SOME NONEXISTENCE THEOREMS ON STABLE MINIMAL SUBMANIFOLDS

BY

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We prove that there exist no stable minimal submanifolds in some n-dimensional ellipsoids, which generalizes J. Simons' result about the unit sphere and gives a partial answer to Lawson-Simons' conjecture.

1. Introduction. In [S], J. Simons proved that there exist no stable minimal submanifolds in the n-dimensional unit sphere $S^n$. In this paper, we establish the following general results.

THEOREM 1. Let $N^n$ be an n-dimensional compact hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the sectional curvature $\overline{K}$ of $N^n$ satisfies

\[ 1/2 < \overline{K} \leq 1, \]

then there exist no stable m-dimensional minimal submanifolds in $N^n$ for each $m$ with $1 \leq m \leq n - 1$.

Remark 1. If $N^n$ is an n-dimensional unit hypersphere $S^n$ in $\mathbb{R}^{n+1}$, then the sectional curvature $\overline{K}$ of $S^n$ is 1, and from Theorem 1 we deduce that there exist no stable m-dimensional minimal submanifolds in $S^n$ for each $m$ with $1 \leq m \leq n - 1$, which was proved by Simons [S].

THEOREM 2. Let $N^n$ be an n-dimensional $(n \geq 4)$ compact submanifold in an $(n + p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Let $R$ and $H$ denote the normalized scalar curvature and the mean curvature functions of $N^n$, respectively. If $R$ satisfies the following pointwise $n(n - 2)/(n - 1)^2$-pinching condition:

\[ \frac{n(n - 2)}{(n - 1)^2} H^2 < R \leq H^2, \]

then there exist no stable m-dimensional minimal submanifolds in $N^n$ for each $m$ with $2 \leq m \leq n - 2$.

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Corollary 1. Let $N^n$ be an $n$-dimensional $(n \geq 4)$ compact hypersurface in $\mathbb{R}^{n+1}$. If all the principal curvatures $k_a$ of $N^n$ satisfy

$$0 < k_a < \sqrt{\frac{1}{n(n-1)}} \sum_{b=1}^{n} k_b, \quad 1 \leq a \leq n,$$

then there exists no $m$-dimensional minimal submanifold in $N^n$ for each $m$ with $2 \leq m \leq n - 2$.

As direct applications of Theorem 1 and Corollary 1, we have

Proposition 1. Let $N^n$ be the following $n$-dimensional $(n \geq 4)$ ellipsoid in $\mathbb{R}^{n+1}$:

$$N^n : \frac{x_1^2}{a_1^n} + \ldots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, \quad 0 < a_1 \leq a_2 \leq \ldots \leq a_{n+1},$$

1. If $1 \leq a_{n+1} < \sqrt{n}$ and $a_1 \geq \sqrt{a_{n+1}}$, then there exist no stable $m$-dimensional minimal submanifolds of $N^n$ for each $m$ with $1 \leq m \leq n - 1$.
2. If $a_{n+1}/a_1 < \sqrt{n/(n-1)}$, then there exist no stable $m$-dimensional minimal submanifolds of $N^n$ for each $m$ with $2 \leq m \leq n - 2$.

Remark 2. It can be proved in a similar way that the above results all keep valid for stable $m$-currents on $N^n$ (for concepts of stable current, see Lawson–Simons [LS]). For example, we can state the counterpart of Theorem 1 as follows:

Theorem 1'. Let $N^n$ be an $n$-dimensional compact hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the sectional curvature $\bar{K}$ of $N^n$ satisfies

$$1/2 < \bar{K} \leq 1,$$

then there exist no stable $m$-currents on $N^n$ for each $m$ with $1 \leq m \leq n - 1$.

Remark 3. Let $N^n$ be an $n$-dimensional compact hypersurface in $\mathbb{R}^{n+1}$ and suppose that every principal curvature $k_a$ of $N^n$ satisfies $\sqrt{\delta} < k_a \leq 1$ $(a = 1, \ldots, n)$. H. Mori [M] and Y. Ohnita [O] proved the conclusion of Theorem 1' under the stronger conditions $\delta > n/(n+1)$ and $\delta > 1/2$, respectively. Our Theorem 1' also gives a partial answer to the following Lawson–Simons' conjecture:

Conjecture ([LS]). Let $N^n$ be a compact $n$-dimensional connected Riemannian manifold with the sectional curvature $\bar{K}$ satisfying

$$1/4 < \bar{K} \leq 1.$$

Then there exist no stable $m$-currents on $N^n$ for each $m$ with $1 \leq m \leq n - 1$.

