STABILITY OF HYPERSURFACES WITH CONSTANT 
\((r + 1)\)-TH ANISOTROPIC MEAN CURVATURE

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Abstract. Given a positive function \(F\) on \(S^n\) which satisfies a convexity condition, we define the \(r\)-th anisotropic mean curvature function \(H^r_F\) for hypersurfaces in \(\mathbb{R}^{n+1}\) which is a generalization of the usual \(r\)-th mean curvature function. Let \(X : M \to \mathbb{R}^{n+1}\) be an \(n\)-dimensional closed hypersurface with \(H^r_{F+1}\) constant, for some \(r\) with \(0 \leq r \leq n - 1\), which is a critical point for a variational problem. We show that \(X(M)\) is stable if and only if \(X(M)\) is the Wulff shape.

1. Introduction

Let \(F : S^n \to \mathbb{R}^+\) be a smooth function which satisfies the following convexity condition:

\[(D^2F + F1)_x > 0 \quad \forall x \in S^n,\]

where \(S^n\) denotes the standard unit sphere in \(\mathbb{R}^{n+1}\), \(D^2F\) denotes the intrinsic Hessian of \(F\) on \(S^n\) and 1 denotes the identity on \(T_xS^n\). \(> 0\) means that the matrix is positive definite. We consider the map

\[\phi : S^n \to \mathbb{R}^{n+1},\]
\[x \mapsto F(x)x + (\text{grad}_{S^n} F)_x,\]

its image \(W_F = \phi(S^n)\) is a smooth, convex hypersurface in \(\mathbb{R}^{n+1}\) called the Wulff shape of \(F\) (see [4], [7]–[9], [11], [14], [18], [19]). We note when \(F \equiv 1\), \(W_F\) is just the sphere \(S^n\).

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Now let $X : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed, orientable hypersurface. Let $\nu : M \to S^n$ denotes its Gauss map, that is $\nu$ is the unit inner normal vector of $M$.

Let $A_F = D^2 F + F1$, $S_F = -d(\phi \circ \nu) = -A_F \circ d\nu$. $S_F$ is called the F-Weingarten operator, and the eigenvalues of $S_F$ are called anisotropic principal curvatures. Let $\sigma_r$ be the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \leq r \leq n).$$

We set $\sigma_0 = 1$. The $r$-th anisotropic mean curvature $H^F_r$ is defined by $H^F_r = \sigma_r / C^n_r$, also see Reilly [16].

For each $r$, $0 \leq r \leq n - 1$, we set

$$A_{r,F} = \int_M F(\nu) \sigma_r \, dA_X.$$

The algebraic $(n+1)$-volume enclosed by $M$ is given by

$$V = \frac{1}{n+1} \int_M \langle X, \nu \rangle \, dA_X.$$

We consider those hypersurfaces which are critical points of $A_{r,F}$ restricted to those hypersurfaces enclosing a fixed volume $V$. By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

$$F_{r,F;\Lambda} = A_{r,F} + \Lambda V(X),$$

where $\Lambda$ is a constant. We will show the Euler–Lagrange equation of $F_{r,F;\Lambda}$ is:

$$(r + 1)\sigma_{r+1} - \Lambda = 0.$$  

So the critical points are just hypersurfaces with $H^F_{r+1} = \text{constant}$.

If $F \equiv 1$, then the function $A_{r,F}$ is just the functional $A_r = \int_M S_r \, dA_X$ which was studied by Alencar, do Carmo, and Rosenberg in [1], where $H_r = S_r / C^n_r$ is the usual $r$-th mean curvature. For such a variational problem, they call a critical immersion $X$ of the functional $A_r$ (that is, a hypersurface with $H_{r+1} = \text{constant}$) stable if and only if the second variation of $A_r$ is nonnegative for all variations of $X$ preserving the enclosed $(n+1)$-volume $V$. They proved the following theorem.

**Theorem 1.1 ([1]).** Suppose $0 \leq r \leq n - 1$. Let $X : M \to \mathbb{R}^{n+1}$ be a closed hypersurface with $H_{r+1} = \text{constant}$. Then $X$ is stable if and only if $X(M)$ is a round sphere.

