SECOND EIGENVALUE OF A JACOBI OPERATOR
OF HYPERSURFACES
WITH CONSTANT SCALAR CURVATURE

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ABSTRACT. Let $x : M \to S^{n+1}(1)$ be an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, in a unit sphere $S^{n+1}(1)$, $n \geq 5$, and let $J_s$ be the Jacobi operator of $M$. In 2004, L. J. Alías, A. Brasil and L. A. M. Sousa studied the first eigenvalue of $J_s$ of the hypersurface with constant scalar curvature $n(n-1)$ in $S^{n+1}(1)$, $n \geq 3$. In 2008, Q.-M. Cheng studied the first eigenvalue of the Jacobi operator $J_s$ of the hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. In this paper, we study the second eigenvalue of the Jacobi operator $J_s$ of $M$ and give an optimal upper bound for the second eigenvalue of $J_s$.

1. Introduction

Let $M$ be an $n$-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$. We denote the components of the second fundamental form of $M$ by $h_{ij}$, and denote the principal curvatures of $M$ by $k_1, \ldots, k_n$. Let $H$, $H_2$ and $H_3$ denote the mean curvature, the 2nd mean curvature and the 3rd mean curvature of $M$ respectively, namely,

$$H = \frac{1}{n} \sum_{i=1}^{n} k_i, \quad H_2 = \frac{2}{n(n-1)} \sum_{1 \leq i_1 < i_2 \leq n} k_{i_1} k_{i_2},$$

$$H_3 = \frac{6}{n(n-1)(n-2)} \sum_{1 \leq i_1 < i_2 < i_3 \leq n} k_{i_1} k_{i_2} k_{i_3}.$$

We denote the square norm of the second fundamental form of $M$ by $S$. A Schrödinger operator $J_m = -\Delta - S - n$, where $\Delta$ stands for the Laplace-Beltrami operator, is called a Jacobi operator. Its spectral behavior is directly related to the
instability of both minimal hypersurfaces and hypersurfaces with constant mean curvature in $S^{n+1}(1)$ (cf. [19] and [3]). The first eigenvalue of the Jacobi operator $J_n$ of such hypersurfaces in $S^{n+1}(1)$ was studied by Simons [19] and Wu [22]. The second eigenvalue of the Jacobi operator $J_m$ of the compact hypersurfaces in $S^{n+1}(1)$ was studied by A. El Soufi and S. Ilias in [20]. They obtained that if $M$ is an $n$-dimensional compact hypersurface in $S^{n+1}(1)$, then the second eigenvalue $\lambda^J_2$ of the Jacobi operator $J_m$ satisfies $\lambda^J_2 \leq 0$, where the equality holds if and only if $M$ is a totally umbilical hypersurface in $S^{n+1}(1)$.

For any $C^2$-function $f$ on $M$, we define a differential operator

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij})f_{ij},$$

where $(f_{ij})$ is the Hessian of $f$. The differential operator $\Box$ is self-adjoint and was introduced by S. Y. Cheng and Yau in [8] in order to study the compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. They proved that if $M$ is an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, and if the sectional curvature of $M$ is non-negative, then $M$ is either a totally umbilical hypersurface $S^n(c)$ or a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, where $S^k(c)$ denotes a sphere of radius $c$. In [12], the first author proved that if $M$ is an $n$-dimensional $(n \geq 3)$ compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, and if $S \leq (n-1)^2 + n\sqrt{1-c^2}$, then $M$ is either a totally umbilical hypersurface or a Riemannian product $S^1(c) \times S^{n-1}(\sqrt{1-c^2})$ with $0 < 1 - c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$. Furthermore, the Riemannian product $S^1(c) \times S^{n-1}(\sqrt{1-c^2})$ has been characterized in [5] and [6].

In [11], Alencar, do Carmo and Colares studied the stability of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. In this case, the Jacobi operator $J_s$ is given by (cf. [11] and [17])

$$J_s = -\Box - \{n(n-1)H + nHS - f_3\},$$

which is associated with the variational characterization of the hypersurfaces with constant scalar curvature in $S^{n+1}(1)$, where $f_3 = \sum_{j=1}^{n} k_j^3$ (cf. [17] and [18]). The spectral behavior of $J_s$ is directly related to the instability of hypersurfaces with constant scalar curvature.

In general, $J_s$ is not an elliptic operator. When $r > 1$, we have $n^2H^2 > S > 0$ and $J_s$ is an elliptic operator (cf. pp. 3310–3311 in [17]). When $r = 1$, if we assume that $H_3 \neq 0$ on $M$, then we have $H \neq 0$ and $J_s$ is an elliptic operator (cf. Proposition 1.5 in [11]).

**Definition 1.1.** We call $\lambda^J_s$ an eigenvalue of $J_s$ if there exists a non-zero function $f$ on $M$ such that $J_s f = \lambda^J_s f$, we call $\lambda^\Box$ an eigenvalue of $\Box$ if there exists a non-zero function $f$ on $M$ such that $\Box f + \lambda^\Box f = 0$, and we call $\lambda^\Delta$ an eigenvalue of $\Delta$ if there exists a non-zero function $f$ on $M$ such that $\Delta f + \lambda^\Delta f = 0$. 
In [7], Q.-M. Cheng studied the first eigenvalue of the Jacobi operator $J_s$ of hypersurfaces with constant scalar curvature $n(n - 1)r$, $r > 1$, in $S^{n+1}(1)$ and derived an optimal upper bound for the first eigenvalue of $J_s$.

**Theorem 1.2** (see Corollary 1.2 in [7]). Let $M$ be an $n$-dimensional compact orientable hypersurface with constant scalar curvature $n(n - 1)r$, $r > 1$, in $S^{n+1}(1)$. Then the Jacobi operator $J_s$ is elliptic and the first eigenvalue of $J_s$ satisfies

$$\lambda_{1s}^J \leq -n(n - 1)r\sqrt{-1},$$

where the equality holds if and only if $M$ is totally umbilical and non-totally geodesic.