We are greatly indebted to P. F. Leung's papers [L1, L2] which motivated us to do this work.
2. Basic formulas and notations. In this paper, we shall make use of the following convention on the ranges of indices:

\[ 1 \leq A, B, C, \ldots \leq n + p; \quad 1 \leq a, b, c, \ldots \leq n; \quad n + 1 \leq \mu, \nu, \ldots \leq n + p; \]
\[ 1 \leq i, j, k \ldots \leq m; \quad m + 1 \leq \alpha, \beta, \gamma \ldots \leq n. \]

Let \( M^m \) and \( N^n \) be Riemannian manifolds of dimension \( m \) and dimension \( n \), respectively. Let \( M^m \) be an \( m \)-dimensional compact minimal submanifold of \( N^n \), \( n > m \). For any normal variation vector field \( U = \sum_a u_a e_a \) of \( M^m \), the second variation of the volume is given by (see [S])

\[
I(U, U) = \int_{M^m} \left[ \sum_{a,i} u_{a,i}^2 - \sum_{a,\beta} (\sigma_{a,\beta} + \overline{R}_{a,\beta} u_a u_\beta) \right] dv,
\]

where \( u_{a,i} \) are the covariant derivatives of \( u_a \),

\[
\sigma_{a,\beta} = \sum_{i,j} h_{ij}^a h_{ij}^{\beta},
\]

\[
\overline{R}_{a,\beta} = \sum_i \overline{R}_{a,\beta,i},
\]

and \( h_{ij}^a \) are the components of the second fundamental form \( h \) of \( M^m \) in \( N^n \).

Now let \( x : N^n \rightarrow \mathbb{R}^{n+p} \) be an \( n \)-dimensional submanifold in the \((n+p)\)-dimensional Euclidean space \( \mathbb{R}^{n+p} \). We choose a local field of orthonormal frames \( e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p} \) in \( \mathbb{R}^{n+p} \) such that, restricted to \( N^n \), the vectors \( e_1, \ldots, e_n \) are tangent to \( N^n \). Their dual coframe fields are \( \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+p} \). Then we have

\[
dx = \sum_a \omega_a e_a,
\]

\[
de_a = \sum_b \omega_{ab} e_b + \sum_{\mu,b} B_{ab}^{\mu} \omega_\mu e_\mu,
\]

\[
de_\mu = -\sum_{a,b} B_{ab}^{\mu} \omega_b e_a + \sum_{\nu} \omega_{\mu \nu} e_\nu,
\]

and the second fundamental form of \( N^n \) in \( \mathbb{R}^{n+p} \) is

\[
B = \sum_{a,b,\mu} B_{ab}^{\mu} \omega_a \otimes \omega_b \otimes e_\mu.
\]

The Gauss equation of \( N^n \) in \( \mathbb{R}^{n+p} \) is

\[
n(n-1)R = n^2 H^2 - S,
\]

where \( R, H \) and \( S \) are the normalized scalar curvature, the mean curvature and the length square of the second fundamental form of \( N^n \) in \( \mathbb{R}^{n+p} \), respectively.
3. An $m$-dimensional minimal submanifold in $N^n$. Let $M^m$ be an $m$-dimensional minimal submanifold in $N^n$, and $N^n$ be an $n$-dimensional submanifold in $\mathbb{R}^{n+p}$. In this case we can choose a local orthonormal basis $e_1, \ldots, e_m, e_{m+1}, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$ in $\mathbb{R}^{n+p}$ such that, restricted to $M^m$, the vectors $e_1, \ldots, e_m$ are tangent to $M^m$, $e_1, \ldots, e_n$ are tangent to $N^n$, $e_{n+1}, \ldots, e_{n+p}$ are normal to $N^n$. Their dual coframe fields are $\omega_1, \ldots, \omega_m$, $\omega_{m+1}, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+p}$. From (10)–(12), restricted to $M^m$, we have