Analogously, we call a critical immersion $X$ of the functional $A_{r,F}$ stable if and only if the second variation of $A_{r,F}$ (or equivalently of $F_{r,F;\Lambda}$) is nonnegative for all variations of $X$ preserving the enclosed $(n+1)$-volume $V$. 


In [14], Palmer proved the following theorem (also see Winklmann [19]).

**Theorem 1.2 ([14])**. Let $X : M \rightarrow \mathbb{R}^{n+1}$ be a closed hypersurface with $H^F_1 = \text{constant}$. Then $X$ is stable if and only if, up to translations and homotheties, $X(M)$ is the Wulff shape.

In this paper, we prove the following theorem.

**Theorem 1.3**. Suppose $0 \leq r \leq n - 1$. Let $X : M \rightarrow \mathbb{R}^{n+1}$ be a closed hypersurface with $H^F_{r+1} = \text{constant}$. Then $X$ is stable if and only if, up to translations and homotheties, $X(M)$ is the Wulff shape.

**Remark 1.4**. In the case $F \equiv 1$, Theorem 1.3 becomes Theorem 1.1. Theorem 1.3 gives an affirmative answer to the problem proposed in [8]. We also note that in the case $F \equiv 1$, our result here gives a new and geometric proof of Theorem 1.1, which is different from [1].

2. Preliminaries

Let $X : M \rightarrow \mathbb{R}^{n+1}$ be a smooth closed, oriented hypersurface with Gauss map $\nu : M \rightarrow S^n$, that is, $\nu$ is the unit inner normal vector field. Let $X_t$ be a variation of $X$, and $\nu_t : M \rightarrow S^n$ be the Gauss map of $X_t$. We define

$$\psi = \left\langle \frac{dX_t}{dt}, \nu_t \right\rangle, \quad \xi = \left( \frac{dX_t}{dt} \right)^\top,$$

where $\top$ represents the tangent component and $\psi, \xi$ are dependent of $t$. The corresponding first variation of the unit normal vector is given by (see [8], [11], [14], [19])

$$\nu_t' = -\text{grad} \psi + d\nu_t(\xi),$$

the first variation of the volume element is (see [2], [3], or [10])

$$\partial_t dA_{X_t} = (\text{div} \xi - nH\psi) dA_{X_t},$$

and the first variation of the volume $V$ is

$$V'(t) = \int_M \psi dA_{X_t},$$

where grad, div, $H$ represents the gradients, the divergence, the mean curvature with respect to $X_t$, respectively.

Let $\{E_1, \ldots, E_n\}$ be a local orthogonal frame on $S^n$, let $e_i = e_i(t) = E_i \circ \nu_t$, where $i = 1, \ldots, n$ and $\nu_t$ is the Gauss map of $X_t$, then $\{e_1, \ldots, e_n\}$ is a local orthogonal frame of $X_t : M \rightarrow \mathbb{R}^{n+1}$.

The structure equations of $x : S^n \rightarrow \mathbb{R}^{n+1}$ are:

$$\begin{aligned}
\text{d}x &= \sum_i \theta_i E_i, \\
\text{d}E_i &= \sum_j \theta_{ij} E_j - \theta_i x, \\
\text{d}\theta_i &= \sum_j \theta_{ij} \wedge \theta_j, \\
\text{d}\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} &= \frac{1}{2} \sum_{k,l} \tilde{R}_{ijkl} \theta_k \wedge \theta_l = -\theta_i \wedge \theta_j,
\end{aligned}$$
where $\theta_{ij} + \theta_{ji} = 0$ and $\tilde{R}_{ijkl} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$.

The structure equations of $X_t$ are (see [12], [13]):

\[
\begin{align*}
\text{d}X_t &= \sum_i \omega_i e_i, \\
\text{d}\nu_t &= -\sum_{i,j} h_{ij} \omega_j e_i, \\
\text{d}e_i &= \sum_j \omega_{ij} e_j + \sum_j h_{ij} \omega_j \nu_t, \\
\text{d}\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \\
\text{d}\omega_{ij} &= -\sum_{k} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l,
\end{align*}
\]

where $\omega_{ij} + \omega_{ji} = 0$, $R_{ijkl} + R_{ijlk} = 0$, and $R_{ijkl}$ are the components of the Riemannian curvature tensor of $X_t(M)$ with respect to the induced metric $dX_t \cdot dX_t$. Here, we have omitted the variable $t$ for some geometric quantities.

