left( e_1 + \frac{n+1}{n-i} w_1 \right) = 2K(\tilde{R}(w_1, w_j) v_1, v_1).
\]

(5.9)

We observe that, for \( 2 \leq j \leq n - i, \)

\[
\tilde{R}(w_1, w_j) e_1 = -K_{w_j} K_{w_j} e_1 + K_{w_j} K_{w_j} e_1 = 0,
\]

(5.10)

\[
\tilde{R}(w_1, w_j) v_1 = -K_{w_j} K_{w_j} v_1 + K_{w_j} K_{w_j} v_1 = -K_{w_j} K_{w_j} v_1 + K_{w_j} \left( \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-i}} v_1 \right) = 0,
\]

(5.11)

\[
\tilde{R}(w_1, w_j) v_1 = -Hw_j - K_{w_j} K_{w_j} w_1 + K_{w_j} K_{w_j} w_1 = \left[ (i+1)(n+1) \frac{\lambda_1}{2} - \mu_j \right] \sqrt{\frac{n+1}{n-i}} v_1 = \left[ \frac{(i+1)(n+1)}{2(n-i)^2} \frac{\lambda_1}{2} - \mu_j \right] \sqrt{\frac{n+1}{n-i}} w_j;
\]

(5.12)

here, in the last step deriving (5.11), we use the facts that \( K_{w_j} v_1 \in D_2 \) and for \( k \neq 1, \)

\[
h(K_{w_j} v_1, v_k) = h(K_{v_1}, w_j, w_k) = 0,
\]

thus \( K_{w_j} v_1 = h(K_{w_j} v_1, v_1) = 0 \),

\[
-K_{w_j} K_{w_j} v_1 = -h(K_{w_j} v_1, v_1) K_{w_j} v_1 = -h(K_{w_j} v_1, v_1) \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-i}} v_1 = \left( i+1 \right) \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-i}} K_{w_j} v_1.
\]

From (5.9)–(5.12), we see that

\[
\frac{(i+1)(n+1)}{2(n-i)^2} \lambda_1 - \mu_j + \frac{2n - 3i - 1}{2(n-i)} \sqrt{\frac{n+1}{n-i}} \lambda_1 = 0, \quad \text{for} \quad 2 \leq j \leq n - i.
\]

This implies that \( \mu_j = \sqrt{\frac{n+1}{n-i}} \lambda_1 \) or \( \mu_j = -\frac{i+1}{2(n-i)} \sqrt{\frac{n+1}{n-i}} \lambda_1. \) If we assume, for \( [\mu_j]_{2 \leq j \leq n-i}, \) that \( \sqrt{\frac{n+1}{n-i}} \lambda_1 \) is of multiplicity \( p \) and \( -\frac{i+1}{2(n-i)} \sqrt{\frac{n+1}{n-i}} \lambda_1 \) is of multiplicity \( q = n - p - i - 1, \) then the trace \( \text{tr}(K(w_j)) = 0 \) reduces to

\[
(i-1) \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-i}} + \frac{2n - 3i - 1}{2(n-i)} \sqrt{\frac{n+1}{n-i}} \lambda_1 + p \sqrt{\frac{n+1}{n-i}} \lambda_1 - (n - p - i - 1) \frac{i+1}{2(n-i)} \sqrt{\frac{n+1}{n-i}} \lambda_1 = 0.
\]

Then \( p = 0 \) and Lemma 5.4 follows. \( \Box \)

Proof of Theorem 5.1. Based on the Lemmas 5.2, 5.3 and 5.4, we choose

\[
\tau = \sqrt{n - i} e_1 + \sqrt{n + 1} w_1, \quad \nu = -\sqrt{n + 1} e_1 + \sqrt{n - 1} w_1.
\]

Then \( \{\tau, v_1, \ldots, v_{i-1}, v, w_2, \ldots, w_{n-i}\} \) forms an orthonormal basis of \( T_p M^n; \) with respect to this basis, the difference tensor \( K \) takes the following form:

\[
\begin{align*}
K_{\tau} t &= 2 \frac{i-1}{n-i} \sqrt{\frac{n+1}{n-i}} \lambda_1 t =: \sigma_1 t, \\
K_{\tau} v &= \frac{\lambda_1}{2} \sqrt{\frac{2n-i+1}{n-i}} w =: \sigma_2 v, \\
K_{\tau} v_j &= \frac{\lambda_1}{2} \sqrt{\frac{n+1}{n-i}} w_j =: \sigma_2 v_j, \quad \text{for} \quad 1 \leq j \leq i - 1, \\
K_{\tau} w_k &= -i \frac{\lambda_1}{n-i} \frac{2n-i+1}{n-i} \lambda_1 w_k =: \sigma_3 w_k, \quad \text{for} \quad 2 \leq k \leq n - i.
\end{align*}
\]

(5.13)
One can easily see that the constants $\sigma_1, \sigma_2, \sigma_3$, defined in (5.13), satisfy
\[
\begin{align*}
\sigma_1 &\neq 2\sigma_2, \quad \sigma_1 \neq 2\sigma_3, \quad \sigma_2 \neq \sigma_3, \quad \sigma_2 + \sigma_3 = \sigma_1, \\
\sigma_2 \sigma_3 &= -\frac{(i + 1)(2n - i + 1)}{4(n - i)^2} \lambda_1^2 = H.
\end{align*}
\]
(5.14)