In [2], L. J. Alías, A. Brasil and L. A. M. Sousa studied the first eigenvalue $\lambda_{1s}^J$ of the Jacobi operator $J_s$ of hypersurfaces with constant scalar curvature $n(n - 1)$ in $S^{n+1}(1)$, $n \geq 3$.

**Theorem 1.3** (see Theorem 2 in [2]). Let $M$ be an $n$-dimensional compact orientable hypersurface with constant scalar curvature $n(n - 1)$, in $S^{n+1}(1)$, $n \geq 3$. Assume that $H_3 \neq 0$. Then the Jacobi operator $J_s$ is elliptic and the first eigenvalue $\lambda_{1s}^J$ of the Jacobi operator $J_s$ satisfies

$$\lambda_{1s}^J \leq -2n(n - 1)\min|H|,$$

where the equality holds if and only if $M$ is the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ with $1 \leq m \leq n - 2$, $c = \sqrt{(n-1)m \pm \sqrt{(n-1)m(n-m)}}/n(n-1)$.

In this paper, we study the second eigenvalue of the Jacobi operator $J_s$ of the hypersurfaces with constant scalar curvature $n(n - 1)r$, $r > 1$, in $S^{n+1}(1)$, $n \geq 5$, and we have the following results.

**Theorem 1.4.** Let $M$ be an $n$-dimensional compact orientable hypersurface with constant scalar curvature $n(n - 1)r$, $r > 1$, in $S^{n+1}(1)$, $n \geq 5$. Then, the Jacobi operator $J_s$ is elliptic and the second eigenvalue $\lambda_{2s}^J$ of the Jacobi operator $J_s$ satisfies

$$\lambda_{2s}^J \leq 0,$$

where the equality holds if and only if $M$ is totally umbilical and non-totally geodesic.

**Theorem 1.5.** Let $M$ be an $n$-dimensional compact orientable hypersurface with constant scalar curvature $n(n - 1)$, in $S^{n+1}(1)$, $n \geq 5$. Assume that $H_3 \neq 0$. Then the Jacobi operator $J_s$ is elliptic and the second eigenvalue $\lambda_{2s}^J$ of the Jacobi operator $J_s$ satisfies

$$\lambda_{2s}^J \leq -\frac{n(n - 1)(n - 2)}{2}\min|H_3|,$$

where the equality holds if and only if $H_3 = \text{constant} \neq 0$ and the position functions of $M$ in $S^{n+1}(1)$ are the second eigenfunctions of $J_s$ corresponding to $\lambda_{2s}^J$. In particular, when $M$ is the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n - 2$, $c = \sqrt{(n-1)m \pm \sqrt{(n-1)m(n-m)}}/n(n-1)$, the equality in (1.3) is attained.
2. Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let \( x : M \rightarrow S^{n+1}(1) \) be an \( n \)-dimensional hypersurface in a unit sphere \( S^{n+1}(1) \). We make the following convention on the range of indices:

\[ 1 \leq i, j, k, l \leq n. \]

We choose a local orthonormal frame \( \{ e_1, \cdots, e_n, e_{n+1} \} \) and the dual coframe \( \{ \omega_1, \cdots, \omega_n, \omega_{n+1} \} \) such that when restricted on \( M \), \( \{ e_1, \cdots, e_n \} \) is a local orthonormal frame on \( M \). Hence we have \( \omega_{n+1} = 0 \) on \( M \) and we have the following structure equations (see [4], [9], [12] and [19]):

\[ dx = \sum_i \omega_i e_i, \]

\[ de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij} \omega_j e_{n+1} - \omega_i x, \]

\[ de_{n+1} = -\sum_{i,j} h_{ij} \omega_j e_i, \]

where \( h_{ij} \) denote the components of the second fundamental form of \( M \).

The Gauss equations are (see [9], [12])

\[ R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + h_{ik} h_{jl} - h_{il} h_{jk}, \]

\[ R_{ik} = (n-1) \delta_{ik} + nh_{ik} - \sum_j h_{ij} h_{jk}, \]

\[ R = n(n-1)r = n(n-1) + n^2 H^2 - S, \]

where \( R \) is the scalar curvature of \( M \), \( r \) is the normalized scalar curvature of \( M \), \( S = \sum_{i,j} h_{ij}^2 \) is the norm square of the second fundamental form, and \( H = \frac{1}{n} \sum_i h_{ii} \) is the mean curvature of \( M \).

The Codazzi equations are given by (see [9], [12])

\[ h_{ijk} = h_{ikj}. \]

Let \( f \) be a smooth function on \( M \). We define its gradient and Hessian by (see [9], [12])

\[ df = \sum_{i=1}^n f_i \omega_i, \]

\[ \sum_{j=1}^n f_{ij} \omega_j = df_i + \sum_{j=1}^n f_j \omega_{ji}. \]

Thus, the Jacobi operator \( J_s \) (see [12]) is defined by

\[ J_s f = -\Box f - \{ n(n-1)H + nHS - f_3 \} f \]

\[ = -\sum_{i,j} (nH \delta_{ij} - h_{ij}) f_{ij} - \{ n(n-1)H + nHS - f_3 \} f. \]
3. Some examples and some lemmas

First of all, we consider the first and second eigenvalues of the Jacobi operator $J_s$ of the totally umbilical and non-totally geodesic hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, $r > 1$, and the Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(1-c^2)$, $1 \leq m \leq n-2$, with constant scalar curvature $n(n-1)$ in $\mathbb{S}^{n+1}(1)$, $n \geq 3$.