\begin{align}
\begin{align}
dx &= \sum_i \omega_i e_i, \\
d e_i &= \sum_j \omega_i e_j + \sum_{\alpha,j} h_{i,j}^\alpha \omega_j e_\alpha + \sum_{\mu,j} B_{i,j}^\mu \omega_j e_\mu, \\
d e_\alpha &= - \sum_{i,j} h_{i,j}^\alpha \omega_i e_j + \sum_\beta \omega_{\alpha\beta} e_\beta + \sum_{\mu,j} B_{\alpha,j}^\mu \omega_j e_\mu, \\
d e_\mu &= - \sum_{i,j} B_{i,j}^\mu \omega_i e_j - \sum_{\alpha,j} B_{\alpha,j}^\mu \omega_j e_\alpha + \sum_\nu \omega_{\mu\nu} e_\nu,
\end{align}\end{align}

where $h = \sum_{i,j,k} h_{i,j}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$ is the second fundamental form of $M^m$ in $N^n$ and $\sum_{k} h_{i,k}^\alpha = 0$ for any $\alpha$, since $M^m$ is a minimal submanifold in $N^n$.

We choose the following normal variation vector field of $M^m$ in $N^n$:

\begin{align}
\begin{align}
U &= \sum_\alpha u_\alpha e_\alpha, \\
u_\alpha &= \langle A, e_\alpha \rangle,
\end{align}\end{align}

where $A$ is a constant vector in $\mathbb{R}^{n+p}$.

Using (15)–(18), a straightforward computation shows

\begin{align}
\begin{align}
u_{\alpha i} &= - \sum_k h_{k,i}^\alpha u_k + \sum_\mu B_{\alpha i}^\mu u_\mu, \\
\sum_{\alpha,i} u_{\alpha i}^2 &= \sum_{\alpha,i} \left[ \sum_{j,k} h_{k,j}^\alpha h_{j,i}^\alpha u_k u_j + \sum_\mu B_{\alpha i}^\mu B_{\alpha i}^\nu u_\mu u_\nu - 2 \sum_{\mu,k} h_{k,i}^\alpha B_{\alpha i}^\mu u_k u_\mu \right],
\end{align}\end{align}

where

\begin{align}
\begin{align}
u_j &= \langle A, e_j \rangle, \\
u_\mu &= \langle A, e_\mu \rangle.
\end{align}\end{align}

Let $E_1, \ldots, E_{n+p}$ be a fixed orthonormal basis of $\mathbb{R}^{n+p}$, and $U_A = \sum_\alpha \langle E_A, e_\alpha \rangle e_\alpha$. Since

\begin{align}
\sum_{A=1}^{n+p} \langle E_A, v \rangle \langle E_B, w \rangle = \langle v, w \rangle
\end{align}

for any vectors $v, w$ in $\mathbb{R}^{n+p}$, putting (21) into (7) and using (22) and (23),

\begin{align}
\begin{align}
\end{align}\end{align}
we obtain

\begin{equation}
\text{trace}(I) \equiv \sum_{A=1}^{n+p} I(U_A, U_A)
\end{equation}

\begin{align*}
&= - \int_{M^m} \left[ - \sum_{\alpha, k, \mu} (B_{ak}^{\mu})^2 + \sum_{\alpha} \bar{R}_{\alpha\alpha} \right] dv \\
&= - \int_{M^m} \sum_{\alpha, k} \left[ - \sum_{\mu} (B_{ak}^{\mu})^2 + \bar{R}_{akak} \right] dv \\
&= - \int_{M^m} \left[ - \sum_{\alpha, \mu, k} B_{\alpha\alpha}^{\mu} B_{kk}^{\mu} + 2 \sum_{\alpha, k} \bar{R}_{akak} \right] dv \\
&= \int_{M^m} \left[ 2 \sum_{\mu, \alpha, k} (B_{ak}^{\mu})^2 - \sum_{\mu, \alpha, k} B_{\alpha\alpha}^{\mu} B_{kk}^{\mu} \right] dv.
\end{align*}

Thus we obtain

**Proposition 2.** Let \( N^n \) be an \( n \)-dimensional compact submanifold in \( \mathbb{R}^{n+p} \). Let \( M^m \) be an \( m \)-dimensional compact minimal submanifold of \( N^n \). If

\begin{equation}
\text{trace}(I) = \int_{M^m} \left[ 2 \sum_{\mu, \alpha, k} (B_{ak}^{\mu})^2 - \sum_{\mu, \alpha, k} B_{\alpha\alpha}^{\mu} B_{kk}^{\mu} \right] dv < 0,
\end{equation}

then \( M^m \) is not a stable minimal submanifold of \( N^n \).