From $\text{d}e_i = d(E_i \circ \nu_t) = \nu_t^* dE_i = \sum_j \nu_t^* \theta_{ij} e_j - \nu_t^* \theta_i \nu_t$, we get

\[
\begin{align*}
\omega_{ij} &= \nu_t^* \theta_{ij}, \\
\nu_t^* \theta_i &= -\sum_j h_{ij} \omega_j,
\end{align*}
\]

where $\omega_{ij} + \omega_{ji} = 0$, $h_{ij} = h_{ji}$.

Let $F : S^n \to \mathbb{R}^+$ be a smooth function, we denote the coefficients of co-variant differential of $F$, $\text{grad}_{S^n} F$ with respect to $\{E_i\}_{i=1,...,n}$ by $F_i, F_{ij}$ respectively.

From (2.7), $d(F(\nu_t)) = \nu_t^* dF = \nu_t^* (\sum_i F_i \theta_i) = -\sum_{i,j} (F_i \circ \nu_t) h_{ij} \omega_j$, thus,

\[
\text{grad}(F(\nu_t)) = -\sum_{i,j} (F_i \circ \nu_t) h_{ij} e_j = \nu_t (\text{grad}_{S^n} F).
\]

Through a direct calculation, we easily get

\[
\begin{align*}
\text{d}\phi &= (D^2 F + F 1) \circ dx = \sum_{i,j} A_{ij} \theta_i E_j,
\end{align*}
\]

where $A_{ij}$ is the coefficient of $A_F$, that is, $A_{ij} = F_{ij} + F \delta_{ij}$.

Taking exterior differential of (2.9) and using (2.5), we get

\[
\begin{align*}
A_{ijk} &= A_{jik} = A_{ikj},
\end{align*}
\]

where $A_{ijk}$ denotes coefficient of the covariant differential of $A_F$ on $S^n$.

We define $(A_{ij} \circ \nu_t)_k$ by

\[
\begin{align*}
d(A_{ij} \circ \nu_t) + \sum_k (A_{kj} \circ \nu_t) \omega_{ki} + \sum_k (A_{ik} \circ \nu_t) \omega_{kj} = \sum_k (A_{ij} \circ \nu_t)_k \omega_k.
\end{align*}
\]

By a direct calculation by using (2.7) and (2.11), we have

\[
\begin{align*}
(A_{ij} \circ \nu_t)_k &= -\sum_l h_{kl} A_{ijl} \circ \nu_t.
\end{align*}
\]
We define $L_{ij}$ by

\[(2.13) \quad \left( \frac{de_i}{dt} \right)^\top = - \sum_j L_{ij} e_j,\]

where $\top$ denote the tangent component, then $L_{ij} = -L_{ji}$ and we have (see [2], [3], or [10])

\[(2.14) \quad h'_{ij} = \psi_{ij} + \sum_k \{h_{ijk} \xi_k + \psi h_{ik} h_{jk} + h_{ik} L_{kj} + h_{jk} L_{ki}\}.\]

Let $s_{ij} = \sum_k (A_{ik} \circ \nu) h_{kj}$, then from (2.7) and (2.9), we have

\[(2.15) \quad d((\phi \circ \nu) t) = \nu^* d\phi = - \sum_{i,j} s_{ij} \omega_j e_i.\]

We define $S_F$ by $S_F = -d(\phi \circ \nu) = -A_F \circ d\nu$, then we have $S_F(e_j) = \sum_i s_{ij} e_i$. We call $S_F$ the $F$-Weingarten operator. From the positive definiteness of $(A_{ij})$ and the symmetry of $(h_{ij})$, we know the eigenvalues of $(s_{ij})$ are all real. We call them anisotropic principal curvatures, and denote them by $\lambda_1, \ldots, \lambda_n$.

Taking exterior differential of (2.15) and using (2.6), we get

\[(2.16) \quad s_{ijk} = s_{ikj},\]

where $s_{ijk}$ denotes coefficient of the covariant differential of $S_F$.