By parallel translation along geodesics through $p$ to a normal neighborhood around $p$ we can extend $\{t, v, v_1, \ldots, v_{i-1}, w_2, \ldots, w_{n-1}\}$ to obtain a local $h$-orthonormal basis
\[
\{T, V, V_1, \ldots, V_{i-1}, W_2, \ldots, W_{n-1}\}
\]
such that
\[
\begin{align*}
K_7 T &= \sigma_1 T; \quad K_7 V = \sigma_2 V, \quad K_7 V_j = \sigma_2 V_j, \quad 1 \leq j \leq i - 1; \\
K_7 W_k &= \sigma_3 W_k, \quad 2 \leq k \leq n - i.
\end{align*}
\]

Now we can apply Theorem 3 of [11] to conclude that $M^n$ is decomposed as the Calabi product of two hyperbolic affine spheres, both with a parallel cubic form. \(\square\)

6. Hypersurfaces with $\dim \mathcal{D}_2 = 2$ and $\dim (\text{Im } L) \geq 2$

In this section, we consider a locally strongly convex affine hypersurface with parallel and non-vanishing cubic form satisfying the condition of the case $C_3$ and, in the notation of Section 3, $\dim (\text{Im } L) \geq 2$. The main results of this section are the following two theorems.

**Theorem 6.1.** Let $M^n$ be a locally strongly convex affine hypersurface which has parallel and non-vanishing cubic form. If $\dim \mathcal{D}_2 = 2$ and $\dim (\text{Im } L) = 2$ then $n = 5$ and, up to a suitable homothetic transformation, $M^n$ is affinely equivalent to an open part of the standard embedding $SL(3, \mathbb{R})/SO(3, \mathbb{R}) \hookrightarrow \mathbb{R}^6$.

**Theorem 6.2.** Let $M^n$ be a locally strongly convex affine hypersurface which has parallel and non-vanishing cubic form. If $\dim \mathcal{D}_2 = 2$ and $\dim (\text{Im } L) \geq 3$ then $n = 6$ and $\dim (\text{Im } L) = 3$ and $n \geq 6$. Moreover, $M^n$ can be decomposed as the Calabi product of a hyperbolic affine sphere with parallel cubic form and a point for $n = 6$, or as the Calabi product of two hyperbolic affine spheres, both with parallel cubic form, for $n \geq 7$.

Let us choose $\{v_1, v_2\}$ as an orthonormal basis of $\mathcal{D}_2$. Since $L$ is bilinear, we see that $\text{Im } L \subseteq \text{Span}(L(v_1, v_1), L(v_1, v_2), L(v_2, v_2))$, and it follows that $\dim (\text{Im } L) \leq 3$. First of all, we have

**Lemma 6.1.** The three vectors $L(v_1, v_1) + L(v_2, v_2), L(v_1, v_2) - L(v_2, v_2)$ and $L(v_1, v_2)$ are mutually orthogonal. Moreover, we have
\[
h(L(v_1, v_1) - L(v_2, v_2), L(v_1, v_1) - L(v_2, v_2)) = 4h(L(v_1, v_2), L(v_1, v_2)),
\]
which implies that both $L(v_1, v_1) - L(v_2, v_2) \neq 0$ and $L(v_1, v_2) \neq 0$.

**Proof.** From the isotropy conditions (3.7) and (3.8), we get the first claim. From (3.7) and (3.9), we easily verify (6.1). Finally, from (6.1) we see that $L(v_1, v_1) - L(v_2, v_2) = 0$ if and only if $L(v_1, v_2) = 0$. Then $\dim \mathcal{D}_2 = 2$ implies the last claim. \(\square\)

According to Lemma 6.1, we can choose orthonormal vectors $w_1, w_2 \in \mathcal{D}_3$ such that
\[
L(v_1, v_1) = b_1 w_1 + b_2 w_3, \quad L(v_1, v_2) = b_1 w_2, \quad L(v_2, v_2) = -b_1 w_1 + b_2 w_3,
\]
where $b_1 \neq 0$, and if $b_2 = 0$ then we choose $w_3 = 0$, if $b_2 \neq 0$ then we choose $w_3 \in \mathcal{D}_3$ as unit vector such that $h(w_1, w_3) = h(w_2, w_3) = 0$. Moreover, the isotropy condition implies that
\[
b_1^2 + b_2^2 = \frac{n + 1}{4(n - 3)} \lambda_1^2.
\]
(6.3)

From Lemmas 3.2 and 3.5, (6.2) and the apolarity we establish the following formulas
\[
\begin{align*}
K_{v_1} e_1 &= \lambda_1 e_1, \quad K_{v_1} v_1 = \frac{1}{4} v_1, \quad K_{v_1} v_2 = \frac{1}{2} v_2, \quad K_{v_1} w_j = -\frac{2 \lambda_1}{n} w_j, \quad j = 1, 2, 3; \\
k_{v_2} v_1 &= \frac{1}{4} e_1 + b_1 w_1 + b_2 w_3, \quad k_{v_2} v_2 = b_1 w_2, \quad k_{v_2} v_3 = \frac{1}{4} e_1 - b_1 w_1 + b_2 w_3; \\
k_{v_1} w_1 &= b_1 v_1, \quad k_{v_2} w_1 = -b_1 v_2, \quad k_{v_1} w_2 = b_1 v_2, \quad k_{v_2} w_2 = b_1 v_1; \\
k_{v_1} w_3 &= b_2 v_1, \quad k_{v_2} w_3 = b_2 v_2.
\end{align*}
\]
(6.4)

In the sequel we will derive the remaining formulas for the difference tensor $K$. 