**Example 3.1.** Let $M$ be a totally umbilical and non-totally geodesic hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $\mathbb{S}^{n+1}(1)$. We can assume $H > 0$. In this case, $\Box = (n-1)H\Delta$, and from $S = nH^2$ and the Gauss equation (2.6) we have $H = \sqrt{r-1}$. By (1.2) we have

$$J_s = -\Box - \{n(n-1)H + nHS - f_3\} = -\{(n-1)H\Delta + n(n-1)H(1 + H^2)\};$$

hence the eigenvalues $\lambda_{J_s}$ of $J_s$ are given by

$$\lambda_{J_s}^1 = (n-1)H\lambda_{\Delta}^1 - n(n-1)H(1 + H^2),$$

where $\lambda_{\Delta}^1$ denotes the eigenvalue of $\Delta$ (see Definition 1.1). It is well-known that $\lambda_{\Delta}^1 = 0$, $\lambda_{\Delta}^2 = nr = n(1 + H^2)$. Hence we have

$$\lambda_{J_s}^1 = -n(n-1)H(1 + H^2) = -n(n-1)r\sqrt{r-1} < 0,$$

(3.1)

$$\lambda_{J_s}^2 = (n-1)H \cdot n(1 + H^2) - n(n-1)H(1 + H^2) = 0.$$  

**Example 3.2.** Let $M$ be the Riemannian product

$$\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(1-c^2), 1 \leq m \leq n-2, c = \sqrt{(n-1)m + \sqrt{(n-1)m(n-m)}}$$

in $\mathbb{S}^{n+1}(1)$, $n \geq 3$. In this case, the position vector is

$$x = (x_1, x_2) \in \mathbb{S}^m(c) \times \mathbb{S}^{n-m}(1-c^2)$$

and the unit normal vector at this point $x$ is given by $e_{n+1} = (\sqrt{1-c^2}x_1, -\frac{c}{\sqrt{1-c^2}}x_2)$. Its principal curvatures are given by

$$k_1 = \cdots = k_m = -\frac{\sqrt{1-c^2}}{c}, k_{m+1} = \cdots = k_n = \frac{c}{\sqrt{1-c^2}}.$$  

(3.2)

Hence $H, S, f_3$ are given by

$$H = \frac{nc^2 - m}{cn\sqrt{1-c^2}},$$

(3.3)

$$S = \frac{m(1-c^2)}{c^2} + \frac{(n-m)c^2}{1-c^2} = n^2H^2,$$

$$f_3 = -\frac{m(1-c^3)^{3/2}}{c^3} + \frac{(n-m)c^3}{(1-c^2)^{3/2}}.$$  

After a long but straightforward computation, we know that $M$ has constant scalar curvature $n(n-1)$ and

$$H_3 = -\frac{2H}{n-2} = -\frac{2(nc^2 - m)}{cn(n-2)\sqrt{1-c^2}} < 0;$$  

(3.4)
hence the Jacobi operator $J_s$ is elliptic (cf. Proposition 1.5 in [11]). We also have

$$(3.5) \quad n(n - 1)H + nHS - f_3 = \frac{(n - 2m)(n - 1)c^4 + 2m(m - 1)c^2 - m(m - 1)}{c^3(1 - c^2)^{3/2}}.$$ 

Thus the Jacobi operator $J_s = -\Box - \{n(n - 1)H + nHS - f_3\}$ becomes

$$(3.6) \quad J_s = -\Box - \frac{(n - 2m)(n - 1)c^4 + 2m(m - 1)c^2 - m(m - 1)}{c^3(1 - c^2)^{3/2}},$$

and hence the eigenvalues $\lambda^J_i$ of $J_s$ are given by

$$(3.7) \quad \lambda^J_i = \lambda_i^\Box - \frac{(n - 2m)(n - 1)c^4 + 2m(m - 1)c^2 - m(m - 1)}{c^3(1 - c^2)^{3/2}},$$

where $\lambda_i^\Box$ denote the eigenvalues of the differential operator $\Box$ (see Definition [11]).

Since the differential operator $\Box$ is self-adjoint and $M$ is compact, we have $\lambda_1^\Box = 0$ and its corresponding eigenfunctions are non-zero constant functions; hence

$$(3.8) \quad \lambda^J_1 = -\frac{(n - 2m)(n - 1)c^4 + 2m(m - 1)c^2 - m(m - 1)}{c^3(1 - c^2)^{3/2}}.$$ 

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal basis of $TM$ with dual basis $\{\omega_1, \cdots, \omega_n\}$ such that $\{e_1, \cdots, e_m\}$ is a local orthonormal basis of $TS^m(c)$ when restricted on $S^m(c)$ and $\{e_{m+1}, \cdots, e_n\}$ is a local orthonormal basis of $TS^{n-m}(\sqrt{1 - c^2})$ when restricted on $S^{n-m}(\sqrt{1 - c^2})$. So we have

$$(3.9) \quad \Box f = \sum_{i=1}^m (nH - k_1)f_{ii} + \sum_{j=m+1}^n (nH - k_n)f_{jj} = (nH - k_1)\Delta_1 f + (nH - k_n)\Delta_2 f,$$

where $\Delta_1$ and $\Delta_2$ denote the Laplacian operators on $S^m(c)$ and $S^{n-m}(\sqrt{1 - c^2})$ respectively. Since $(nH - k_1) = \frac{(n-1)c^2 - (m-1)}{c\sqrt{1 - c^2}} > 0$, $(nH - k_n) = \frac{(n-1)c^2 - m}{c\sqrt{1 - c^2}} > 0$, we conclude that

$$(3.10) \quad \lambda^\Box = \min \{ (nH - k_1)\lambda_2^{\Delta_1}, (nH - k_n)\lambda_2^{\Delta_2} \},$$

where $\lambda_2^{\Delta_1}$ and $\lambda_2^{\Delta_2}$ are the second eigenvalues (or the first non-zero eigenvalues) of $\Delta_1$ and $\Delta_2$ which are given by

$$(3.11) \quad \lambda_2^{\Delta_1} = \frac{m}{c^2}, \quad \lambda_2^{\Delta_2} = \frac{n - m}{1 - c^2}.$$ 