We have $n$ invariants, the elementary symmetric function $\sigma_r$ of the anisotropic principal curvatures:

\[(2.17) \quad \sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \leq r \leq n).\]

For convenience, we set $\sigma_0 = 1$ and $\sigma_{n+1} = 0$. The $r$-th anisotropic mean curvature $H^F_r$ is defined by

\[(2.18) \quad H^F_r = \sigma_r / C^n_r, \quad C^n_r = \frac{n!}{r!(n-r)!}.\]

We have, by use of (2.2) and (2.6),

\[(2.19) \quad \sum_{i,j} \frac{d((A_{ij} E_i \otimes E_j) \circ \nu_t)}{dt} = \sum_{i,j} \langle (D(A_{ij} E_i \otimes E_j)) \nu_t, \nu'_t \rangle = - \sum_{i,j,k} A_{ijk} \left( \psi_k + \sum_l h_{kl} \xi_l \right) e_i \otimes e_j,\]

where $D$ is the Levi-Civita connection on $S^n$. 

On the other hand, we have

$$\sum_{i,j} \frac{d((A_{ij} E_i \otimes E_j) \circ \nu_t)}{dt} = \sum_{i,j} \left\{ A'_{ij} e_i \otimes e_j + A_{ij} \left( \frac{de_i}{dt} \right)^\top \otimes e_j + A_{ij} e_i \otimes \left( \frac{de_j}{dt} \right)^\top \right\}. \tag{2.20}$$

By use of (2.13), we get from (2.19) and (2.20)

$$\frac{d(A_{ij} \circ \nu_t)}{dt} = A'_{ij}(t) = \sum_k \left\{ -A_{ijk} \psi_k - \sum_l A_{ijk} h_{kl} \xi_l + A_{ik} L_{kj} + A_{jk} L_{ki} \right\}. \tag{2.21}$$

By (2.12), (2.14), (2.21) and the fact $L_{ij} = -L_{ji}$, through a direct calculation, we get the following lemma.

**Lemma 2.1.**

$$\frac{ds_{ij}}{dt} = s'_{ij}(t) = \sum_k \left\{ (A_{ik} \psi_k)_{ij} + s_{ijk} \psi_{ij} + s_{ikj} L_{ki} + s_{ijk} L_{kj} \right\}. \tag{2.22}$$

As $M$ is a closed oriented hypersurface, one can find a point where all the principal curvatures with respect to $\nu$ are positive. By the positiveness of $A_F$, all the anisotropic principal curvatures are positive at this point. Using the results of Gårding [5], we have the following lemma.

**Lemma 2.2.** Let $X : M \to \mathbb{R}^{n+1}$ be a closed, oriented hypersurface. Assume $H_{r+1}^F > 0$ holds at every point of $M$, then $H_k^F > 0$ holds on every point of $M$ for every $k = 1, \ldots, r$.

Using the characteristic polynomial of $S_F$, $\sigma_r$ is defined by

$$\det(tI - S_F) = \sum_{r=0}^n (-1)^r \sigma_r t^{n-r}. \tag{2.23}$$

So, we have

$$\sigma_r = \frac{1}{r!} \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r} \delta^{j_1 \cdots j_r}_{i_1 \cdots i_r} s_{i_1 j_1 \cdots s_{i_r j_r}},$$

where $\delta^{j_1 \cdots j_r}_{i_1 \cdots i_r}$ is the usual generalized Kronecker symbol, i.e., $\delta^{j_1 \cdots j_r}_{i_1 \cdots i_r}$ equals $+1$ (resp. $-1$) if $i_1 \cdots i_r$ are distinct and $(j_1 \cdots j_r)$ is an even (resp. odd) permutation of $(i_1 \cdots i_r)$ and in other cases it equals zero.

We introduce two important operators $P_r$ and $T_r$ by

$$P_r = \sigma_r I - \sigma_{r-1} S_F + \cdots + (-1)^r S_r^F, \quad r = 0, 1, \ldots, n, \tag{2.24}$$

$$T_r = P_r A_F, \quad r = 0, 1, \ldots, n - 1. \tag{2.25}$$
Obviously, $P_n = 0$ and we have

\[(2.26) \quad P_r = \sigma_r I - P_{r-1} S_F = \sigma_r I + T_{r-1} \, d\nu, \quad r = 1, \ldots, n.\]

From the symmetry of $A_F$ and $d\nu$, $S_F A_F$ and $d\nu \circ S_F$ are symmetric, so $T_r = P_r A_F$ and $d\nu \circ P_r$ are also symmetric for each $r$.