Lemma 6.2. For \( \{w_1, w_2, w_3\} \) that defined by (6.2), we have
\[
\begin{align*}
K_{w_1}w_1 &= -\frac{2}{n-3}\lambda_1e_1 + 2b_2w_3, \\
K_{w_1}w_3 &= 2b_2w_1, \\
b_2K_{w_1}w_3 &= [2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2]w_1, \\
b_2K_{w_3}w_3 &= -\frac{2}{n-3}b_2\lambda_1e_1 + [2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2]w_3.
\end{align*}
\] (6.5)

Proof. We use (2.1) and (6.4) and calculate the following:
\[
\begin{align*}
\hat{R}(e_1, v_1)v_1 &= -\frac{(n+1)^2}{4(n-3)^2}\lambda_1^2e_1 + \frac{n+1}{2(n-3)}\lambda_1(b_1w_1 + b_2w_3), \\
\hat{R}(e_1, v_2)v_2 &= -\frac{(n+1)^2}{4(n-3)^2}\lambda_1^2e_1 + \frac{n+1}{2(n-3)}\lambda_1(-b_1w_1 + b_2w_3), \\
\hat{R}(e_1, v_1)w_1 &= -\frac{n+1}{2(n-3)}\lambda_1b_1v_1, \\
\hat{R}(e_1, v_2)w_1 &= \frac{n+1}{2(n-3)}\lambda_1b_1v_2, \\
\hat{R}(e_1, v_1)w_3 &= -\frac{n+1}{2(n-3)}\lambda_1b_2v_1, \\
\hat{R}(e_1, v_2)w_3 &= -\frac{n+1}{2(n-3)}\lambda_1b_2v_2.
\end{align*}
\] (6.6)–(6.9)

Applying (6.4) and inserting (6.6)–(6.9) into the following equations:
\[
\begin{align*}
\hat{R}(e_1, v_1)K(v_1, w_1) &= K(\hat{R}(e_1, v_1)v_1, w_1) + K(v_1, \hat{R}(e_1, v_1)w_1), \\
\hat{R}(e_1, v_2)K(v_2, w_1) &= K(\hat{R}(e_1, v_2)v_2, w_1) + K(v_2, \hat{R}(e_1, v_2)w_1), \\
\hat{R}(e_1, v_1)K(v_1, w_3) &= K(\hat{R}(e_1, v_1)v_1, w_3) + K(v_1, \hat{R}(e_1, v_1)w_3), \\
\hat{R}(e_1, v_2)K(v_2, w_3) &= K(\hat{R}(e_1, v_2)v_2, w_3) + K(v_2, \hat{R}(e_1, v_2)w_3);
\end{align*}
\]
we obtain
\[
\begin{align*}
b_2K(w_3, w_3) + b_1K(w_1, w_3) &= -\frac{2}{n-3}\lambda_1b_2e_1 + \left[2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2\right]w_3 + 2b_1b_2w_1, \\
b_2K(w_3, w_3) - b_1K(w_1, w_3) &= -\frac{2}{n-3}\lambda_1b_2e_1 + \left[2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2\right]w_3 - 2b_1b_2w_1, \\
b_2K(w_1, w_3) + b_1K(w_1, w_1) &= -\frac{2}{n-3}\lambda_1b_1e_1 + 2b_1b_2w_3 + \left[2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2\right]w_1, \\
b_2K(w_1, w_3) - b_1K(w_1, w_1) &= \frac{2}{n-3}\lambda_1b_1e_1 - 2b_1b_2w_3 + \left[2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2\right]w_1.
\end{align*}
\]
From these equations we get the assertion in (6.5). \( \square \)

Lemma 6.3. For \( \{w_1, w_2, w_3\} \) that defined by (6.2), we have
\[
\begin{align*}
K_{w_1}w_2 &= 0, \\
K_{w_2}w_2 &= -\frac{2}{n-3}\lambda_1e_1 + 2b_2w_3, \\
b_2K_{w_2}w_2 &= \left[2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2\right]w_2.
\end{align*}
\] (6.10)

Proof. Consider an orthogonal transformation for the basis of \( D_2 \):
\[
\begin{align*}
\tilde{v}_1 &= \cos tv_1 + \sin tv_2, \\
\tilde{v}_2 &= -\sin tv_1 + \cos tv_2.
\end{align*}
\] (6.11)

In analogy to (6.2) we implicitly define \( \tilde{w}_1, \tilde{w}_2, \tilde{w}_3 \) by:
\[
\begin{align*}
L(\tilde{v}_1, \tilde{v}_1) &= b_1\tilde{w}_1 + b_2\tilde{w}_3, \\
L(\tilde{v}_1, \tilde{v}_2) &= b_1\tilde{w}_2, \\
L(\tilde{v}_2, \tilde{v}_2) &= -b_1\tilde{w}_1 + b_2\tilde{w}_3.
\end{align*}
\]
The triples \( (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \) and \( (w_1, w_2, w_3) \) are related by
\[
\begin{align*}
\tilde{w}_1 &= \cos 2tw_1 + \sin 2tw_2, \\
\tilde{w}_2 &= -\sin 2tw_1 + \cos 2tw_2, \\
\tilde{w}_3 &= w_3.
\end{align*}
\] (6.12)

In analogy to (6.5), for any \( t \), we have
\[
\begin{align*}
K_{\tilde{w}_1}\tilde{w}_1 &= -\frac{2}{n-3}\lambda_1e_1 + 2b_2\tilde{w}_3, \\
b_2K_{\tilde{w}_1}\tilde{w}_3 &= \left[2b_2^2 - \frac{n+1}{(n-3)^2}\lambda_1^2\right]\tilde{w}_1.
\end{align*}
\] (6.13)

As \( t \) is arbitrary, the assertion (6.10) follows from (6.12) and (6.13). \( \square \)
Lemma 6.4. The constant $b_2$, defined in (6.2), is zero if and only if $n = 5$. 