Therefore, from (3.7), (3.8), (3.10) and (3.11), after a direct computation, we have

$$(3.12) \quad \lambda^J_2 = \min \{ \frac{2m}{c^2}, \frac{n - m}{1 - c^2} - \lambda_1^J \}$$

$$= \min \left\{ \frac{(n - m)[(1 - n)c^2 + m]}{c(1 - c^2)^{3/2}}, \frac{-m[(n - 1)c^2 - (m - 1)]}{c^3(1 - c^2)^{3/2}} \right\}.$$
Since \( c = \sqrt{(n-1)m + \sqrt{(n-1)m(n-m)}} \), we have

\[
\frac{(n - m)[(1 - n)c^2 + m]}{c(1 - c^2)^{3/2}} - \frac{m[(n - 1)c^2 - (m - 1)]}{c^3(1 - c^2)^{1/2}}
\]

\[\tag{3.13}\]

\[- \frac{n(n - 1)c^4 + 2m(1 - n)c^2 + m(m - 1)}{c^3(1 - c^2)^{3/2}} = 0.\]

It follows from (3.12) and (3.13) that

\[
\lambda_2^J = \frac{(n - m)[(1 - n)c^2 + m]}{c(1 - c^2)^{3/2}} < 0.
\]

On the other hand, we also have

\[
- \frac{(n - 2m)(n - 1)c^4 + 2m(m - 1)c^2 - m(m - 1)}{c^3(1 - c^2)^{3/2}} + 2n(n - 1)H
\]

\[\tag{3.15}\]

\[- \frac{(2c^2 - 1)(n(n - 1)c^4 + 2m(1 - n)c^2 + m(m - 1))}{c^3(1 - c^2)^{3/2}} = 0,\]

\[\tag{3.17}\]

\[- \frac{n(n - 1)c^4 + 2m(1 - n)c^2 + m(m - 1)}{c^3(1 - c^2)^{3/2}} = 0,\]

and

\[
\frac{(n - m)[(1 - n)c^2 + m]}{c(1 - c^2)^{3/2}} - \frac{n(n - 1)(n - 2)}{2}H_3
\]

\[\tag{3.18}\]

\[- \frac{(n(n - 1)c^4 + 2m(1 - n)c^2 + m(m - 1))(c^2(2n - 1) - 2m + 1)}{c^3(1 - c^2)^{3/2}} = 0.\]

Hence, from (3.12), (3.13), (3.15), (3.16) and (3.17), we have

\[
\lambda_1^J = -2n(n - 1)H < \lambda_2^J = -n(n - 1)H = \frac{n(n - 1)(n - 2)}{2}H_3 < 0.
\]

In the following we will assume that \( x : M \to S^{n+1}(1) \) is an \( n \)-dimensional compact orientable hypersurface with constant scalar curvature \( n(n - 1)r \), \( r \geq 1 \), in \( S^{n+1}(1) \), \( n \geq 5 \), when \( r = 1 \). We assume moreover that \( H_3 \neq 0 \). When \( r > 1 \), we have \( n^2H^2 > S > 0 \). When \( r = 1 \), since \( H_3 \neq 0 \), we have \( H \neq 0 \). Hence, we can assume that \( H > 0 \) (cf. [2] and [11]).

Let \( a \) be a fixed vector in \( \mathbb{R}^{n+2} \). We define functions \( f^a : M \to \mathbb{R} \) and \( \tilde{g}^a : M \to \mathbb{R} \) by

\[
f^a = <a, x>, \quad \tilde{g}^a = <a, e_{n+1}>,
\]

where \( x \) is the position vector and \( e_{n+1} \) is the unit normal vector. By using the structure equations and the definitions of the covariant derivatives, we have the following result.
Lemma 3.3 (see [4]). The gradient and the second derivative of the functions \( f^a \) and \( \tilde{g}^a \) are given by

\[
\begin{align*}
(f^a)_{i} &= \langle a, e^i \rangle, & (f^a)_{ij} &= \tilde{g}^a_{ij} - f^a \delta_{ij}, & \tilde{g}^a_{jk} &= -\sum_{i=1}^{n} \langle a, e^i \rangle h^a_{ijk} + f^a h_{jk}, \\
\end{align*}
\]

(3.20)

Proof. By (2.1) we have

\[
\sum_{i=1}^{n} \langle a, e^i \rangle \omega^i = \sum_{i=1}^{n} \langle a, e^i \rangle = \langle a, \omega \rangle.
\]

Thus from (2.8) we have

\[
(f^a)_{i} = \langle a, e^i \rangle.
\]

(3.21)

From (2.1) and (3.21) we have

\[
\sum_{j=1}^{n} (f^a)_{ij} \omega^j = \sum_{j=1}^{n} (f^a)_{ij} \omega^j = \langle a, d x \rangle = \langle a, \omega \rangle + \sum_{j=1}^{n} \langle a, e^j \rangle \omega^j.
\]

From (2.2) and (3.21) we have

\[
\sum_{j=1}^{n} (f^a)_{ij} \omega^j = \sum_{j=1}^{n} (f^a)_{ij} \omega^j = \langle a, d x \rangle = \langle a, \omega \rangle + \sum_{j=1}^{n} \langle a, e^j \rangle \omega^j.
\]

hence we have

\[
(f^a)_{ij} = \langle a, e^{n+1} \rangle h_{ij} - \langle a, x \rangle \delta_{ij} = \tilde{g}^a_{ij} - f^a \delta_{ij}.
\]

(3.22)

After an analogous argument, we have

\[
\tilde{g}^a_{jk} = -\sum_{i=1}^{n} \langle a, e^i \rangle h^a_{ijk} - \sum_{i=1}^{n} \langle a, e^i \rangle h^a_{ijk} + f^a h_{jk}.
\]

(3.23)

We will use a technique which was introduced by Li and Yau in [13] and was later used by other authors (see [14], [16] and [21]).

