The Gauss-Bonnet-Grotemeyer Theorem in spaces of constant curvature *

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Abstract

In 1963, K.P. Grotemeyer proved an interesting variant of the Gauss-Bonnet Theorem. Let $M$ be an oriented closed surface in the Euclidean space $\mathbb{R}^3$ with Euler characteristic $\chi(M)$, Gauss curvature $G$ and unit normal vector field $\vec{n}$. Grotemeyer’s identity replaces the Gauss-Bonnet integrand $G$ by the normal moment $(\vec{a} \cdot \vec{n})^2 G$, where $a$ is a fixed unit vector: $\int_M (\vec{a} \cdot \vec{n})^2 G dv = \frac{2\pi}{3} \chi(M)$. We generalize Grotemeyer’s result to oriented closed even-dimensional hypersurfaces of dimension $n$ in an $(n+1)$-dimensional space form $N^{n+1}(k)$.

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1. Introduction

In 1963, K.P. Grotemeyer proved the following interesting result:

**Theorem 1 [Gr]** Let $M$ be an oriented closed surface in 3-dimensional Euclidean space $\mathbb{R}^3$ with Gauss curvature $G$ and a unit normal vector field $\vec{n}$. Then for any fixed unit vector $\vec{a}$ in $\mathbb{R}^3$, we have

$$\int_M (\vec{a} \cdot \vec{n})^2 G dv = \frac{2\pi}{3} \chi(M),$$

(1.1)

where $\vec{a} \cdot \vec{n}$ denotes the inner product of $\vec{a}$ and $\vec{n}$, $\chi(M)$ is the Euler characteristic of $M$.

**Remark 1.1** Let $\{E_1, E_2, E_3\}$ be a fixed orthogonal frame in $\mathbb{R}^3$ and choose $\vec{a} = E_i$. We have

$$\int_M (E_i \cdot \vec{n})^2 G dv = \frac{2\pi}{3} \chi(M), \quad i = 1, 2, 3$$

(1.2)

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Noting that \( \sum_i (E_i \cdot \vec{n})(E_i \cdot \vec{n}) = \vec{n} \cdot \vec{n} = 1 \), we obtain the following Gauss-Bonnet formula via summation of (1.2) over \( i \) from 1 to 3:

**Corollary 1** (Gauss-Bonnet Theorem). Under the same hypothesis of Theorem 1, we have

\[
\int_M Gdv = 2\pi \chi(M). \tag{1.3}
\]

Thus we can consider Grotemeyer’s Theorem 1 as an extended form of the Gauss-Bonnet Theorem.

Let \( n \) be even and let \( N^{n+1}(k) \) be an \((n+1)\)-dimensional simply connected Riemannian manifold of constant sectional curvature \( k \). That is, \( N^{n+1}(k) = \mathbb{R}^{n+1} \) if \( k = 0 \); \( N^{n+1}(k) = S^{n+1}(\frac{1}{\sqrt{k}}) \), an \((n+1)\)-dimensional sphere space with radius \( \frac{1}{\sqrt{k}} \) if \( k > 0 \); \( N^{n+1}(k) = H^{n+1}(\frac{1}{\sqrt{-k}}) \), an \((n+1)\)-dimensional hyperbolic space with, as Bolyai would say, radius \( \sqrt{-1}/\sqrt{k} \) if \( k < 0 \). We will often call \( N^{n+1}(k) \) a space form. We will view \( N^{n+1}(k) \) as standardly imbedded in an appropriate linear space \( L_{n+1}(k) \) (\( \mathbb{R}^{n+2} \) if \( k > 0 \), \( \mathbb{R}^{n+1,1} \) if \( k < 0 \) and \( \mathbb{R}^{n+1} \) if \( k = 0 \)).