**Lemma 2.3.** The matrix of $P_r$ is given by:

\[(2.27) \quad (P_r)_{ij} = \frac{1}{r!} \sum_{i_1, \ldots, i_r; j_1, \ldots, j_r} \delta_{i_1 \cdots i_r}^{i_1 \cdots i_r} s_{i_1 j_1} \cdots s_{i_r j_r}.\]

*Proof.* We prove Lemma 2.3 inductively. For $r = 0$, it is easy to check that (2.27) is true.

Assume (2.27) is true for $r = k$, then from (2.26),

\[(P_{k+1})_{ij} = \sigma_{k+1} \delta_{i}^{i} - \sum_{l} (P_k)_{il} s_{lj} = \frac{1}{(k+1)!} \sum_{l} \left( \delta_{i_1 \cdots i_{k+1}}^{i_1 \cdots i_{k+1} j_l} - \delta_{i_1 \cdots i_{k+1} j_l}^{i_1 \cdots i_{k+1} j_l} \delta_{j_1 \cdots j_r}^{j_1 \cdots j_r} s_{i_1 j_1} \cdots s_{i_k j_k} \delta_{j_1 \cdots j_r}^{j_1 \cdots j_r} \right)\]

\[\times s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}} = \frac{1}{(k+1)!} \sum_{i_1, \ldots, i_{k+1}; j_1, \ldots, j_{k+1}} \delta_{i_1 \cdots i_{k+1} j_1}^{i_1 \cdots i_{k+1} j_1} s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}}.\]

**Lemma 2.4.** For each $r$, we have

(i) $\sum_j (P_r)_{jj} = 0$,

(ii) $\text{tr}(P_r S_F) = (r + 1) \sigma_{r+1}$,

(iii) $\text{tr}(P_r) = (n - r) \sigma_r$,

(iv) $\text{tr}(P_r S_F^2) = \sigma_1 \sigma_{r+1} - (r + 2) \sigma_{r+2}$.

*Proof.* We only prove (i), the others are easily obtained from (2.23), (2.26), and (2.27).

Noting $s_{i_1 j_1} \cdots s_{i_r j_r} = s_{i_1 j_1} \cdots s_{i_r j_r}$ by (2.16) and $\delta_{i_1 \cdots i_r}^{i_1 \cdots i_r} = -\delta_{i_1 \cdots i_r}^{i_1 \cdots i_r}$, we have

\[\sum_j (P_r)_{jj} = \frac{1}{(r-1)!} \sum_{i_1, \ldots, i_r; j_1, \ldots, j_r} \delta_{i_1 \cdots i_r j_1}^{i_1 \cdots i_r j_1} s_{i_1 j_1} \cdots s_{i_r j_r} = 0.\]

**Remark 2.5.** When $F = 1$, Lemma 2.4 was a well-known result (for example, see Barbosa–Colares [2], Reilly [15], or Rosenberg [17]).

Since $P_{r-1} S_F$ is symmetric and $L_{ij}$ is anti-symmetric, we have

\[(2.28) \quad \sum_{i,j,k} (P_{r-1})_{ji} (s_{kj} L_{ki} + s_{ik} L_{kj}) = 0.\]
From (2.16), (2.26), and (i) of Lemma 2.4, we get

\[(\sigma_r)_k = \sum_j (\sigma_r \delta_{jk})_j = \sum_j (P_r)_{jk} + \sum_{j,l} [(P_{r-1})_{jl} s_{lk}]_j \]
\[= \sum_i (P_{r-1})_{ji} s_{ijk}. \]  

3. First and second variation formulas of $F_r,F;\Lambda$

Define the operator $L_r : C^\infty(M) \to C^\infty(M)$ as follows:

\[(3.1) L_r f = \sum_{i,j} [(T_r)_{ij} f]_i. \]

**Lemma 3.1.**

\[\frac{d\sigma_r}{dt} = \sigma'_r(t) = L_{r-1} \psi + \psi (T_{r-1} \circ d\nu_t, d\nu_t) + \langle \text{grad} \sigma_r, \xi \rangle. \]