Let \( B^{n+2} \) be the open unit ball in \( \mathbb{R}^{n+2} \). For each point \( g \in B^{n+2} \), we consider the map

\[
F_g(p) = \frac{n + (\mu \langle p, g \rangle + \lambda)g}{\lambda (\langle p, g \rangle + 1)}, \ \forall \ p \in S^{n+1}(1) \subset \mathbb{R}^{n+2},
\]

where \( \lambda = (1 - \|g\|^2)^{-1/2}, \mu = (\lambda - 1)\|g\|^{-2} \) and \( \langle , \rangle \) denotes the inner product on \( \mathbb{R}^{n+2} \). A direct computation (see [14], [21]) shows that \( F_g \) is a conformal transformation from \( S^{n+1}(1) \) to \( S^{n+1}(1) \) and that the differential map \( dF_g \) of \( F_g \) is given by

\[
dF_g(v) = \lambda^{-2}(\langle p, g \rangle + 1)^{-2} \{ \lambda(\langle p, g \rangle + 1)v - \lambda < v, g > p \\
+ < v, g > (1 - \lambda)\|g\|^{-2} g \},
\]

where \( v \) is a tangent vector to \( S^{n+1} \) at the point \( p \). Hence, for two vectors \( v, w \in T_p S^{n+1} \) we have (see [14], [16] and [21])

\[
< dF_g(v), dF_g(w) > = \frac{1 - \|g\|^2}{(\langle p, g \rangle + 1)^2} < v, w > .
\]
By use of the technique in Li-Yau [13], we have the following result:

**Lemma 3.4** (see [14], [16] and [21]). Let \( x : M \to S^{n+1} \) be a compact hypersurface in \( S^{n+1} \) with constant scalar curvature \( n(n-1)r, \ r \geq 1 \), and \( u \) be a positive first eigenfunction of the Jacobi operator \( J_s \) on \( M \). Then there exists \( g \in B^{n+2} \) such that \( \int_M u(F_g \circ x) dv = (0, \ldots, 0) \).

Let \( \{E^A\}_{A=1}^{n+2} \) be a fixed orthonormal basis of \( \mathbb{R}^{n+2} \). For a fixed point \( g \in B^{n+2} \), we define functions \( f^A : M \to \mathbb{R} \) by

\[
(3.25) \quad f^A = \langle E^A, F_g \circ x \rangle = \frac{\langle E^A, x \rangle + (\mu < x, g > + \lambda) < g, E^A >}{\lambda ( < x, g > + 1 )}.
\]

**Lemma 3.5.** The gradient of \( f^A \) is given by

\[
(3.26) \quad f^A_i = \frac{\langle E^A, e_i \rangle}{\lambda ( < x, g > + 1 )} + \frac{\langle g, e_i \rangle}{\lambda ( < x, g > + 1 )^2} \cdot ( - < E^A, x > + \frac{1 - \lambda}{\lambda ||g||^2} < g, E^A > )
\]

**Proof.** By applying Lemma 3.3, we have

\[
f^A_i = \frac{\langle E^A, e_i \rangle + \mu < g, e_i > < g, E^A >}{\lambda ( < x, g > + 1 )} - f^A < g, e_i > < x, g > + 1
\]

\[
= \frac{\langle E^A, e_i \rangle}{\lambda ( < x, g > + 1 )} + \frac{\langle g, e_i \rangle}{\lambda ( < x, g > + 1 )^2} [ < E^A, x > + (\mu - \lambda) < g, E^A > ]
\]

\[
= \frac{\langle E^A, e_i \rangle}{\lambda ( < x, g > + 1 )} + \frac{\langle g, e_i \rangle}{\lambda ( < x, g > + 1 )^2} ( < E^A, x > + \frac{1 - \lambda}{\lambda ||g||^2} < g, E^A > ).
\]

We also need Lemma 3.6, Lemma 3.7 and Lemma 3.8 to estimate the second eigenvalue \( \lambda_2^J \) of the Jacobi operator \( J_s \) on \( M \).

**Lemma 3.6.** Let \( M \) be an \( n \)-dimensional compact hypersurface with constant scalar curvature \( n(n-1)r, \ r \geq 1 \), in \( S^{n+1} \). Let \( f^A(1 \leq A \leq n+2) \) be the functions given by (3.25). We have

\[
(3.27) \quad \sum_{A=1}^{n+2} \int_M (J_s f^A \cdot f^A) dv = \int_M \frac{n(n-1)H(1-||g||^2)}{( < x, g > + 1 )^2} dv
\]

\[
- \int_M \left( \frac{n(n-1)}{2} (2H - (n-2)H_3 + nHH_3) \right) dv.
\]
Proof. By the divergence theorem and Lemma 3.5 we have

\begin{align*}
&\sum_{A=1}^{n+2} \int_M (\Box f^A \cdot f^A) \, dv \\
&= \sum_{A,i,j} \int_M (nH \delta_{ij} - h_{ij}) f_i^A f_j^A \, dv \\
&= \sum_{A,i,j} \int_M (nH \delta_{ij} - h_{ij}) \cdot \left[ \frac{<E^A, e_i>}{\lambda(<x, g >+1)} + \frac{<g, e_i>}{\lambda(<x, g >+1)^2} \right. \\
&\quad \cdot (-<E^A, x > + \frac{1 - \lambda}{\lambda ||g||^2} <g, E^A>) \cdot \left[ \frac{<E^A, e_j>}{\lambda(<x, g >+1)} \\
&\quad + \frac{<g, e_j>}{\lambda(<x, g >+1)^2} (-<E^A, x > + \frac{1 - \lambda}{\lambda ||g||^2} <g, E^A>) \right] \, dv \\
&= \int_M \left\{ \sum_{i,j} (nH \delta_{ij} - h_{ij}) \cdot \frac{\delta_{ij}}{\lambda^2(<x, g >+1)^2} + \frac{<g, e_i> <g, e_j>}{\lambda^2 ||g||^2(<x, g >+1)^2} \cdot \left[ 2(1 - \lambda)\lambda(<x, g >+1) + \lambda^2 ||g||^2 - 2(1 - \lambda)\lambda(<x, g >+1 - \lambda)^2 \right] \right\} \, dv \\
&= \int_M \sum_{i,j} (nH \delta_{ij} - h_{ij}) \cdot \frac{\delta_{ij}}{\lambda^2(<x, g >+1)^2} \, dv \\
&= \int_M \frac{n(n-1)H(1-||g||^2)}{\lambda^2(<x, g >+1)^2} \, dv,
\end{align*}

where we use the fact that \( \sum_{A=1}^{n+2} <E_A, X > <E_A, Y > = <X, Y > (\forall X, Y \in \mathbb{R}^{n+2}) \) in the third equality. By Newton’s formula, we have

\begin{align*}
f_3 &= n^3 H^3 + \frac{n(n-1)(n-2)}{2} H_3 - \frac{3n^2(n-1)}{2} HH_2, \\
S &= n^2 H^2 - n(n-1)H_2.
\end{align*}

