CURVATURE PINCHING FOR MINIMAL
SUBMANIFOLDS IN A SPHERE

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Abstract

By use of the modified Simon's method, a pinching theorem for minimal submanifolds in a sphere is obtained. Our theorem is the improvement of Simon's theorem and Shen Yibing's theorem.

AMS Mathematics Subject Classification (1991): 53C40, 53C20

Key words and phrases: pinching theorem, minimal submanifolds.

1. Introduction

Let $M^n$ be an $n$-dimensional compact minimal submanifold in a Euclidean unit sphere $S^{n+p}$ of dimension $n+p$. It is well known, that if the length square $||\sigma||^2$ of the second fundamental form of $M^n$ satisfies

$$||\sigma||^2 \leq n/(2 - \frac{1}{p})$$

everywhere, then either $||\sigma||^2 = 0$ (i.e. $M^n$ is totally geodesic) or $||\sigma||^2 = n/(2 - \frac{1}{p})$. In the latter case $M^n$ is either a Clifford hypersurface or a Veronese surface in $S^4$ (15, 2)). In [7], Shen Yibing modified Simon's method and improved the above Simon's pinching constant. In fact, he obtained

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This research has been supported by the National Science Foundation of China.
Theorem A (Theorem 1.1 of [7]). Let $M^n$ be an $n$-dimensional compact minimal submanifold in $S^{n+p}$ with

$$|\sigma|^2 \leq n/(1 + \sqrt{\frac{n-1}{2n}}).$$

Then $M^n$ is either a totally geodesic submanifold or a Veronese in $S^4$.

In this paper, we greatly improve the above Theorem A. In fact, we establish the following result:

Theorem 1. Let $M^n$ be an $(n \geq 2)$-dimensional compact minimal submanifold in $S^{n+p}$ with

$$|\sigma|^2 \leq \frac{n(3n-2)}{5n-4}$$

Then $M^n$ is either a totally geodesic submanifold or a Veronese surface in $S^4$.

Remark 1. Our pinching constant $\frac{3n-2}{5n-4}$ is independent of the codimension $p$ of $M^n$ and is not less than Simons' constant $n/(2 - \frac{1}{n})$ in the case $p \geq 2 + 2/n$ (i.e. $n = 2, p \geq 2, n \geq 3, p \geq 3$). Our pinching constant is not less than Shen Yibing's pinching constant $n/(1 + \sqrt{\frac{n-1}{2n}})$.

Remark 2. In [4], Saaki, M. studied the same problem. By Gauss equation his condition of Theorem 1 in [4] is equivalent to $|\sigma|^2 \leq n/(2 - \frac{\sqrt{2n}}{5})$. Obviously our condition (3) is better than his.

Remark 3. When $n = 2$, our Theorem 1, Theorem A and Saaki's Theorem all have the same result. In this case, the condition of the Theorem is equivalent to the Gauss curvature $K \geq \frac{1}{3}$. This result has been proved by Benio-Rotho-Senn-Mu-Simon ([1]) and Roh [3] independently.

In [3] and [6], it was proved that if $M^n$ is a compact minimal submanifold of $S^{n+p}$ with a sectional curvature $\geq \min\left(\frac{1}{2n-1}, \frac{1}{2(n+1)}\right)$, then $M^n$ is totally geodesic. A further problem is as follows: Can we find a pinching constant $c_1$ depending on the dimension $n$ and the codimension $p$ such that $c_1 \leq \min\left(\frac{1}{2n-1}, \frac{1}{2(n+1)}\right)$? By use of our Theorem 1, we give a partial answer to this problem.

Theorem 2. Let $M^n$ be a compact minimal submanifold in $S^{n+p}$ with the constant scalar curvature. If the sectional curvature of $M^n$ is not less than
\[ \frac{1}{2} - \frac{3(n-2)}{5p(n-4)} \text{ everywhere, then } M^n \text{ is either a totally geodesic submanifold or a Verneuse surface in } S^4. \]

**Remark 4.** It is clear that the pinching constant \( \frac{1}{2} - \frac{3(n-2)}{5p(n-4)} \leq \min\left( \frac{1}{2(n-2)}, \frac{n}{2(n+1)} \right) \) if \( 3 - 2/n \leq p \leq \frac{2(n-2)(n+1)}{5(n-4)} \).