**Proof.** Using (2.23), (2.28), (2.29), Lemma 2.1, Lemma 2.3, and (i) of Lemma 2.4, we have

\[\sigma'_r = \frac{1}{(r-1)!} \sum_{i_1,\ldots,i_r} \delta^{i_1\ldots i_r}_{i_1\ldots i_r} s_{i_1 j_1} \cdots s_{i_{r-1} j_{r-1}} s_{i_r j_r} \]
\[= \sum_{i,j} (P_{r-1})_{ji} s'_{ij} \]
\[= \sum_{i,j,k} (P_{r-1})_{ji} [(A_{ik} \psi_k)_j + \psi s_{ik} h_{kj} + s_{ijk} \xi_k + s_{ik} L_{ki} + s_{ik} L_{kj}] \]
\[= \sum_{i,j,k} [(P_{r-1})_{ji} A_{ik} \psi_k]_j + \psi \sum_{i,j,k,l} (P_{r-1})_{ji} A_{il} h_{lk} h_{kj} + \sum_k (\sigma_r)_{k} \xi_k \]
\[= \sum_{j,k} [(T_{r-1})_{jk} \psi_k]_j + \psi \sum_{i,j,k} (T_{r-1})_{ji} h_{lk} h_{kj} + \sum_k (\sigma_r)_{k} \xi_k \]
\[= L_{r-1} \psi + \psi (T_{r-1} \circ d\nu_t, d\nu_t) + \langle \text{grad} \sigma_r, \xi \rangle. \]

**Lemma 3.2.** For each $0 \leq r \leq n$, we have

\[(3.2) \text{div}(P_r(\text{grad}_{S_n} F) \circ \nu_t) + F(\nu_t) \text{tr}(P_r \circ d\nu_t) = -(r+1)\sigma_{r+1} \]

and

\[(3.3) \text{div}(P_r X^\top) + \langle X, \nu_t \rangle \text{tr}(P_r \circ d\nu_t) = (n-r)\sigma_r. \]

**Proof.** From (2.6), (2.15), and Lemma 2.4,

\[\text{div}(P_r(\text{grad}_{S_n} F) \circ \nu_t) = \text{div}(P_r(\phi \circ \nu_t)^\top) \]
\[= \sum_{i,j} ((P_r)_{ji} (\phi \circ \nu_t, e_i))_j \]
Thus, the conclusion follows. \qed

**Theorem 3.3** (First variational formula of $\mathcal{A}_{r,F}$).

(3.4) \[ \mathcal{A}'_{r,F}(t) = -(r + 1) \int_M \psi \sigma_{r+1} \, dA_{X_t}. \]

**Proof.** We have $(F(\nu_t))' = \langle \nabla_{S^n} F, \nu_t \rangle$, so by use of Lemma 3.1, Lemma 3.2, (2.2), (2.3), (2.8), (2.26), and Stokes formula, we have

\[
\mathcal{A}_{r,F}(t) = \int_M \left( F(\nu_t) \sigma_r + (F(\nu_t))' \sigma_r \right) \, dA_{X_t} + F(\nu_t) \sigma_r \partial_t dA_{X_t}
\]

\[
= \int_M \left( F(\nu_t) \nabla(T_{r-1} \nabla \psi) + F(\nu_t) \nabla(T_{r-1} \circ dv_t, dv_t) + F(\nu_t) (\nabla \sigma_r, \xi) + \langle \sigma_r (\nabla_{S^n} F) \circ \nu_t, - \nabla \psi + dv_t (\xi) \rangle + F(\nu_t) \sigma_r (-nH \psi + dv) \right) \, dA_{X_t}
\]

\[
= \int_M \left( -\langle \nabla(F(\nu_t)), T_{r-1} \nabla \psi \rangle + F(\nu_t) \nabla(T_{r-1} \circ dv_t, dv_t) + F(\nu_t) (\nabla \sigma_r, \xi) + \psi \nabla(\sigma_r (\nabla_{S^n} F) \circ \nu_t) + F(\nu_t) \nabla \sigma_r + F(\nu_t) \sigma_r \nabla \xi \right) \, dA_{X_t}
\]

\[
= \int_M \left( -\langle T_{r-1} \nabla(F(\nu_t)), \nabla \psi \rangle + F(\nu_t) \nabla(T_{r-1} \circ dv_t, dv_t) \right) \psi \nabla(\sigma_r (\nabla_{S^n} F) \circ \nu_t) - nH \psi F(\nu_t) \sigma_r \, dA_{X_t}
\]