This will enable us to define functions on \( M \) such as \( (\vec{a} \cdot \vec{n}) \), where \( \vec{a} \) is a fixed vector in the ambient linear space, \( \vec{n} \) is a normal vector field on \( M \), and \( (\cdot) \) denotes the inner product on the ambient linear space. The generalized Grotemeyer Theorem we have in mind can be stated as follows:

**Theorem 2** Let \( n \) even, \( n \geq 2 \). Let \( \vec{x} : M \to N^{n+1}(k) \) be an immersed \( n \)-dimensional oriented closed hypersurface in the \((n+1)\)-dimensional space form \( N^{n+1}(k) \), with Euler characteristic \( \chi(M) \), Gauss-Kronecker curvature \( G \) and unit normal vector field \( \vec{n} \). Assume that \( N^{n+1}(k) \) is standardly imbedded in the linear space \( L_{n+1}(k) \). Then for any fixed unit vector \( \vec{a} \) in \( L_{n+1}(k) \) we have

\[
\int_M (\vec{a} \cdot \vec{n})^2 Gdv = \frac{1}{n+1} [\frac{\text{vol}S^n(1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2i} dv] + \frac{1}{n+1} \int_M (\vec{a} \cdot \vec{n})(\vec{a} \cdot \vec{x}) K_{n-1} dv - \frac{k}{n+1} \int_M (\vec{a} \cdot \vec{x})^2 Gdv, \tag{1.4}
\]

where the \( c_i \) are constants that depend only on the dimension \( n \) and \( K_i \) is the \( i \)-th mean curvature of \( M \).

In the case \( n = 2 \) in the Theorem above, we obtain

**Corollary 2** Let \( M \) be an oriented closed surface in the 3-dimensional space form \( N^3(k) \) with extrinsic curvature \( G \) and unit normal vector field \( \vec{n} \). Then for any fixed unit vector \( \vec{a} \) in the linear space \( L_3(k) \) we have

\[
\int_M (\vec{a} \cdot \vec{n})^2 Gdv = \frac{2\pi}{3} \chi(M) - \frac{k}{3} \text{vol}(M) + \frac{k}{2} \int_M (\vec{a} \cdot \vec{n})(\vec{a} \cdot \vec{x}) K_1 dv - \frac{k}{2} \int_M (\vec{a} \cdot \vec{x})^2 Gdv, \tag{1.5}
\]

where \( K_1 \) is the mean curvature of \( M \) and \( \chi(M) \) is the Euler characteristic of \( M \).
Remark 1.2 Our Corollary reduces to Grotemeyer’s original theorem in the case $k = 0$.

Remark 1.3 In the case $k = 0$ and $n \geq 3$, Theorem 2 was proved by B. -Y. Chen in [Ch] by a different method.

Remark 1.4. We can recover the standard Gauss-Bonnet Theorem from our Theorem as follows. Let $m$ be the dimension of the linear space $L_{n+1}(k)$. (Thus $m = n + 1$ in the flat case, $m = n + 2$ in the positive and negatively curved cases.) Let $\{E_1, \cdots, E_m\}$ be a fixed orthonormal frame in $L_{n+1}(k)$; choose $\vec{a} = E_i$. Then

\[
\int_M (E_i \cdot \vec{n})^2 G dv = \frac{1}{n+1} \left[ \frac{\text{vol}^n(S^1)}{2} x(M) - \sum_i c_i k^i \int_M K_{n-2} dv \right] + \frac{k}{n+1} \int_M G(E_i \cdot x)^2 dv, \quad (i = 1, 2, \ldots) \tag{1.6}
\]

Noting $\sum_i (E_i \cdot \vec{n})(E_i \cdot \vec{n}) = \vec{n} \cdot \vec{n} = 1$ and $\sum_i (E_i \cdot \vec{n})(E_i \cdot x) = \vec{n} \cdot x = 0$, we obtain the following Gauss-Bonnet formula by summing of (1.6) over all appropriate $i$:

**Corollary 3** (Gauss-Bonnet Theorem). Under the same hypothesis of Theorem 2, we have

\[
\int_M G dv = \frac{\text{vol}^n(S^1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2} dv \tag{1.7}
\]

where $\chi(M)$ is the Euler characteristic of $M$, constants $c_i$ depends only on dimension $n$, and $K_i$ is the $i$-th mean curvature of $M$.

So we can view Theorem 2 as an extended form of the Gauss-Bonnet Theorem.