Proof. If $b_2 = 0$ then the last equation of (6.10) gives that $b_2^2 = \frac{n-1}{2(n-3)}$. Combining this with (6.3), we get $n = 5$. If $n = 5$ then the condition $\dim D_2 = 2$ implies that $\dim(\text{Im } L) \leq n - 3 = 2$. From this last fact and (6.2) we get the assertion $b_2 = 0$. □

Now, we will separate the discussion into two cases, namely $b_2 = 0$ and $b_2 \neq 0$.

Case 6.1. $b_2 = 0$. In this case, we have (6.4), (6.5) and (6.10). This proves the following:

Proposition 6.1. If $b_2 = 0$, then $n = 5$. Furthermore, with respect to the above chosen orthonormal basis $\{e_1, v_1, v_2, w_1, w_2\}$ at $p \in M^5$, the difference tensor $K$ takes the following form

$$
\begin{align*}
K_{e_1} e_1 &= \lambda_1 e_1, \quad K_{e_1} v_1 = \frac{\lambda_1}{2} v_1, \quad K_{e_1} v_2 = \frac{\lambda_1}{2} v_2, \quad K_{e_1} w_1 = -\lambda_1 w_1, \quad K_{e_1} w_2 = -\lambda_1 w_2; \\
K_{v_1} v_1 &= \frac{\lambda_1}{2} e_1 + \frac{\sqrt{2}}{2} \lambda_1 w_1, \quad K_{v_1} v_2 = \frac{\sqrt{2}}{2} \lambda_1 w_2, \quad K_{v_1} v_2 = \frac{\lambda_1}{2} e_1 - \frac{\sqrt{2}}{2} \lambda_1 w_1; \\
K_{v_2} v_1 &= -\sqrt{2} \lambda_1 v_1, \quad K_{v_2} v_2 = -\lambda_1 v_2, \quad K_{v_2} v_2 = \sqrt{2} \lambda_1 v_1; \\
K_{w_1} w_1 &= -\lambda_1 e_1, \quad K_{w_1} w_1 = 0, \quad K_{w_2} w_2 = -\lambda_1 e_1.
\end{align*}
$$

(6.14)

Case 6.2. $b_2 \neq 0$. In this case we have $n \geq 6$. From (6.5) and (6.3) the constants $b_1, b_2$ satisfy

$$
\begin{align*}
b_1 &= \frac{\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1, \\
b_2 &= \frac{\sqrt{2} (n+1) (n-5)}{4(n-3)} \lambda_1.
\end{align*}
$$

(6.15)

where we have assumed $b_1 > 0$ and $b_2 > 0$; otherwise we can change the direction of $\{w_1, w_2\}$ or $w_3$, resp.

Insert (6.15) into (6.4), (6.5) and (6.10); we have

Proposition 6.2. If $b_2 \neq 0$, then $n \geq 6$. Furthermore, with respect to the above chosen orthonormal vectors $\{e_1, v_1, v_2, w_1, w_2, w_3\}$ at $p \in M^n$, the difference tensor $K$ takes the following form

$$
\begin{align*}
K_{e_1} e_1 &= \lambda_1 e_1, \quad K_{e_1} v_1 = \frac{\lambda_1}{2} v_1, \quad K_{e_1} v_2 = \frac{\lambda_1}{2} v_2, \quad K_{e_1} w_j = -\frac{2\lambda_1}{n-3} w_j, \quad j = 1, 2, 3; \\
K_{v_1} v_1 &= \frac{\lambda_1}{2} e_1 + \frac{\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 w_1 + \frac{\sqrt{2} (n+1) (n-5)}{4(n-3)} \lambda_1 w_3, \\
K_{v_2} v_2 &= \frac{\lambda_1}{2} e_1 - \frac{\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 w_1 + \frac{\sqrt{2} (n+1) (n-5)}{4(n-3)} \lambda_1 w_3, \\
K_{v_1} v_1 &= \frac{\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 w_2, \quad K_{v_1} w_1 = \frac{\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 v_1, \\
K_{v_2} v_2 &= \frac{2\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 v_2, \quad K_{v_2} w_2 = \frac{2\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 v_2, \\
K_{w_1} w_1 &= -\frac{\sqrt{2} (n+1) (n-1)}{4(n-3)} \lambda_1 v_2, \quad K_{w_1} w_2 = \frac{\sqrt{2} (n+1) (n-5)}{4(n-3)} \lambda_1 v_2, \quad K_{w_1} w_2 = 0, \\
K_{w_2} w_2 &= \frac{\sqrt{2} (n+1) (n-5)}{4(n-3)} \lambda_1 v_2, \quad K_{w_2} w_3 = \frac{\sqrt{2} (n+1) (n-5)}{4(n-3)} \lambda_1 v_2, \quad K_{w_1} w_3 = 0, \\
K_{w_1} w_1 &= -\frac{2\sqrt{2} (n+1) (n-5)}{2(n-3)} \lambda_1 w_1, \quad K_{w_1} w_1 = -\frac{2\sqrt{2} (n+1) (n-5)}{2(n-3)} \lambda_1 w_1, \\
K_{w_2} w_2 &= -\frac{2\sqrt{2} (n+1) (n-5)}{2(n-3)} \lambda_1 w_1, \quad K_{w_2} w_3 = -\frac{2\sqrt{2} (n+1) (n-5)}{2(n-3)} \lambda_1 w_1, \quad K_{w_2} w_3 = 0, \\
K_{w_3} w_3 &= -\frac{2\sqrt{2} (n+1) (n-5)}{2(n-3)} \lambda_1 w_1, \quad K_{w_3} w_3 = -\frac{2\sqrt{2} (n+1) (n-5)}{2(n-3)} \lambda_1 w_1, \quad K_{w_3} w_3 = 0.
\end{align*}
$$