\[
= \int_M \left( \psi \nabla((\sigma_r (\nabla_{S^n} F) \circ \nu_t) + \nabla(T_{r-1} \nabla(F(\nu_t)))) + F(\nu_t) \nabla(T_{r-1} \circ dv_t, dv_t) - nH \psi (\nu_t) \sigma_r \right) \, dA_{X_t}
\]

\[
= \int_M \left( \psi \nabla((\sigma_r + T_{r-1} \circ dv_t)(\nabla_{S^n} F) \circ \nu_t) \right) + F(\nu_t) \nabla([T_{r-1} \circ dv_t + \sigma_r I] \circ dv_t) \, dA_{X_t}
\]
\[ \psi \{ \text{div} (P_r(\text{grad}_{S^r} F) \circ \nu_t) + F(\nu_t) \text{tr}(P_r \circ d\nu_t) \} \, dA_X, \]
\[ = -(r+1) \int_M \psi \sigma_{r+1} \, dA_X. \]

**Remark 3.4.** When \( F = 1 \), Lemma 4.1 and Theorem 3.3 were proved by R. Reilly [15] (also see [2], [3]).

From (1.6), (2.4), and (3.4), we get

**Proposition 3.5 (The first variational formula).** For all variations of \( X \), we have

\[ (3.5) \quad \mathcal{F}'_{r,F;\Lambda}(t) = - \int_M \psi \{(r+1)\sigma_{r+1} - \Lambda\} \, dA_X. \]

Hence, we obtain the Euler–Lagrange equation of \( \mathcal{F}_{r,F;\Lambda} \):

\[ (3.6) \quad (r+1)\sigma_{r+1} - \Lambda = 0. \]

**Theorem 3.6 (The second variational formula).** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional closed hypersurface, which satisfies (3.6), then for all variations of \( X \) preserving \( V \), the second variational formula of \( \mathcal{A}_{r,F} \) at \( t = 0 \) is given by

\[ (3.7) \quad \mathcal{A}''(0) = \mathcal{F}''_{r,F;\Lambda}(0) = -(r+1) \int_M \psi \{ L_r \psi + \psi \langle T_r \circ d\nu, d\nu \rangle \} \, dA_X, \]

where \( \psi \) satisfies

\[ (3.8) \quad \int_M \psi \, dA_X = 0. \]

**Proof.** Differentiating (3.5), we get (3.7) by use of (3.6) and Lemma 3.1.

We call \( X : M \to \mathbb{R}^{n+1} \) to be a stable critical point of \( \mathcal{A}_{r,F} \) for all variations of \( X \) preserving \( V \), if it satisfies (3.6) and \( \mathcal{A}_{r}(0) \geq 0 \) for all \( \psi \) with condition (3.8).

### 4. Proof of Theorem 1.3

Firstly, we prove that if \( X(M) \) is, up to translations and homotheties, the Wulff shape, then \( X \) is stable.

From \( d\phi = (D^2 F + F1) \circ dx \), \( d\phi \) is perpendicular to \( x \). So \( \nu = -x \) is the unit inner normal vector. We have

\[ (4.1) \quad d\phi = -A_F \circ d\nu = \sum_{i,j,k} A_{jk} h_{ki} \omega_i e_j. \]

On the other hand,

\[ (4.2) \quad d\phi = \sum_i \omega_i e_i, \]
so we have

\[(4.3) \quad s_{ij} = \sum_k A_{ik} h_{kj} = \delta_{ij}.\]

From this, we easily get \(\sigma_r = C_n^r\) and \(\sigma_{r+1} = C_{n+1}^r\), thus, the Wulff shape satisfies (3.6) with \(\Lambda = (r+1)C_{n+1}^r\). Through a direct calculation, we easily know for Wulff shape,

\[(4.4) \quad A''_r(0) = -(r+1)C_{n-1}^r \int_M [\text{div}(A_F \text{grad} \psi) + \psi \langle A_F \circ d\nu, d\nu \rangle] dA_X,
\]

and \(\psi\) satisfies

\[(4.5) \quad \int_M \psi dA_X = 0.\]

From Palmer [14] (also see Winklmann [19]), we know \(A''_r(0) \geq 0\), that is the Wulff shape is stable.