2. Reilly’s operator and its properties

In order to prove Theorem 2, we need to recall Reilly’s operator and its properties. Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold, let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field in $M$ with dual coframe field $\{\theta_1, \cdots, \theta_n\}$. Given a symmetric tensor $\phi = \sum_{i,j} \phi_{ij} \theta_i \theta_j$ defined on $M$ we define a second order differential operator

\[
\Box \equiv \Box_\phi : C^\infty(M) \to C^\infty(M), \quad \Box f = \sum_{i,j} \phi_{ij} f_{ij} \tag{2.1}
\]

where $f_{ij}$ are the components of the second covariant differential of $f$, as follows:

\[
df = \sum_i f_i \theta_i, \quad df_i + \sum_j f_j \theta_{ji} = \sum_j f_{ij} \theta_j, \tag{2.2}
\]
where \( \{ \theta_{ij} \} \) is the Levi-Civita connection of \( g \).

For the following criterion for self-adjointness of the operator \( \Box \) see Cheng-Yau [CY] or Li [L1],[L2].

**Proposition 2.1** Let \( M \) be a closed orientable Riemannian manifold with symmetric tensor \( \phi = \sum_{i,j} \phi_{ij} \theta_i \theta_j \). Then \( \Box \) is a selfadjoint operator if and only if

\[
\sum_{j=1}^{n} \phi_{ij,j} = 0, \quad 1 \leq i \leq n.
\]  

(2.3)

Here \( \phi_{ij,k} \) is the derivative of the tensor \( \phi_{ij} \) in the direction \( e_k \).

**Remark 2.1** We call \( \Box \) the Cheng-Yau operator. It was introduced by S.Y. Cheng and S.T. Yau in 1977 [CY]. If \( \phi = \sum_{i,j} \phi_{ij} \theta_i \theta_j \) satisfies the Cheng-Yau condition (2.3), then

\[
\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (\phi_{ij} f_i)_j = \text{div}(\phi \nabla f).
\]

Let \( x : M \to N^{n+1}(k) \) be an \( n \)-dimensional closed hypersurface in an \( (n + 1) \)-dimensional space form of constant sectional curvature \( k \). Let \( (h_{ij}) \) be the components of the second fundamental form of \( M \). We recall the Reilly operator, which is a second order differential operator \( L_r : C^\infty(M) \to C^\infty(M) \) defined by

\[
L_r f = \sum_{i,j} T_{ij}^r f_{ij}, \quad f \in C^\infty(M),
\]  

(2.4)

where \( T_{ij}^r \) is given by

\[
T_{ij}^0 = \delta_{ij}, \quad T_{ij}^r = K_r \delta_{ij} - \sum_k h_{ik} T_{kj}^{r-1}, \quad r = 1, 2, \ldots, n.
\]  

(2.5)

(See Reilly [Re], Rosenberg [Ro] or Barbosa-Colares [BC].)

Denote the \( r \)-th mean curvature of \( M \) by

\[
K_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad B = (h_{ij}) = (k_i \delta_{ij}).
\]  

(2.6)

We note that the Gauss-Kronecker curvature of \( M \) is \( G \equiv K_n \).

**Definition 2.1** ([Re]) The \( r \)-th Newton transformation, \( r \in \{0, 1, \ldots, n\} \) is the linear transformation

\[
T_r = K_r I - K_{r-1} B + \cdots + (-1)^r B^r,
\]  

(2.7)

i.e.,

\[
T_{ij}^r = K_r \delta_{ij} - K_{r-1} h_{ij} + \cdots + (-1)^r \sum_{j_1, \ldots, j_r} h_{ij_1} h_{j_1 j_2} \cdots h_{j_r j_r}.
\]  

(2.7)′
If \( I \equiv i_1, \ldots, i_q \) and \( J \equiv j_1, \ldots, j_q \) are multi-indices of integers between 1 and \( n \), define

\[
\delta_J^I = \begin{cases} 
1, & \text{if } i_1, \ldots, i_q \text{ are distinct and } J \text{ is an even permutation of } I \\
-1, & \text{if } i_1, \ldots, i_q \text{ are distinct and } J \text{ is an odd permutation of } I \\
0, & \text{otherwise}
\end{cases}
\]