(6.16)

Now we define

$$
\tau := \sqrt{\frac{n-5}{3(n-1)}} e_1 + \sqrt{\frac{2(n+1)}{3(n-1)}} w_3, \quad \tau^* := -\sqrt{\frac{2(n+1)}{3(n-1)}} e_1 + \sqrt{\frac{n-5}{3(n-1)}} w_3.
$$
**Proposition 6.3.** In the notations as above, we have

\[
\begin{align*}
K_t t &= -\frac{n-1}{n-3} \frac{n-1}{3(n-5)} \lambda_1 t, \\
K_t u &= -\frac{5}{n-3} \frac{n-1}{3(n-5)} \lambda_1 u, \quad u = v_1, v_2, t^*, w_1, w_2.
\end{align*}
\]

(6.17)

Moreover, if \( \text{Im}(L) \neq D_3 \) then for any \( w \in D_3 \) with \( w \perp \text{Im}(L) = \text{Span}\{w_1, w_2, w_3\} \), we have

\[
K_t w = -\frac{6}{n-3} \frac{n-1}{3(n-5)} \lambda_1 w.
\]

(6.18)

**Proof.** (6.17) can be checked directly by using the formulas in **Proposition 6.2**. To prove (6.18), we see that

\[
K_t w = \sqrt{\frac{n-5}{3(n-1)}} K_t v, \quad K_t v + \sqrt{\frac{2(n+1)}{3(n-1)}} K_t w = -\frac{2}{n-3} \frac{n-1}{3(n-1)} \lambda_1 w + \sqrt{\frac{2(n+1)}{3(n-1)}} K_t w,
\]

then we use (3.12) and the fact

\[
K_t w = \frac{1}{2b_2} K(L(v_1, v_1) + L(v_2, v_2), w) = -\frac{n+1}{(n-3)^2b_2} \lambda_1 w
\]

\[
= -\frac{2}{n-3} \frac{2(n+1)}{n-5} \lambda_1 w. \quad \square
\]

**Proof of Theorem 6.2 for \( n = 6 \).** For \( n = 6 \) we see that \( \text{Im} L = D_3 \) and \( \{e_1, v_1, v_2, w_1, w_2, w_3\} \) is an orthonormal basis of \( T_p M^6 \), whereas **Proposition 6.2** has established all information for the difference tensor \( K \). Moreover, for \( t, t^* \) defined by (6.16), (6.7) reduces to

\[
\begin{align*}
K_t t &= -\frac{5\sqrt{15}}{9} \lambda_1 t =: \sigma_1 t, \\
K_t u &= \frac{\sqrt{15}}{9} \lambda_1 u =: \sigma_2 u, \quad \text{for} \ u = v_1, v_2, t^*, w_1, w_2.
\end{align*}
\]

(6.20)

where \( \sigma_1, \sigma_2 \) and \( H \) are related by \( \sigma_1 \sigma_2 - \sigma_2^2 = -\frac{10}{9} \lambda_1^2 = H \).

By parallel translation along geodesics through \( p \) to a normal neighborhood around \( p \), we can extend \( \{t, v_1, v_2, t^*, w_1, w_2\} \) to obtain a local \( h \)-orthonormal basis \( \{T; V_1, V_2, V_3, V_4, V_5\} \) such that

\[
K_T \sigma_1 T, \quad K_T V_j = \sigma_2 V_j, \quad 1 \leq j \leq 5; \quad \sigma_1 \sigma_2 - \sigma_2^2 = H.
\]

Then we can apply Theorem 1 of [11] to conclude that \( M^6 \) is decomposed as the Calabi product of a hyperbolic affine sphere with parallel cubic form and a point. \( \square \)

**Proof of Theorem 6.2 for \( n \geq 7 \).** In this general case, we choose \( \{w_4, \ldots, w_{n-3}\} \subset D_3 \) such that \( \{w_1, \ldots, w_{n-3}\} \) is an orthonormal basis of \( D_3 \). Then, according to **Proposition 6.3**, for \( t, t^* \) defined by (6.16), we have such that

\[
K_t T = \sigma_1 T, \quad K_T V_j = \sigma_2 V_j \quad \text{for} \ 1 \leq j \leq 5,
\]

\[
K_t W_k = \sigma_3 W_k \quad \text{for} \ 1 \leq k \leq n-6,
\]

with coefficients satisfying \( \sigma_1 \neq 2\sigma_2, \sigma_1 \neq 2\sigma_3 \) and \( \sigma_2 \neq \sigma_3 \). From this we can apply Theorem 3 of [11] to conclude that \( M^n \) is decomposed as the Calabi product of two hyperbolic affine spheres, both with parallel cubic form. \( \square \)

**Proof of Theorem 6.1.** To complete the proof of **Theorem 6.1**, we look at the homogeneous space \( \text{SL}(m, \mathbb{R})/\text{SO}(m) \) and recall its affine invariants in more detail than what has been given in [6].