**4. The proof of Theorem 1.** Let \( N^n \) be an \( n \)-dimensional hypersurface in \( \mathbb{R}^{n+1} \) and \( M^m \) be an \( m \)-dimensional compact minimal submanifold in \( N^n \). At a given point \( p \in M^m \) in \( N^n \), we can choose a local orthonormal frame field \( e_1^*, \ldots, e_n^*, \bar{n} \) in \( \mathbb{R}^{n+1} \) such that \( e_1^*, \ldots, e_n^* \) are tangent to \( N^n \) and at \( p \in M^m \),

\begin{equation}
B_{\alpha\beta}^* = \langle B(e_\alpha^*, e_\beta^*), \bar{n} \rangle = k_\alpha \delta_{\alpha\beta}, \quad 1 \leq a, b \leq n,
\end{equation}

where the \( k_\alpha \) are the principal curvatures of \( N^n \) in \( \mathbb{R}^{n+1} \).

Since \( M^m \) is an \( m \)-dimensional compact minimal submanifold in \( N^n \), at a given point \( p \in M^m \) in \( N^n \), we can also choose a local orthonormal frame field \( e_1, \ldots, e_m, e_{m+1}, \ldots, e_n \) in \( N^n \) such that \( e_1, \ldots, e_m \) are tangent to \( M^m \). Noting that \( e_1, \ldots, e_n \) and \( e_1^*, \ldots, e_n^* \) are two local orthonormal frame fields in a neighborhood of \( p \in M^m \), we can set

\begin{align*}
&\ e_i = \sum_{b=1}^{n} A_{i}^{b} e_b^*, \quad 1 \leq i \leq m, \\
&\ e_\alpha = \sum_{b=1}^{n} A_{\alpha}^{b} e_b^*, \quad m + 1 \leq \alpha \leq n,
\end{align*}
where $(A^b_a) \in SO(n)$, i.e.

$$(29) \quad \sum_{a=1}^{n} A^b_a A^a_c = \delta_{bc}, \quad \sum_{a=1}^{n} A^b_a A^c_a = \delta_{bc}. $$

It is a direct verification that at $p \in M^m$, by use of (26)–(29) and (1),

$$(30) \quad \sum_{\alpha, k} B_{\alpha k} B_{kk} = \sum_{\alpha, k} (B(e_{\alpha}, e_{\alpha}), B(e_{k}, e_{k}))$$

$$= \sum_{\alpha, k, a, b, c, d} A^a_{\alpha} A^b_{\alpha} A^c_{k} A^d_{k} (B(e^*_a, e^*_b), B(e^*_c, e^*_d))$$

$$= \sum_{\alpha, k, a, c} k_{\alpha} k_{c} (A^a_{\alpha})^2 (A^c_{k})^2$$

$$= \sum_{\alpha, c} \bar{R}_{\alpha c c} (A^a_{\alpha})^2 (A^c_{k})^2$$

$$\leq \sum_{\alpha, c, \alpha, k} (A^a_{\alpha})^2 (A^c_{k})^2 = m(n - m),$$

where $\bar{R}_{\alpha c c} = k_{\alpha} k_{c}$ is the sectional curvature of $N^n$. From (1), we also have

$$(31) \quad -2 \sum_{\alpha, k} \bar{R}_{\alpha k k} < -2 \cdot \frac{1}{2} m(n - m) = -m(n - m).$$

Putting (30) and (31) into (24), we obtain $\text{trace}(I) < 0$. From Proposition 2, we infer that $M^m$ is not a stable minimal submanifold of $N^n$.