Thus \( J_s \) becomes

\begin{align*}
J_s &= -\Box - \{ n(n-1)H + nH(n^2 H^2 - n(n-1)H_2) \\
&\quad - (n^3 H^3 + \frac{n(n-1)(n-2)}{2} H_3 - \frac{3n^2(n-1)}{2} HH_2) \} \\
&= -\Box - n(n-1)H - \frac{n^2(n-1)}{2} HH_2 + \frac{n(n-1)(n-2)}{2} H_3 \\
&= -\Box - \frac{n(n-1)}{2} (2H - (n-2)H_3 + nHH_2).
\end{align*}

Then by using the fact that

\begin{align*}
\sum_{A=1}^{n+2} f^A \cdot f^A &= \sum_{A=1}^{n+2} <E^A, F_g \circ x > <E^A, F_g \circ x > = <F_g \circ x, F_g \circ x > = 1,
\end{align*}

we immediately get (3.27).

\( \square \)
For a fixed point $g \in B^{n+2}$, let
\[(3.32) \quad f = \langle x, g \rangle, \quad \tilde{g} = \langle e_{n+1}, g \rangle, \quad \rho = -\ln \lambda - \ln (1 + f),\]

where $\lambda = (1 - ||g||^2)^{-1/2}$, $x$ is the position vector and $e_{n+1}$ is the unit normal vector. We have
\[(3.33) \quad e^{2\rho} = \frac{1}{\lambda^2(1 + f)^2} = \frac{1 - ||g||^2}{(< x, g > + 1)^2}, \quad \rho_i = \frac{-f_i}{1 + f}, \quad \rho_{ij} = \frac{-f_{ij}}{1 + f} + \frac{f_if_j}{(1 + f)^2}.\]

**Lemma 3.7.** Let $x : M \to \mathbb{S}^{n+1}(1)$ be an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, in $\mathbb{S}^{n+1}(1)$. When $r = 1$, we assume moreover that $H_3 \neq 0$. Then we have $H \neq 0$; hence we can assume $H > 0$. Let $\rho$ be the function defined by (3.32). We have
\[(3.34) \quad \int_M H(1 - ||g||^2) dv \leq \int_M (H + \frac{H^2}{H}) dv - \int_M [H \nabla \rho]^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij}) \rho_i \rho_j dv,
\]
and the equality holds if and only if $H_2 + \frac{\rho H}{1+r} \equiv 0$ on $M$.

**Proof.** Under the hypothesis of the lemma, we can assume $H > 0$ (cf. [7] and [2]). We have
\[(3.35) \quad \sum_{i,j} (nH\delta_{ij} - h_{ij}) \rho_i \rho_j = \sum_{i,j} (nH\delta_{ij} - h_{ij}) \frac{f_if_j}{(1 + f)^2} = \frac{nH\|\nabla f\|^2}{(1 + f)^2} - \sum_{i,j} \frac{h_{ij}f_if_j}{(1 + f)^2},\]
and
\[(3.36) \quad \Box \rho = \sum_{i,j} (nH\delta_{ij} - h_{ij}) \rho_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) \left(\frac{-f_{ij}}{1 + f} + \frac{f_if_j}{(1 + f)^2}\right) = -\frac{\Delta f nH}{1 + f} + \frac{nH\|\nabla f\|^2}{(1 + f)^2} + \sum_{i,j} \frac{h_{ij}f_if_j}{1 + f} - \sum_{i,j} \frac{h_{ij}f_if_j}{(1 + f)^2}.\]
From (3.33), (3.35) and (3.36) and by using Lemma 3.3, we have

\[
(\Box \rho - \sum_{i,j} (nH \delta_{ij} - h_{ij}) \rho_i \rho_j) \cdot \frac{2}{n(n-1)} + \frac{H(1 - \|g\|^2)}{(1 + f)^2} \]

\[
= \left(\frac{-\Delta f nH}{1 + f} + \sum_{i,j} \frac{h_{ij}f_{ij}}{1 + f}\right) \cdot \frac{2}{n(n-1)} + \frac{H(1 - \|g\|^2)}{(1 + f)^2} \]

\[
= \left(\frac{-nH(n\tilde{g} - nf)}{1 + f} + \sum_{i,j} \frac{h_{ij}(\tilde{g} h_{ij} - f \delta_{ij})}{1 + f}\right) \cdot \frac{2}{n(n-1)} + \frac{H(1 - \|g\|^2)}{(1 + f)^2} \]

\[
= \frac{2Hf - 2H_2 \tilde{g}}{1 + f} + \frac{H(1 - f^2 - \sum_i f_i^2 - \tilde{g}^2)}{(1 + f)^2} \]

\[
= H - \sum_i \frac{H f_i^2}{(1 + f)^2} - \frac{H \tilde{g}^2}{(1 + f)^2} - \frac{2H_2 \tilde{g}}{1 + f} \]

\[
= H - \sum_i \frac{H f_i^2}{(1 + f)^2} + \frac{H^2}{H} - \frac{(H_2 + \tilde{g} H \frac{\tilde{g}}{1 + f})^2}{H} \]

\[
= H + \frac{H^2}{H} - H \|\nabla \rho\|^2 - \frac{(H_2 + \tilde{g} H \frac{\tilde{g}}{1 + f})^2}{H},
\]

which immediately implies

\[
\int_M \frac{H(1 - \|g\|^2)}{(1 + f)^2} dv = \int_M [H + \frac{H^2}{H} - H \|\nabla \rho\|^2 + \frac{2}{n(n-1)} \sum_{i,j} (nH \delta_{ij} - h_{ij}) \rho_i \rho_j - \frac{(H_2 + \tilde{g} H \frac{\tilde{g}}{1 + f})^2}{H}] dv.
\]

(3.37)

Hence we get the inequality (3.34), and the equality holds if and only if \(H_2 + \tilde{g} H \frac{\tilde{g}}{1 + f} \equiv 0\) on \(M\).