Let \( s(m) \) be the set of real symmetric \((m, m)\)-matrices, \( \text{SL}(m, \mathbb{R}) \) be the set of real \((m, m)\)-matrices of determinant 1, and \( \text{SO}(m) \) be the set of orthogonal \((m, m)\)-matrices with determinant 1. Let \( \sigma \) be the action of \( \text{SL}(m, \mathbb{R}) \) on \( s(m) \) as follows

\[
\sigma : \text{SL}(m, \mathbb{R}) \times s(m) \to s(m) \quad \text{s.t.} \quad (A, X) \mapsto \sigma_A(X) = AXA^T.
\]

Let \( F : s(m) \to \mathbb{R} \) be given by \( F(X) := \det(X) \). Consider the hypersurface of \( s(m) \) satisfying the equation \( \det(X) = 1 \); we take the connected component \( M \) that lies in the open set of \( s(m) \) consisting of all positive definite symmetric matrices. Then the mapping \( f : \text{SL}(m, \mathbb{R}) \to s(m), \) defined by \( f(A) := AA^T \), is a submersion onto \( M \), and it satisfies \( f(AB) = \sigma_A(f(B)) \), hence \( f \) is equivariant. \( M \) is the orbit of \( I \) under the action \( \sigma \). The isotropy group is \( \text{SO}(m) \). Hence \( M \) is diffeomorphic to \( \text{SL}(m, \mathbb{R})/\text{SO}(m) \). It is known ([18], pp. 110–113; [13], Chapter XI) that this is an irreducible, homogeneous, symmetric space of non-compact type, and the involution at \( I \) is given by \( A \mapsto (A^{-1})^T \). We denote this symmetric space by \( M^n \).
Clearly \( f(A) = f(B) \) if and only if \( B^{-1}A \in SO(m) \), then the map \( f : SL(m, \mathbb{R}) \to s(m) \) induces an embedding \( f : SL(m, \mathbb{R})/SO(m) \to s(m) \). Let \( \pi : SL(m, \mathbb{R}) \to M' \) be the natural projection, then there is an immersion \( f' : M' \to s(m) \) such that \( f = f' \circ \pi \). Now we consider
\[
f : SL(m, \mathbb{R})/SO(m) \to \mathbb{R}^n = s(m), \quad n = \frac{1}{2}m(m + 1),
\]
with a transversal vector field \( \xi_A = f(A) \) for any \( A \in SL(m, \mathbb{R})/SO(m) \). Then \( \xi \) is equiaffine and equivariant.

Consider the Cartan decomposition of the Lie algebra \( sl(m, \mathbb{R}) = s_0 \oplus o(m) \), where \( o(m) \) denotes the set of skew-symmetric \((m, m)\)-matrices and \( s_0 := \{ X \in s(m) \mid \text{tr}(X) = 0 \} \). If \( X \in s_0 \) then \( f_s(X) = X \). Now \( s_0 \) can be considered as the tangent space of \( M' \) at \( \pi(I) \).

Since \( f \) is equivariant, it is sufficient to compute the invariant objects of the immersed hypersurface \( M' \) in terms of \( s_0 \).

The embedding \( f : SL(m, \mathbb{R})/SO(m) \to \mathbb{R}^n = s(m) \) with \( \xi = 4f \) has a Blaschke structure (see e.g. \([18, 20]\)) that can be expressed algebraically in terms of the Lie algebra as follows:
\[
\begin{align*}
\nabla_X Y &= XY + YX - \frac{2}{m} \text{tr}(XY) I_m, \\
h(X, Y) &= \frac{4}{m} \text{tr}(XY), \quad S = -I_m.
\end{align*}
\]
Here \( h \) is the natural Riemannian metric on the symmetric space \( M' \); this implies that the Levi-Civita connection of \( h \) is given by \( \nabla_X Y = \frac{1}{2}[X, Y] \). From this it follows easily that the difference tensor \( K \) satisfies \( (\nabla_X K)(X, X) = 0 \). As \( M = f'(M') \) is an affine sphere, we get that \( \nabla K \) is totally symmetric [2]; then it follows from \( (\nabla_X K)(X, X) = 0 \) and polarization of the multilinear symmetric expression over \( T_p(M) \) at \( p \in M \) that \( \nabla K = 0 \).

Now assume \( m = 3 \). Let us choose an \( h \)-orthonormal basis of \( SL(3, \mathbb{R})/SO(3) \) at \( I \) as follows:
\[
\begin{align*}
e_1 &= \sqrt{\frac{1}{2}} \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{pmatrix}, \\
e_2 &= \sqrt{\frac{\sqrt{6}}{4}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \\
e_3 &= \sqrt{\frac{\sqrt{6}}{4}} \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \\
e_4 &= \sqrt{\frac{\sqrt{6}}{4}} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
e_5 &= \sqrt{\frac{\sqrt{6}}{4}} \begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]

Since \( \nabla e_i = 0 \mod (o(3)) \), we can use the formula
\[
K_X Y = XY + YX - \frac{2}{3} \text{tr}(XY) I
\]
to calculate the difference tensor at \( I \); it takes the following form
\[
\begin{align*}
K_{e_1} e_1 &= \frac{\sqrt{2}}{2} e_1, \quad K_{e_1} e_2 = \frac{\sqrt{2}}{4} e_2, \quad K_{e_1} e_3 = \frac{\sqrt{2}}{4} e_3, \quad K_{e_1} e_4 = -\frac{\sqrt{2}}{4} e_4, \quad K_{e_1} e_5 = -\frac{\sqrt{2}}{4} e_5; \\
K_{e_2} e_1 &= \frac{\sqrt{2}}{4} e_1 + \frac{\sqrt{2}}{4} e_4, \quad K_{e_2} e_3 = \frac{\sqrt{2}}{4} e_3, \quad K_{e_2} e_4 = \frac{\sqrt{2}}{4} e_4, \quad K_{e_2} e_5 = \frac{\sqrt{2}}{4} e_5; \\
K_{e_3} e_1 &= \frac{\sqrt{2}}{4} e_1 - \frac{\sqrt{2}}{4} e_4, \quad K_{e_3} e_4 = -\frac{\sqrt{2}}{4} e_3, \quad K_{e_3} e_5 = \frac{\sqrt{2}}{4} e_4; \\
K_{e_4} e_4 &= -\frac{\sqrt{2}}{4} e_1, \quad K_{e_4} e_5 = 0; \quad K_{e_5} e_5 = \frac{\sqrt{2}}{4} e_1.
\end{align*}
\]