5. The proof of Theorem 2. We first establish the following algebraic lemma in order to prove our Theorem 2:

**Lemma 1.** Let

$$1 \leq a, b \leq n; \quad 1 \leq i, j \leq m; \quad m + 1 \leq \alpha, \beta \leq n,$$

and consider the symmetric $n \times n$ matrix

$$\begin{bmatrix} T_{ij} & T_{i\alpha} \\ T_{\beta j} & T_{\beta \alpha} \end{bmatrix}$$

such that

$$(32) \quad \sum_{i=1}^{m} T_{ii} + \sum_{\alpha=m+1}^{n} T_{\alpha \alpha} = D, \quad \sum_{a,b=1}^{n} T_{ab}^2 = S.$$

Then:

1. If $m = 1$ or $m = n - 1$, we have

$$(33) \quad \left( \sum_{i} T_{ii} \right)^2 - D \sum_{i} T_{ii} + 2 \sum_{i, \alpha} (T_{i\alpha})^2 \leq S + \frac{n-5}{2} D^2.$$
If $2 \leq m \leq n - 2$, we have

\[
\left( \sum_{i} T_{ii} \right)^2 - D \sum_{i} T_{ii} + 2 \sum_{i, \alpha} (T_{i\alpha})^2 \leq \frac{m(n - m)}{n^2} S + \frac{|(2m - n)D|}{n^2} \sqrt{m(n - m)(S_m - D^2)} - \frac{2m(n - m)D^2}{n^2}.
\]

Proof. We apply the Lagrange multiplier method to the problem (cf. P. F. Leung [L1, L2])

\[
\left( \sum_{i} X_{ii} \right)^2 - D \sum_{i} X_{ii} + 2 \sum_{i, \alpha} (X_{i\alpha})^2 = \max!
\]

subject to the constraints

\[
\sum_{i} X_{ii} + \sum_{\alpha} X_{\alpha\alpha} = D
\]

and

\[
\sum_{i} (X_{ii})^2 + \sum_{\alpha} (X_{\alpha\alpha})^2 + 2 \sum_{i < j} (X_{ij})^2 + 2 \sum_{\alpha < \beta} (X_{\alpha\beta})^2 + 2 \sum_{i, \alpha} (X_{i\alpha})^2 = S,
\]

where $S = \sum_{a,b}(T_{ab})^2$ and the $X_{ab}$ form a symmetric $n \times n$ matrix

\[
\begin{bmatrix}
X_{ij} & X_{i\alpha} \\
X_{\beta j} & X_{\beta\alpha}
\end{bmatrix}.
\]

We consider the function

\[
f = \left( \sum_{i} X_{ii} \right)^2 - D \sum_{i} X_{ii} + 2 \sum_{i, \alpha} (X_{i\alpha})^2 + \lambda \left( \sum_{i} X_{ii} + \sum_{\alpha} X_{\alpha\alpha} - D \right) + \mu \left[ \sum_{i} (X_{ii})^2 + \sum_{\alpha} (X_{\alpha\alpha})^2 + 2 \sum_{i, \alpha} (X_{i\alpha})^2 - S \right],
\]

where $\lambda$, $\mu$ are the Lagrange multipliers.

Differentiating with respect to each variable and equating to zero, we obtain

\[
2 \sum_{j} X_{jj} - D + \lambda + 2\mu X_{ii} = 0,
\]

\[
\lambda + 2\mu X_{\alpha\alpha} = 0,
\]

\[
4X_{i\alpha} + 4\mu X_{i\alpha} = 0,
\]

\[
4\mu X_{ij} = 0, \quad i < j,
\]

\[
4\mu X_{\alpha\beta} = 0, \quad \alpha < \beta.
\]
Hence (with the numbers standing for the corresponding left hand sides)
\[
\sum_i X_{ii} (38) + \sum_{\alpha} X_{\alpha \alpha} (39) + \sum_i X_{ii \alpha} (40) + \sum_{i<j} X_{ij} (41) + \sum_{\alpha<\beta} X_{\alpha \beta} (42) = 0
\]
gives
\[
(43) \quad 2 \left( \sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 4 \sum_{i,\alpha} (X_{i \alpha})^2 = -(\lambda D + 2\mu S).
\]

(1) Case \( \mu = 0 \). It is easy to see in this case
\[
(44) \quad \left( \sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i,\alpha} (X_{i \alpha})^2 = -\frac{D^2}{4}.
\]

(2) Case \( \mu = -1 \). First we suppose \( m(n - m) > n \), and putting \( X_{\alpha \alpha} = \lambda/2, \sum_i X_{ii} = D - (n - m)\lambda/2 \) into (38), we have
\[
\lambda = \frac{(m - 2)D}{m(n - m) - n}, \quad X_{ii} = \frac{(n - m - 2)D}{2[m(n - m) - n]},
\]
\[
(45) \quad X_{\alpha \alpha} = \frac{(m - 2)D}{2[m(n - m) - n]},
\]
and
\[
(46) \quad \left( \sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i,\alpha} (X_{i \alpha})^2 = S - \frac{m(n - m) - 4D^2}{4|m(n - m) - n|} D^2
\]
is another critical value.