Lemma 3.8. Let \(M\) be an \(n\)-dimensional compact hypersurface with constant scalar curvature \(n(n-1)r, r \geq 1, \) in \(S^{n+1}(1), n \geq 5\). When \(r = 1\), we assume moreover that \(H_3 \neq 0\). Then we have \(H \neq 0\); hence we can assume \(H > 0\). We have

\[
\int_M [H \|\nabla \rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH \delta_{ij} - h_{ij}) \rho_i \rho_j] dv \geq 0.
\]

(3.38)

Proof. Under the hypothesis of the lemma, we can assume \(H > 0\) (cf. [7] and [2]). For all \(p \in M\), let \(k_1, \ldots, k_n\) denote the principal curvatures of \(M\) at \(p\). We choose an orthonormal basis such that \(h_{ij} = \delta_{ij} k_i\). By the Gauss equation (2.4), we have

\[
n^2 H^2 - \sum_i k_i^2 = n(n-1)(r - 1) \geq 0,
\]

(3.39)

which leads to

\[
nH \geq |k_i|, \forall 1 \leq i \leq n.
\]

(3.40)
As $n \geq 5$, we have $\frac{n(n-3)}{2}H \geq nH$, so we have

$$H\|\nabla \rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j$$

$$= H\sum_i \rho_i^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - \delta_{ij}k_i)\rho_i\rho_j$$

$$= H\sum_i \rho_i^2 - \sum_i \frac{2}{n(n-1)}(nH - k_i)\rho_i^2$$

$$= \frac{2}{n(n-1)} \sum_i \rho_i^2 \left( \frac{n(n-3)}{2}H + k_i \right)$$

$$\geq \frac{2}{n(n-1)} \sum_i \rho_i^2 (nH - |k_i|) \geq 0.$$

Hence, we get that $H\|\nabla \rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j \geq 0$ holds at each point of $M$, which immediately implies (3.38).

4. Proofs of Theorem 1.4 and Theorem 1.5

**Proof of Theorem 1.4.** Since $r > 1$, we have that $\Box$ is an elliptic operator and $H \neq 0$. Hence, we can assume $H > 0$ (see [7]). Let $u$ be a first eigenfunction of $J_s$; we can assume $u$ is positive on $M$. By Lemma 3.4 there exists $g \in B^{n+2}$ such that

$$\int_M u(F_g \circ x)dv = (0, \ldots, 0),$$

which implies that the functions $\{f^A, 1 \leq A \leq n+2\}$ given by (3.25) are perpendicular to the function $u$, i.e., $\int_M u \cdot f^Adv = 0, \forall 1 \leq A \leq n+2$. Then by using the min-max characterization of eigenvalues for elliptic operators, we have

$$\lambda_{2s} \cdot \int_M (f^A \cdot f^A)dv \leq \int_M (J_s f^A \cdot f^A)dv, \forall 1 \leq A \leq n+2.$$

Summing up and using the fact that $\sum_{A=1}^{n+2} f^A \cdot f^A = 1$ (see (3.31)), we obtain

$$\lambda_{2s} \cdot Vol(M) \leq \sum_{A=1}^{n+2} \int_M (J_s f^A \cdot f^A)dv.$$

From Lemma 3.6 and (4.3) we have

$$\lambda_{2s} \cdot Vol(M) \leq \int_M \frac{n(n-1)H(1 - \|g\|^2)}{<x, g > + 1}dv$$

$$- \int_M \frac{n(n-1)}{2} (2H - (n-2)H_3 + nHH_2)dv.$$
Then by (4.4), Lemma 3.7 and Lemma 3.8, we have

$$\lambda_J^2 \cdot \text{Vol}(M) \leq n(n-1) \cdot \int_M \left( H + \frac{H_2^2}{H} \right) dv$$

$$= \int_M \frac{n(n-1)}{2}(2H - (n-2)H_3 + nHH_2) dv$$

$$= n(n-1) \cdot \int_M \left( \frac{H_2^2}{H} + \frac{n-2}{2}H_3 - \frac{nHH_2}{2} \right) dv.$$

From the definition of $H_2$ and the Gauss equation (2.6) we have

$$H_2 = r - 1 = \text{constant} > 0.$$

So we have $H_3 \leq \frac{H_2^2}{H}$ and $H_2 \leq H^2$ (see [10], p. 52), and hence

$$\lambda_J^2 \cdot \text{Vol}(M) \leq n(n-1) \cdot \int_M \left( H + \frac{H_2^2}{H} + \frac{n-2}{2}H_3 - \frac{nHH_2}{2} \right) dv$$

$$= n(n-1) \cdot \int_M \left( \frac{H_2^2}{H} + \frac{n-2}{2}H_3 - \frac{nHH_2}{2} \right) dv$$

$$= n(n-1) \cdot \int_M \frac{nH_2^2}{2} \left( \frac{H_2}{H} - H \right) dv \leq 0;$$

therefore we get $\lambda_J^2 \leq 0$. When $\lambda_J^2 = 0$, all the inequalities become equalities. When the equality in (4.7) holds, we have $H_2 \equiv H^2$ on $M$. Since $H_2$ is a positive constant, we get that $M$ is a totally umbilical and non-totally geodesic hypersurface with constant scalar curvature $n(n-1)r$. On the other hand, if $M$ is a totally umbilical and non-totally geodesic hypersurface with constant scalar curvature $n(n-1)r$, from Example 3.1, we know that $\lambda_J^2 = 0$. □