If we identify \((e_1, e_2, e_3, e_4, e_5)\) in (6.24) with \((e_1, v_1, v_2, w_1, w_2)\) in (6.14), we see that (6.24) is exactly the same as (6.14) corresponding to the value \( \lambda_1 = \frac{\sqrt{2}}{2} \), or equivalently \( H = -1 \).

Now, for a locally strongly convex affine hypersurface \( M^5 \) in \( \mathbb{R}^6 \), satisfying \( c_3 \), we see from the above discussion the following (if necessary, we apply a homothetic transformation to make \( H = -1 \)): \( M^5 \) and \( SL(3, \mathbb{R})/SO(3) \) have the affine metric \( h \) and the cubic form \( C \) with identical affine invariant properties, resp. Then the fundamental uniqueness theorem of affine geometry in its classical version (see e.g. \([4, \text{Theorem 3.5}]\) and \([25], \text{Section 4.12.3}\)) states that \( M^5 \) and \( SL(3, \mathbb{R})/SO(3) \) are locally affinely equivalent. This proves Theorem 6.1. \( \square \)

7. Hypersurfaces with \( \dim \mathcal{D}_2 \geq 3 \) and \( \dim(\text{Im} L) \geq 2 \)

In this section, we assume that \( M^n \) is a locally strongly convex hypersurface with parallel and non-vanishing cubic form satisfying the conditions of the case \( c_i \). Moreover, we assume that, in the notation of Section 3, \( \dim \mathcal{D}_2 \geq 3 \) and \( \dim(\text{Im} L) \geq 2 \). The main result is the following:

**Theorem 7.1.** For any integer \( n \), there does not exist a locally strongly convex affine hypersurface in \( \mathbb{R}^{n+1} \) with parallel cubic form such that \( \dim \mathcal{D}_2 = 3 \) and \( \dim(\text{Im} L) = 3 \).

To prove this theorem, we first establish the following lemma for arbitrary dimension \( n \).
Lemma 7.1. If \( \dim D_2 \geq 3 \) and \( \dim(\text{Im} L) \geq 2 \) then \( \dim(\text{Im} L) \geq 3 \).

Proof. For a pair \( v_1, v_2 \in D_2 \) of orthonormal vectors we define a function \( g \) by

\[
g(v_1, v_2) = h(L(v_1, v_2), L(v_1, v_2)).
\]

We choose \( (v_1, v_2) \) such that the absolute maximum for \( g \) is attained. We extend \( (v_1, v_2) \) to get an orthonormal basis \( \{v_1, v_2, \ldots, v_{n-1}\} \) of \( D_2 \). Observe that, for all \( k \geq 3 \), we have

\[
\left. \frac{d}{dt} \right|_{t=0} g(v_1, \cos tv_2 + \sin tv_k) = 0, \quad \left. \frac{d}{dt} \right|_{t=0} g(\cos tv_1 + \sin tv_k, v_2) = 0.
\]

This implies:

\[
h(L(v_1, v_2), L(v_1, v_k)) = 0 = h(L(v_1, v_2), L(v_2, v_k)), \quad \forall k \geq 3.
\]

We discuss the two cases: (i) \( g \equiv 0 \) and (ii) \( g \neq 0 \).

(i) If \( g \equiv 0 \), then \( L(v_j, v_k) = 0 \) for \( j \neq k \). From (7.7) and (7.9) we see that, for \( j \neq k \),

\[
h(L(v_j, v_j), L(v_j, v_j)) = h(L(v_k, v_k), L(v_k, v_k)) = h(L(v_j, v_j), L(v_j, v_k)) = \frac{n+1}{4(n-i)} \lambda_1^2.
\]

From (7.3) and the Cauchy Schwarz inequality we see that \( L(v_j, v_j) = L(v_k, v_k) \) for all \( j \neq k \). It follows that \( \dim(\text{Im} L) \leq 1 \), a contradiction.

(ii) If \( g \neq 0 \), then \( h(L(v_1, v_2), L(v_1, v_2)) > 0 \). Assume on the contrary that \( \dim(\text{Im} L) = 2 \). Then it follows from (7.8) that \( \{L(v_1, v_1), L(v_1, v_2)\} \) is an orthogonal basis of \( \text{Im}(L) \). From (7.8) we see that

\[
h(L(v_1, v_1), L(v_1, v_1)) = h(L(v_1, v_2), L(v_2, v_2)) = 0, \quad \forall k \geq 3.
\]

From (7.7) and (7.9), we get \( L(v_1, v_k) = L(v_2, v_k) = 0, \forall k \geq 3 \). Insert this into (7.9), we obtain

\[
h(L(v_1, v_1), L(v_1, v_k)) = h(L(v_2, v_2), L(v_k, v_k)) = h(L(v_k, v_k), L(v_k, v_k)) = \frac{n+1}{4(n-i)} \lambda_1^2.
\]

We apply the Cauchy Schwarz inequality again to obtain \( L(v_1, v_1) = L(v_2, v_2) = L(v_k, v_k) \). We use this and choose \( X = v_1, Y = v_2 \) in (7.9), then we get \( L(v_1, v_2) = 0 \). This is also a contradiction.