Now suppose \( m(n - m) = n \), i.e. \( n = 4, m = 2 \). If \( \mu = -1 \), then
\[
(47) \quad X_{ii} = \frac{1}{2}(D - \lambda), \quad X_{\alpha \alpha} = \frac{\lambda}{2},
\]
\[
(48) \quad \left( \sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i,\alpha} (X_{i \alpha})^2 = S - \frac{D^2}{2},
\]
that is, equality holds in (34) in this case.

(3) Case \( \mu \neq 0, -1 \). Let \( X = \sum_i X_{ii} \). Then
\[
(49) \quad X_{\alpha \alpha} = -\frac{\lambda}{2\mu}, \quad 2\mu(X - D) = (n - m)\lambda,
\]
\[
(50) \quad \lambda = D - 2 \left( 1 + \frac{\mu}{m} \right) X.
\]
Substituting (50) into the second formula of (49), we get
\[
(51) \quad \mu = \frac{m(n - m)(D - 2X)}{2(nX - mD)}, \quad \frac{\lambda}{\mu} = \frac{2}{n - m}(X - D).
\]
From (43), we have

\[(52) \quad \frac{X(D - 2X)}{\mu} = \frac{\lambda}{\mu} D + 2S.\]

Putting (51) into (52), we get

\[X^2 - \frac{2mD}{n} X - \left( \frac{m(n - m)}{n} S - \frac{m}{n} D^2 \right) = 0,
\]

that is,

\[(53) \quad X = \frac{m}{n} D \pm \sqrt{\frac{m(n - m)}{n} \left( S - \frac{D^2}{n} \right)}.
\]

The critical value is

\[(54) \quad \left( \sum_i X_{ii} \right)^2 - D \sum_i X_{ii} + 2 \sum_{i, \alpha} (X_{i\alpha})^2
\]

\[= \frac{m(n - m)}{n} S + \frac{|(2m - n)D|}{n^2} \sqrt{m(n - m)(Sn - D^2)} - \frac{2m(n - m)D^2}{n^2}.
\]

Hence, the critical values are

\[-\frac{D^2}{4}, \quad S - \frac{m(n - m) - 4}{4[m(n - m) - n]} D^2,
\]

\[\frac{m(n - m)}{n} S + \frac{|(2m - n)D|}{n^2} \sqrt{m(n - m)(Sn - D^2)} - \frac{2m(n - m)D^2}{n^2}.
\]

It can be verified directly by calculation that if \(m = 1\) or \(m = n - 1\), then \(m(n - m) = n - 1\) and the maximum is \(S + \frac{n+5}{4} D^2\); if \(2 \leq m \leq n - 2\), the maximum is (cf. [L1])

\[\frac{m(n - m)}{n} S + \frac{|(2m - n)D|}{n^2} \sqrt{m(n - m)(Sn - D^2)} - \frac{2m(n - m)D^2}{n^2}.
\]

This completes the proof of Lemma 1.

**PROPOSITION 3.** Let \(N^n\) be an \(n\)-dimensional \((n \geq 4)\) compact submanifold in \(\mathbb{R}^{n+p}\). Let \(S\) be the length square of the second fundamental form. If

\[(55) \quad S < 2nH^2 - |(2m - n)H| \sqrt{\frac{n}{m(n - m)} (S_H - nH^2)},\]

then there exist no stable \(m\)-dimensional minimal submanifolds of \(N^n\) for each \(m\) with \(2 \leq m \leq n - 2\), where \(S_H\) is the length square of the second fundamental form in the direction of the mean curvature vector of \(N^n\).