**Remark 4.1.** We notice that from (4.7) we can get a more precise upper bound for $\lambda_J^2$; that is,

$$\lambda_J^2 \leq n(n-1) \left( \frac{H_2^2}{\min H} + \frac{n-2}{2} \max H_3 - \frac{nH_2}{2} \min H \right)$$

$$= n(n-1) \left( \frac{(r-1)^2}{\min H} + \frac{n-2}{2} \max H_3 - \frac{n(r-1)}{2} \min H \right).$$

**Proof of Theorem 1.5.** Since $r = 1$, from (4.6) we have $H_2 = 0$. Since we assume that $H_3$ does not vanish on $M$, we have that $J_s$ is elliptic and the mean curvature $H$ does not vanish on $M$ (cf. Proposition 1.5 in [11]). Hence, we can assume $H > 0$. Thus $H_3 \leq \frac{H_2^2}{H} = 0$. Since we assume that $H_3 \neq 0$ on $M$, we get $H_3 < 0$. As Lemma 3.6, Lemma 3.7 and Lemma 3.8 hold for both the case $r > 1$ and the case $r = 1$, after an analogous argument with the proof of Theorem 1.4, we know that
still hold in this case. Hence we have
\[
\lambda_2 J^J \cdot Vol(M) \leq n(n-1) \cdot \int_M \left( \frac{H_2^2}{H} + \frac{n - 2}{2} H_3 - \frac{HH_2}{2} \right) dv
\]
\[
= \frac{n(n-1)(n-2)}{2} \cdot \int_M H_3 dv
\]
\[
\leq \frac{n(n-1)(n-2)}{2} \max H_3 \cdot Vol(M)
\]
\[
= -\frac{n(n-1)(n-2)}{2} \min |H_3| \cdot Vol(M).
\]

Hence, we get
\[
\lambda_2 J^J \leq -\frac{n(n-1)(n-2)}{2} \min |H_3|.
\]

When \( \lambda_2 J^J = -\frac{n(n-1)(n-2)}{2} \min |H_3| \), the inequalities in (4.3), (4.2) and (4.9) become equalities. The equality in (4.9) holds implies that \( \tilde{H} = 0 \), otherwise, we have the fact that \( \tilde{g} = 0 \) on \( M \). We claim that \( \tilde{g} \) must be 0, otherwise, we have the fact that \( H_3 \) is a hyperbolic (see Theorem 1 in [15]); hence \( M \) is totally umbilical. Since \( H_2 = 0 \), we immediately get that \( M \) is totally geodesic, which is a contradiction with \( H_3 \neq 0 \). Hence we have \( g \equiv 0 \). From (3.25) we get \( f^A = \langle E^A, F_g \circ x \rangle = \langle E^A, x \rangle \), which means that \{\( f^A, 1 \leq A \leq n + 2 \}\} are the position functions of \( x : M \rightarrow S^{n+1}(1) \). Since the equality in (4.2) holds, it follows that the position functions \{\( f^A = \langle E^A, x \rangle \), \( 1 \leq A \leq n + 2 \}\} must be the second eigenfunctions of \( J_\nu \) corresponding to \( \lambda_2 J^J \).

On the other hand, we assume that \( H_3 = \text{constant} \neq 0 \) and the position functions \{\( \tilde{f}^A = \langle E^A, x \rangle \), \( 1 \leq A \leq n + 2 \}\} are the second eigenfunctions of \( J_\nu \) corresponding to \( \lambda_2 J^J \). Since \( H_3 \neq 0 \), we have \( H \neq 0 \). Hence, we can assume \( H > 0 \), \( H_3 < 0 \) (cf. Proposition 1.5 in [11]). Since \( H_2 = 0 \), by using (1.1) and (3.20), we get
\[
\Box \tilde{f}^A = n(n-1)H_2 < E^A, e_{n+1} > -n(n-1)H \tilde{f}^A = -n(n-1)H \tilde{f}^A, \forall 1 \leq A \leq n+2.
\]

Then from (1.2) and (3.29) we have
\[
J_\nu \tilde{f}^A = (n-1)H \tilde{f}^A - \{n(n-1)H + nHS - f_3 \} \tilde{f}^A
\]
\[
= (f_3 - nHS) \tilde{f}^A
\]
\[
= \left[ (n^3 H^3 + \frac{n(n-1)(n-2)}{2} H_3 - \frac{3n^2(n-1)}{2} HH_2) - (n^3 H^3 - n^2(n-1)HH_2) \right] \tilde{f}^A
\]
\[
= \frac{n(n-1)(n-2)}{2} H_3 \tilde{f}^A, \forall 1 \leq A \leq n + 2;
\]

hence we get \( \lambda_2 J^J = \frac{n(n-1)(n-2)}{2} H_3 = -\frac{n(n-1)(n-2)}{2} \min |H_3| \).

In particular, when \( M \) is the Riemannian product \( S^m(c) \times S^{n-m}(\sqrt{1-c^2}) \), \( 1 \leq m \leq n-2 \) with \( c = \sqrt{(n-1)m + \sqrt{(n-1)m(n-m)}} \), from Example 3.3, we know that the equality in (4.10) is attained.

Remark 4.2. We notice that in the proof of Theorem 1.4, if we choose \( u \equiv 1 \), all the arguments stay true. Hence, we shall see that the totally umbilical and non-totally geodesic hypersurface is the only stable compact orientable hypersurface
with constant scalar curvature \( n(n - 1)r \), \( r > 1 \), in \( S^{n+1}(1) \), \( n \geq 5 \). Hence, we give a new proof of a stability result in [1].

**Remark 4.3.** Since Lemma 3.8 does not hold when \( n = 3 \) and \( n = 4 \), we cannot prove Theorem 1.3 and Theorem 1.4 by our technique in \( n = 3 \) and \( n = 4 \). Therefore, it is an interesting problem to study the estimate for the second eigenvalue of the Jacobi operator \( J_s \) of the hypersurface \( x : M^n \to S^{n+1}(1) \) when \( n = 3 \) and \( n = 4 \).

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