Lemma 7.1 is proved. \( \square \)

Now we assume that \( \dim D_2 = 3 \) and \( \dim(\text{Im} L) = 3 \). This implies that \( i = 4 \). Since \( L : D_2 \times D_2 \to D_3 \) is isotropic, we can apply Lemma 3.1(c) of L. Vrancken [27] to obtain the following:

Lemma 7.2. If \( \dim D_2 = 3 \) and \( \dim(\text{Im} L) = 3 \), then there exist an orthonormal basis \( \{v_1, v_2, v_3\} \) of \( D_2 \) and orthonormal vectors \( \{w_1, w_2, w_3\} \subset D_3 \) such that

\[
\begin{align*}
L(v_1, v_1) &= -L(v_2, v_2) = L(v_3, v_3) = \frac{\lambda_1}{2} \frac{n+1}{n-4} w_1, \\
L(v_1, v_2) &= \frac{\lambda_1}{2} \frac{n+1}{n-4} w_2, \quad L(v_1, v_3) = 0, \quad L(v_2, v_3) = \frac{\lambda_1}{2} \frac{n+1}{n-4} w_3.
\end{align*}
\]

Proof of Theorem 7.1. Assume on the contrary that a hypersurface exists satisfying case \( C_4 \) with \( \dim(\text{Im} L) = 3 \). Then we can easily get the following formulas for the difference tensor from Lemmas 7.2 and 3.5:

\[
\begin{align*}
K_{t_1} e_1 &= \lambda_1 e_1; & K_{t_1} v_j &= \frac{\lambda_1}{2} v_j & \text{for } 1 \leq j \leq 3; \\
K_{t_1} w_k &= -\frac{5}{2(n-3)} w_k & \text{for } 1 \leq k \leq 3; \\
K_{v_1} v_1 &= \frac{\lambda_1}{2} e_1 + \frac{\lambda_1}{2} \frac{n+1}{n-4} w_1; & K_{v_1} v_2 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} w_2. \\
K_{v_1} v_2 &= \frac{\lambda_1}{2} e_1 - \frac{\lambda_1}{2} \frac{n+1}{n-4} w_1; & K_{v_1} v_3 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} w_3. \\
K_{v_1} v_3 &= 0; & K_{v_3} v_1 &= \frac{\lambda_1}{2} e_1 + \frac{\lambda_1}{2} \frac{n+1}{n-4} w_1; \\
K_{v_1} w_1 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} v_1; & K_{v_2} w_1 &= -\frac{\lambda_1}{2} \frac{n+1}{n-4} v_2; & K_{v_3} w_1 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} v_3; \\
K_{v_1} w_2 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} v_2; & K_{v_2} w_2 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} v_1; & K_{v_3} w_2 &= 0; \\
K_{v_1} w_3 &= 0; & K_{v_2} w_3 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} v_3; & K_{v_3} w_3 &= \frac{\lambda_1}{2} \frac{n+1}{n-4} v_2.
\end{align*}
\]
From (7.6) and (2.1), we have the following calculation
\[
\hat{R}(e_1, v_1)v_3 = -K_{e_1}K_{v_1}v_3 + K_{v_1}K_{e_1}v_3 = 0,
\]
\[
\hat{R}(e_1, v_1)v_2 = -K_{v_1}K_{v_1}v_2 + K_{v_1}K_{v_1}v_2 = -\frac{\lambda^2}{4} \left( \frac{n+1}{n-4} \right)^{3/2} v_2.
\]
and from (7.6) and (7.7), the equation
\[
\hat{R}(e_1, v_1)K(v_3, v_2) = K(\hat{R}(e_1, v_1)v_3, v_2) + K(v_3, \hat{R}(e_1, v_1)v_2)
\]
becomes equivalent 
\[-\frac{\lambda^2}{4} \left( \frac{n+1}{n-4} \right)^{3/2} K_{v_1}v_3 = 0.\]
This is a contradiction which completes the proof of Theorem 7.1. \(\Box\)

**Remark 2.** If \(n = 7\), then \(n = 4\). From this and Lemma 7.1 we see that, if \(\dim D_2 \geq 3\) and \(\dim(\text{Im} L) \geq 2\) then \(\dim D_2 = 3\) and \(\dim(\text{Im} L) = 3\). Therefore, as an immediate consequence of Theorem 7.1, we have:

**Corollary 7.1.** There does not exist a locally strongly convex affine hypersurface in \(\mathbb{R}^8\) with parallel cubic form such that \(\dim D_2 \geq 3\) and \(\dim(\text{Im} L) \geq 2\).

**Concluding remarks.** We summarize the known cases with respect to \(\dim D_2\) and \(\dim(\text{Im} L)\) as follows:

- The case \(\dim D_2 = 0\) is known by Theorem 4.1.
- The case \(\dim(\text{Im} L) = 1\) is known by Theorem 5.1; \(\dim D_2 = 1\) belongs to this case.
- The case \(\dim D_2 = 2\) is known by Theorems 6.1 and 6.2.
- The case \(\dim D_2 \geq 3\) and \(\dim(\text{Im} L) = 2\) does not occur by Lemma 7.1.
- The case \(\dim D_2 = 3\) and \(\dim(\text{Im} L) = 3\) does not occur by Theorem 7.1.

From these known cases, we have completed the proof of our Classification Theorem. The other cases, namely that \(\dim D_2 \geq 3\) and \(\dim(\text{Im} L) \geq 4\) for dimension \(n \geq 8\), are much more complicated; we will discuss them in forthcoming papers.

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**References**