Then we have (see Reilly [Re])

\[
K_r = \frac{1}{r!} \sum \delta_{i_1 \cdots i_r}^{j_1 \cdots j_r} h_{i_1 j_1} \cdots h_{i_r j_r}. \tag{2.8}
\]

**Proposition 2.2** The matrix of \( T_r \) is given by

\[
T_{ij}^r = \frac{1}{r!} \sum \delta_{i_1 \cdots i_r}^{j_1 \cdots j_r} h_{i_1 j_1} \cdots h_{i_r j_r}. \tag{2.9}
\]

**Proposition 2.3** For each \( r \), we have

1. \( \text{div} T_r = \sum_j T_{ij,j}^r = 0 \),
2. Newton’s formula: \( \text{trace}(BT_r) = (r + 1)K_{r+1} \),
3. \( \text{trace}(T_r) = (n - r)K_r \)

**Proposition 2.4** Let \( \bar{x} : M \to N^{n+1}(k) \) be an \( n \)-dimensional hypersurface with unit normal vector field \( \vec{n} \). Then we have

\[
x_i = e_i, \quad \vec{n}_i = -\sum_j h_{ij} e_j, \quad x_{ij} = h_{ij} \vec{n} - kx \delta_{ij}. \tag{2.10}
\]

\[
L_r \bar{x} = (r + 1)K_{r+1} \vec{n} - (n - r)kK_r \bar{x}, \tag{2.11}
\]

**Proof.** Let \( \vec{a} \) be a fixed vector in \( L_n(k) \). Write

\[
f = \vec{n} \cdot \vec{a}, \quad g = \bar{x} \cdot \vec{a}. \tag{2.12}
\]

Then (2.11) is equivalent to

\[
L_r g = (r + 1)K_{r+1} f - (n - r)kK_r g. \tag{2.11}'
\]

Choosing an orthonormal frame \( \{e_1, \ldots, e_n, \vec{n}\} \) and their dual frame \( \{\theta_1, \ldots, \theta_n, \theta_{n+1}\} \) along \( M \) in \( N^{n+1}(k) \), we have the structure equations

\[
dx = \sum_i \theta_i e_i, \quad de_i = \sum_j \theta_{ij} e_j + \sum_j h_{ij} \theta_j \vec{n} - kx \theta_i, \quad d\vec{n} = -\sum_{i,j} h_{ij} \theta_j e_i. \tag{2.13}
\]
Here we have sometimes abbreviated \( \bar{x} \) as merely \( x \), for simplicity. By use of (2.13) and through a direct calculation we get

\[
g_i = e_i \cdot \bar{a}, \quad g_{ij} = fh_{ij} - kg_{ij}.
\]

(2.14)

By use of proposition 2.3 and (2.14), we get

\[
L_rg = \sum_{i,j} T_{rij}g_{ij} = \sum_{ij} T_{ij}h_{ij}f - kg \sum_{i,j} T_{ij}^r \delta_{ij} = (r + 1)K_{r+1}f - k(n - r)gK_r.
\]

Thus we have proved (2.11)', which is equivalent to (2.11).

Similarly, from definitions of \( f_i \), we get by use of (2.13)

\[
f_i = -\sum_j h_{ij} (e_j \cdot \bar{a}).
\]

(2.15)

Because \( \bar{a} \) is arbitrary, we have proved (2.10) from (2.14) and (2.15).

**Proposition 2.5** Let \( M \) be an \( n \)-dimensional oriented closed hypersurface in \((n + 1)\)-dimensional space form \( N^{n+1}(k) \). Then for any smooth functions \( f \) and \( g \) on \( M \) we have

\[
\int_M gL_{n-1}fdv = \int_M fL_{n-1}gdv, \quad \int_M L_{n-1}fdv = 0.
\]

(2.16)

**Proof.** Choosing \( r = n - 1 \) in (1) of proposition 2.3, and using the criterion from proposition 2.1, we know that the operator \( L_{n-1} \) is a self-adjoint operator. Thus we obtain (2.16).