**Remark 5.** Theorem 2 has improved Proposition 4.1 of [7].

2. Preliminaries

In this paper, we shall use the same notation as in [7]. Let \( M^n \) be an \( n \)-dimensional compact minimal submanifold in a unit sphere \( S^{n+p} \) of the dimension \( n + p \). We choose a local field of orthonormal frames \( e_1, \ldots, e_{n+p} \) in \( S^{n+p} \) in such a way that, when restricted to \( M^n \), \( e_1, \ldots, e_n \) are tangent to \( M^n \). Let \( \omega^1, \ldots, \omega^{n+p} \) be their dual coframes. The following convention on the range of indices will be used

\[ 1 \leq A, B, C, \ldots \leq n + p; \quad 1 \leq i, j, k, \ldots \leq n; \]

\[ n + 1 \leq \alpha, \beta, \ldots \leq n + p. \]

The second fundamental form of \( M^n \) in \( S^{n+p} \) is

\[
\sigma = \sum_{i,j} h_{ij} \omega^i \epsilon_j^\alpha \epsilon_\alpha
\]

and

\[
||\sigma||^2 = \sum_{i,j} (h_{ij})^2
\]

is the square of the length of \( \sigma \).

Let

\[ UM = \bigcup_{s \in M} UM_s \quad \text{and} \quad UM_s = \{ u \in TM_s : ||u|| = 1 \}. \]

Thus \( UM \to M \) is the unit tangent bundle over \( M^n \). We define \( \sigma \)-function \( f : UM \to R \) by

\[ f(u) = ||\sigma(u, u)||^2 = \sum_{i,j} (h_{ij} u^i u^j)^2, \]
for $x = \sum v_i e_i \in U M$. Since $U M$ is compact, $f$ attain its maximum at the vector in $U M$. Suppose that this vector is $v = \sum v_i e_i \in U M_{x_0}$, $x_0 \in M^n$. Assume $e_i = v$ at $x_0$ and

$$b_{ij} = \sum_\alpha h_{1i}^\alpha h_{1j}^\alpha.$$  

From the maximum conditions of the function $f$ at $v$, we have at $x_0$ (see Lemma 1.1 of [7]):

$$f(v) = b_{11} = \max_{u \in U M} \{||\sigma(u, u)||^2\},$$

$$\sum_\alpha (h_{1i}^\alpha)^2 + \sum_\alpha h_{1j}^\alpha h_{1i}^{\alpha^T} \leq 0,$$

$$b_{ij} = 0 \quad (i \neq j),$$

$$2 \sum_\alpha (h_{ik}^\alpha)^2 + b_{ik} - f(v) \leq 0 \quad (k \neq 1).$$

3. Proofs

Proof of Theorem 1. All the calculations below will be made out at the point $x_0$. Summing up over $i$ in (6), by the Ricci identity and (7), we get (see (2.3) of [7]):

$$0 \geq n f(v) + 2 \sum_{o, \alpha \neq 1} b_{o\alpha} (h_{o\alpha}^\alpha)^2 - 2 f(v) \sum_{o, \alpha \neq 1} (h_{o\alpha}^\alpha)^2 - \sum_{o, \alpha \neq 1} b_{o\alpha} (h_{o\alpha}^\alpha)^2,$$

which is the fundamental inequality used below. The key problem is how to estimate from below the second term on the right-hand side: $2 \sum_{o, \alpha \neq 1} b_{o\alpha} (h_{o\alpha}^\alpha)^2$. We obtain a better result because we give a better lower bound.

In [7], Shen Yibing gave the following estimation $2 \sum_{o, \alpha \neq 1} b_{o\alpha} (h_{o\alpha}^\alpha)^2$ (see (2.7) of [7]):

$$2 \sum_{o, \alpha \neq 1} b_{o\alpha} (h_{o\alpha}^\alpha)^2 \geq -\frac{1}{d} f(v) \sum_{o, \alpha \neq 1} (h_{o\alpha}^\alpha)^2 - a f(v) \sum_{o, \alpha \neq 1} (h_{o\alpha}^\alpha)^2,$$

where $a > 0$ is an arbitrary real number.
On the other hand, by $b_{\alpha \beta} \leq f(\nu) \sum_\alpha (h_{\alpha \nu}^2) \leq f(\nu)^2$, we have $(f(\nu) + b_{\alpha \beta})(f(\nu) - b_{\alpha \beta}) \geq 0$. Combining this with (8), we get $f(\nu) + b_{\alpha \beta} \geq 0$, i.e. $b_{\alpha \beta} \geq -f(\nu)$. Therefore we have the following estimation:

$$2 \sum_{\alpha, \nu \neq \lambda} b_{\alpha \nu} (h_{\nu \lambda}^2) \geq -2f(\nu) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2).$$

(11)

Combining (10) with (11), we obtain

$$2 \sum_{\alpha, \nu \neq \lambda} b_{\alpha \nu}^2 (h_{\nu \lambda}^2) = b \sum_{\alpha, \nu \neq \lambda} b_{\alpha \nu} (h_{\nu \lambda}^2) + (2 - b) \sum_{\alpha, \nu \neq \lambda} b_{\alpha \nu} (h_{\nu \lambda}^2) \geq -\frac{b f(\nu)}{2a} \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - \frac{ab f(\nu)}{2} \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - (2 - b) f(\nu) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2,$$

(12)

where $a > 0$ and $0 \leq b \leq 2$.