**Proof.** We choose a local orthonormal frame field \(e_1, \ldots, e_{n+p}\) in \(\mathbb{R}^{n+p}\) with \(e_1, \ldots, e_n\) tangent to \(N^n\) and \(e_{n+1}, \ldots, e_{n+p}\) normal to \(N^n\). Let \(e_{n+1}\)
be parallel to the mean curvature vector \( \overline{H} \) and

\[
B(X, Y) = \sum_{\mu=n+1}^{n+p} B^\mu(X, Y)e_\mu,
\]

then

\[
\sum_a B^{n+1}(e_a, e_a) = nH, \quad \sum_a B^\mu(e_a, e_a) = 0, \quad n + 2 \leq \mu \leq n + p.
\]

Moreover,

\[
\sum_{i, \alpha} [2\|B(e_i, e_\alpha)\|^2 - \langle B(e_i, e_i), B(e_\alpha, e_\alpha) \rangle] = \left( \sum_i B^{n+1}(e_i, e_i) \right)^2 + 2 \sum_{i, \alpha} (B^{n+1}(e_i, e_\alpha))^2 - nH \sum_i B^{n+1}(e_i, e_i) + \sum_{\mu=n+2}^{n+p} \left[ \left( \sum_i B^\mu(e_i, e_i) \right)^2 + 2 \sum_{i, \alpha} (B^\mu(e_i, e_\alpha))^2 \right].
\]

For each symmetric \( n \times n \)-matrix \( (B^{n+1}(e_a, e_b)) \) and \( (B^\mu(e_a, e_b)), 1 \leq a, b \leq n, n + 1 \leq \mu \leq n + p, \) applying Lemma 1, we have

\[
\left( \sum_i B^{n+1}(e_i, e_i) \right)^2 + 2 \sum_{i, \alpha} (B^{n+1}(e_i, e_\alpha))^2 - nH \sum_i B^{n+1}(e_i, e_i) \leq \frac{m(n-m)}{n} S_H + 2(2m-n)H \sqrt{\frac{m(n-m)}{n}(S_H - nH^2)} - 2m(n-m)H^2
\]

and

\[
\left( \sum_i B^\mu(e_i, e_i) \right)^2 + 2 \sum_{i, \alpha} (B^\mu(e_i, e_\alpha))^2 \leq \frac{m(n-m)}{n} \sum_{a, b} (B^\mu(e_a, e_b))^2.
\]

Combining (58), (59) with (60), from assumption (55) we get

\[
\sum_{i, \alpha} [2\|B(e_i, e_\alpha)\|^2 - \langle B(e_i, e_i), B(e_\alpha, e_\alpha) \rangle] \leq \frac{m(n-m)}{n} S - 2m(n-m)H^2 + 2(2m-n)H \sqrt{\frac{m(n-m)}{n}(S_H - nH^2)} < 0.
\]

This completes the proof of Proposition 3.

Proof of Theorem 2. Let \( N^n \) be an \( n \)-dimensional \( (n \geq 4) \) compact submanifold in \( \mathbb{R}^{n+p} \). By the Gauss equation (14) and the fact that \( S \geq nH^2 \), we know that condition (2) is equivalent to

\[
S < \frac{n^2H^2}{n-1}.
\]
But (62) is equivalent to

\[ \sqrt{S - nH^2} < \sqrt{\frac{n}{n-1}} |H| = \frac{1}{2} \sqrt{\frac{n}{n-1}} n|H| - \frac{1}{2} (n-2) \sqrt{\frac{n}{n-1}} |H|. \]

Now (63) is equivalent to

\[ \left( \sqrt{S - nH^2} + \frac{1}{2} (n-2) \sqrt{\frac{n}{n-1}} |H| \right)^2 < \left( \frac{1}{2} \sqrt{\frac{n}{n-1}} n|H| \right)^2, \]

that is,

\[ S < 2nH^2 - (n-2) \sqrt{\frac{n}{n-1}} |H| \sqrt{S - nH^2}. \]

Since \( |2m-n| \sqrt{n/(m(m-n))} \leq (n-2) \sqrt{n/(n-1)} \) and \( S_H \leq S \), we see that (65) implies (55) for each \( m \) with \( 2 \leq m \leq n-2 \). Therefore, Theorem 2 follows from Proposition 3 directly.