**Proposition 2.6** Let \( M \) be an \( n \)-dimensional hypersurface in \((n + 1)\)-dimensional space form \( N^{n+1}(k) \). Then we have

\[
G\delta_{ij} - \sum_k h_{ik}T_{kj}^{n-1} = 0.
\]

(2.17)

**Proof.** Choosing \( r = n - 1 \) in (2.5) and noting that \( G = K_n \), we have

\[
T_{ij}^n = G\delta_{ij} - \sum_k h_{ik}T_{kj}^{n-1}.
\]

(2.18)

From the definition of \( T_{ij}^n \) in (2.9) and the definition of \( \delta_{i_1 \cdots i_n}^{j_1 \cdots j_n} \), we have

\[
T_{ij}^n = 0.
\]

(2.19)

Now (2.17) follows from (2.18) and (2.19).
3. Proof of Theorem 2

**Proposition 3.1** Let \( x : M \to N^{n+1}(k) \) be an \( n \)-dimensional oriented closed hypersurface in \((n+1)\)-dimensional space form \( N^{n+1}(k) \). Assume \( M \) has Gauss-Kronecker curvature \( G = K_n \) and a unit normal vector \( \vec{n} \). Then for any fixed unit vector \( \vec{a} \) in \( L_{n+1}(k) \), we have

\[
0 = (n + m) \int_M (\vec{a} \cdot \vec{n})^{m+1} Gdv - m \int_M (\vec{a} \cdot \vec{n})^{m-1} Gdv + k \int_M (\vec{a} \cdot \vec{n})^m (\vec{a} \cdot \vec{x}) K_{n-1}dv + mk \int_M (\vec{a} \cdot \vec{n})^{m-1}(\vec{a} \cdot \vec{x})^2 Gdv,
\]

(3.1)

where \( K_{n-1} \) is the \((n-1)\)-th mean curvature of \( M \).

**Proof.** Write

\[
f = q^m x, \quad q = \vec{a} \cdot \vec{n}.
\]

(3.2)

By definition of the first derivative and the second derivative of \( f \) (see (2.2)), we have

\[
f_i = (q^m)_i x + q^m x_i,
\]

(3.3)

\[
f_{ij} = (q^m)_{ij} x + (q^m)_i x_j + (q^m)_j x_i + q^m x_{ij}.
\]

(3.4)

By definition of operator \( L_{n-1} \), we have

\[
L_{n-1}(f) = xL_{n-1}(q^m) + 2 \sum_{i,j} T_{ij}^{n-1}(q^m)_i x_j + q^m L_{n-1} x.
\]

(3.5)

Let \( r = n - 1 \) in (2.11). We have

\[
L_{n-1} x = nG\vec{n} - kK_{n-1} x.
\]

(3.6)

By Proposition 2.5, (3.6), (2.10) and proposition 2.6, we get by integrating (3.5) over \( M \)

\[
0 = 2 \int_M q^m (L_{n-1} x) dv + 2 \int_M \sum_{i,j} T_{ij}^{n-1}(q^m)_i x_j dv
\]

\[
= 2 \int_M q^m (nG\vec{n} - kK_{n-1} x) dv + 2 \int_M \sum_{i,j,k} T_{ijk}^{n-1} m q^{m-1} [-h_{ik}(\vec{a} \cdot e_k) e_j] dv
\]

\[
= 2 \int_M q^m (nG\vec{n} - kK_{n-1} x) dv - 2m \int_M q^{n-1} G \sum_j (\vec{a} \cdot e_j) e_j dv
\]

\[
= 2 \int_M q^m (nG\vec{n} - kK_{n-1} x) dv - 2m \int_M q^{n-1} G[\vec{a} - (\vec{a} \cdot \vec{n})\vec{n} - k(\vec{a} \cdot \vec{x})\vec{x}] dv,
\]

(3.7)

that is, we obtain for \( m = 1, 2, 3, \cdots \)

\[
0 = (n + m) \int_M q^m G\vec{n} dv - m \int_M q^{m-1} G\vec{a} dv
\]

\[
- k \int_M q^m K_{n-1} x dv + mk \int_M q^{m-1}(x \cdot \vec{a}) G x dv.
\]

(3.8)

Taking the scalar product of \( \vec{a} \) with both sides of (3.8), we get Proposition 3.1.
Remark 3.1 Equation (3.1) was proved by Bang-Yen Chen in the case $k = 0$ by a different method.