Substituting (12) into (9) and using $b_{\alpha \nu} \leq f(\nu) \sum_\alpha (h_{\nu \lambda}^2)$, we get

$$0 \geq n f(\nu) - (1 + \frac{ab}{2}) f(\nu) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - \frac{b}{2a} \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - f(\nu) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2.$$

(13)

(5) implies that

$$\frac{b}{2an} \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 \leq \frac{n(n-1)b}{2an} \sum_{\alpha} (h_{\nu \lambda}^2)^2,$$

(14)

therefore we have from (13)

$$0 \geq n f(\nu) - (4 - b + \frac{ab}{2}) f(\nu) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - \frac{b}{2a} \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - f(\nu) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 =$$

$$= f(\nu)(n - (4 - b + \frac{ab}{2}) \sum_{\alpha, \nu \neq \lambda} (h_{\nu \lambda}^2)^2 - (1 + \frac{n(n-1)b}{2an}) \sum_{\alpha} (h_{\nu \lambda}^2)^2).$$

(15)
Let \( 4b + ab/2 = 2(1 - (n - 1)b/2an) \), i.e. \( b = \frac{4an}{(3n-2)-(n-1)^2} \). Noting
\[
||\sigma||^2 = \sum_{\alpha,\beta}(h^\alpha_\beta)^2 \geq \sum_{\alpha}(h^\alpha_\alpha)^2 + 2 \sum_{\alpha,\beta \neq \alpha}(h^\alpha_\beta)^2,
\]
we obtain from (15)
\[
0 \geq f(\nu)[n - (1 + \frac{(n - 1)b}{2an}) ||\sigma||^2(x_0)]
\]
(16)
\[
= f(\nu)[n - (1 + \frac{2(n - 1)}{(3n - 2) - n(n - 1)^2}) ||\sigma||^2(x_0)].
\]
Let \( a = 1 \), then (16) becomes
\[
0 \geq f(\nu)[n - \frac{5n - 4}{3n - 2} ||\sigma||^2(x_0)].
\]
(17)
Thus, it follows from (3) and (17) that either \( f(\nu) = 0 \) or
\[
||\nu||^2(x_0) = \frac{n(3n - 2)}{5n - 4}.
\]
(18)
If \( f(\nu) = 0 \), then \( ||\sigma(u, \nu)||^2 = 0 \) for any \( u \in UM \), so that \( M^u \) is totally geodesic. If \( f(\nu) \neq 0 \), then (18) holds, so that (9)-(17) are all equalities in the case \( a = 1 \) and \( b = -\frac{4an}{3n-2} \). Making the same discussion as in [7], we conclude that \( n = 2 \) and Theorem 2 follows directly from Theorem B of [1].

Theorem 1 is proved completely. \( \Box \)

**Proof of Theorem 2.** Let \( K_M \) be the infimum of the sectional curvatures of \( M^u \). It is easy to see that (cf. [6])
\[
2 \sum_{\alpha,\beta \neq \alpha}(h^\alpha_\beta)^2(k^{\alpha}_\beta R_{\alpha\beta} + h^\alpha_\beta R_{\alpha\beta}) \geq 2nK_M ||\sigma||^2.
\]
(19)
By Lemma 1.2 in [7] (i.e. (1.19) in [7]), we have
\[
0 \geq \int_{M^n} ||\sigma||^2(2nK_M - \frac{1}{p} ||\sigma||^2 - n) dv.
\]
(20)
Since the scalar curvature of \( M^u \) is a constant, then \( ||\sigma||^2 \) is const. Now we assume that
\[
||\sigma||^2 > \frac{n(3n - 2)}{5n - 4}.
\]
(21)
Substituting (21) into (20) and using the condition of Theorem 2, we have

\[ 0 > \int_{M^*} \|\sigma\|^2 \left( \frac{3n-2}{2p(5n-4)} \right) dv > 0. \]

This contradiction implies that (21) is false. Therefore, \[ \|\sigma\|^2 \leq \frac{n(n-2)}{m-4}, \] so that Theorem 2 follows from Theorem 1. \( \Box \)

Acknowledgment. The author has done this work during his stay as a visiting scholar at the Department of Mathematics of the University of Sarajevo. He expresses his thanks to Prof. dr. Avdispahić for his encouragement and many helpful comments and suggestions.

References


REZIME

GRANIČNA KRIVINA ZA MINIMALNU PODMNOGOSTROŠKOST NA SFERI

U radu je dokazana granična teorema za minimalnu podmnogostrukost na sferi korišćenjem modifikovanog Simonovog metoda. Ova teorema je poboljšanje Simonove teoreme i Shen Yilingove teoreme.

Received by the editors February 28, 1991