### 6. The proof of Corollary 1 and Proposition 1

**Proof of Corollary 1.** Let \( N^n \) be an \( n \)-dimensional compact hypersurface in \( \mathbb{R}^{n+1} \) and let the principal curvatures be \( k_a, 1 \leq a \leq n \). By assumption (3), we have

\[ S = \sum_i k_i^2 < \frac{n^2H^2}{n-1}. \]

By the Gauss equation (14) and the fact \( S \geq nH^2 \), (66) is equivalent to (2). Now Corollary 1 follows from Theorem 2 directly.

**Proof of Proposition 1.** Let \( N^n \) be the following \( n \)-dimensional \((n \geq 4)\) ellipsoid in \( \mathbb{R}^{n+1} \):

\[ N^n : \frac{x_1^2}{a_1^2} + \ldots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, \quad 0 < a_1 \leq a_2 \leq \ldots \leq a_{n+1}. \]

It is not difficult to verify by a direct computation that the maximum and minimum of the principal curvatures are

\[ k_{\text{max}} = \frac{a_{n+1}}{a_1^2}, \quad k_{\text{min}} = \frac{a_1}{a_{n+1}^2}, \]

respectively.

(1) If \( 1 \leq a_{n+1} < \sqrt{2} \) and \( a_1 \geq \sqrt{a_{n+1}} \), then the sectional curvature \( K \) of \( N^n \) satisfies

\[ \frac{1}{2} < \frac{a_1^2}{a_{n+1}^2} = k_{\text{min}}^2 \leq K \leq k_{\text{max}}^2 = \frac{a_{n+1}^2}{a_1^2} \leq 1. \]

Thus the conclusion of Proposition 1 follows from Theorem 1.
(2) If \( a_{n+1}/a_1 < \sqrt[6]{n/(n-1)} \), then
\[
k_n - \sqrt{\frac{1}{n(n-1)}} \sum_{b=1}^{n} k_b \leq \frac{a_{n+1}}{a_1^2} - \sqrt{\frac{n}{n-1}} \frac{a_1}{a_{n+1}^2} < 0.
\]
Thus the conclusion of Proposition 1 follows from Corollary 1.

7. Some remarks. Let \( N^n \) be an \( n \)-dimensional compact submanifold in an \((n + p)\)-dimensional unit sphere \( S^{n+p} \) and \( B \) the second fundamental form of \( N^n \). By a reduction as in the proof of (24) (cf. (2.11) of Pan–Shen [PS]) we have

\[
\text{trace}(I) = - \int_{M^n} \left[ - \sum_{\alpha, k, \mu} (B_{\alpha k}^\mu)^2 + \sum_\alpha \bar{R}_{\alpha \alpha} \right] \, dv
\]
\[
= \int_{M^n} \left[ - m(n-m) + 2 \sum_{\mu, \alpha, k} (B_{\alpha k}^\mu)^2 - \sum_{\mu, \alpha, k} B_{\alpha \alpha}^\mu B_{\alpha k}^\mu \right] \, dv.
\]

We can prove the following counterparts of Theorems 1 and 2 by making use of (67):

THEOREM 3. Let \( N^n \) be an \( n \)-dimensional compact hypersurface in an \((n + 1)\)-dimensional unit sphere \( S^{n+1} \). If the sectional curvature \( K \) of \( N^n \) satisfies

\[
1/2 < K \leq 1,
\]
then there exist no stable \( m \)-dimensional minimal submanifolds in \( N^n \) for each \( m \) with \( 1 \leq m \leq n-1 \).

THEOREM 4. Let \( N^n \) be an \( n \)-dimensional \((n \geq 4)\) compact submanifold in an \((n + p)\)-dimensional Euclidean sphere \( S^{n+p} \). Let \( S \) and \( H \) be the length square of the second fundamental form and the mean curvature of \( N^n \), respectively. If

\[
S < n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1) H^2},
\]
then there exist no stable \( m \)-dimensional minimal submanifolds in \( N^n \) for each \( m \) with \( 2 \leq m \leq n-2 \).

Remark 4. From the main theorem of [L2], we can prove that condition (2) or (69) implies \( \text{Ric}(N^n) > 0 \).

Remark 5. These conclusions keep valid for stable currents (see Lawson–Simons [LS] or Federer–Fleming [FF]).
REFERENCES


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