Proof of Theorem 2 Choosing $m = 1$ in Proposition 3.1, we have

$$
(n + 1) \int_M (\vec{a} \cdot \vec{n})^2 G dv = \int_M G dv + k \int_M (\vec{a} \cdot \vec{n})(\vec{a} \cdot \vec{x}) K_{n-1} dv
$$

(3.9)

Because $M$ is a closed hypersurface in $N^{n+1}(k)$, the Gauss-Bonnet Theorem states in this case that

$$
\int_M G dv = \frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_{i} c_i k i \int_M K_{n-2} dv,
$$

(3.10)

where $\chi(M)$ is the Euler characteristic of $M$, the constants $c_i$ depend only on dimension $n$, and $K_i$ is the $i$-th mean curvature of $M$. (See p. 1105 of [So]; c.f. [C1], [C2].) Inserting (3.10) into (3.9), we have proved our Theorem 2. On the other hand, by choosing $m = 0$ in (3.1), we have

Corollary 3.1 (Bivens [Bi]) Let $x : M \to N^{n+1}(k)$ be an $n$-dimensional closed oriented hypersurface in $N^{n+1}(k)$. Then

$$
\int_M [n(\vec{a} \cdot \vec{n})G - k(\vec{a} \cdot \vec{x})K_{n-1}] dv = 0,
$$

(3.11)

where $\vec{a}$ is any fixed unit vector in the linear space $L_{n+1}(k)$, $G$ is the Gauss-Kronecker curvature of $M$ and $K_{n-1}$ is the $(n-1)$-th mean curvature of $M$.

Remark 3.2 Write $q = \vec{a} \cdot \vec{n}$, from Proposition 3.1, we have

$$
\int_M q^m G dv = \frac{m - 1}{n + m - 1} [\int_M q^{m-2} G dv - k \int_M q^{m-2}(x \cdot \vec{a})^2 G dv + \frac{k}{m - 1} \int_M q^{m-1}(x \cdot \vec{a}) K_{n-1} dv].
$$

(3.12)

By a direct calculation using (3.12), (3.10) and Corollary 3.1, we obtain

Proposition 3.2 Let $n$ and $m$ be even. Under the same hypothesis of Proposition 3.1, we have

$$
\int_M \frac{q^m G dv}{(m-1)(n-1)} = \frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_{i} c_i k i \int_M K_{n-2} dv
$$

$$
- k \int_M \frac{m-1}{n+m-1} q^{m-2} + \frac{(m-1)(n-3)}{(n+m-1)(n+m-3)} q^{m-4} + \cdots + \frac{(m-1)(n-3) \cdots 2}{(n+m-1)(n+m-3) \cdots 2} (x \cdot \vec{a})^2 G dv
$$

$$
+ k \int_M \frac{1}{n+m-1} q^{m-1} + \frac{1}{n+m-1} q^{m-3} + \cdots + \frac{1}{n+m-1} q^1 (x \cdot \vec{a}) K_{n-1} dv.
$$

(3.13)

Also, for $n$ even and $m$ odd, we have

$$
\int_M \frac{q^m G dv}{(m-1)(n-1)} = - k \int_M \frac{m-1}{n+m-1} q^{m-2} + \frac{(m-1)(n-3)}{(n+m-1)(n+m-3)} q^{m-4} + \cdots + \frac{(m-1)(n-3) \cdots 2}{(n+m-1)(n+m-3) \cdots 2} (x \cdot \vec{a})^2 G dv
$$

$$
+ k \int_M \frac{1}{n+m-1} q^{m-1} + \frac{1}{n+m-1} q^{m-3} + \cdots + \frac{1}{n+m-1} q^1 (x \cdot \vec{a}) K_{n-1} dv.
$$

(3.14)
Note: In the case $k = 0$, Proposition 3.2 was proved by Bang-Yen Chen; see Theorem 2 in [Ch].

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