Next, we prove that if \(X\) is stable, then up to translations and homotheties, \(X(M)\) is the Wulff shape. We recall the following lemmas.

**Lemma 4.1** ([7], [8]). *For each \(r = 0, 1, \ldots, n-1\), the following integral formulas of Minkowski type hold:*

\[(4.6) \quad \int_M (H_r^F F(\nu) + H_{r+1}^F \langle X, \nu \rangle) dA_X = 0, \quad r = 0, 1, \ldots, n-1.\]

**Lemma 4.2** ([7], [8], [14]). *If \(\lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{const} \neq 0\), then up to translations and homotheties, \(X(M)\) is the Wulff shape.*

From Lemmas 4.1 and (3.8), we can choose \(\psi = \alpha F(\nu) + H_{r+1}^F \langle X, \nu \rangle\) as the test function, where \(\alpha = \int_M F(\nu) H_r^F dA_X / \int_M F(\nu) dA_X\). For every smooth function \(f : M \rightarrow \mathbb{R}\), and each \(r\), we define:

\[(4.7) \quad I_r[f] = L_r f + f \langle T_r \circ d\nu, d\nu \rangle.\]

Then we have from (3.7)

\[(4.8) \quad A''_r(0) = -(r+1) \int_M \psi I_r[\psi] dA_X.\]

**Lemma 4.3.** *For each \(0 \leq r \leq n-1\), we have*

\[(4.9) \quad I_r[F \circ \nu] = -\langle \text{grad} \sigma_{r+1}, (\text{grad}_{S_n} F) \circ \nu \rangle + \sigma_1 \sigma_{r+1} - (r+2) \sigma_{r+2}\]

and

\[(4.10) \quad I_r[\langle X, \nu \rangle] = -\langle \text{grad} \sigma_{r+1}, X^T \rangle - (r+1) \sigma_{r+1}.\]

**Proof.** From (2.8) and (2.26),

\[
I_r[F \circ \nu] = \text{div}\{T_r \text{grad}(F(\nu))\} + F(\nu) \langle T_r \circ d\nu, d\nu \rangle \\
= \text{div}\{T_r \circ d\nu (\text{grad}_{S_n} F) \circ \nu \} + F(\nu) \langle T_r \circ d\nu, d\nu \rangle \\
= \text{div}(P_{r+1} (\text{grad}_{S_n} F) \circ \nu) + F(\nu) \text{tr}(P_{r+1} d\nu)
\]
Therefore, we obtain from Lemma 4.1 \((\text{recall } 1312)\) Y. HE AND H. LI
\[
\langle \text{grad } \sigma_{r+1}, (\text{grad}_{S^n} F) \circ \nu \rangle
\]
\[
- \sigma_{r+1} \{ \text{div} (P_0 (\text{grad}_{S^n} F) \circ \nu) + F(\nu) \text{ tr}(P_0 d\nu) \},
\]
\[
I_r[\langle X, \nu \rangle] = \text{div}(T_r \text{ grad}(X, \nu)) + \langle X, \nu \rangle \langle T_r \circ d\nu, d\nu \rangle
\]
\[
= \text{div}(T_r \circ d\nu X^\top) + \langle X, \nu \rangle \langle T_r \circ d\nu, d\nu \rangle
\]
\[
= \text{div}(P_{r+1} X^\top) + \langle X, \nu \rangle \text{ tr}(P_{r+1} d\nu) - \langle \text{grad } \sigma_{r+1}, X^\top \rangle
\]
\[
- \sigma_{r+1} \{ \text{div}(P_0 X^\top) + \langle X, \nu \rangle \text{ tr}(P_0 d\nu) \}.
\]
So the conclusions follow from Lemma 3.2. \(\square\)

As \(H^F_{r+1}\) is a constant, from (4.9) and (4.10), we have
\[
(4.11) \quad I_r[\psi] = \alpha I_r[F \circ \nu] + H^F_{r+1} I_r[\langle X, \nu \rangle]
\]
\[
= \alpha (\sigma_1 \sigma_{r+1} - (r+2) \sigma_{r+2}) - (r+1) H^F_{r+1} \sigma_{r+1}
\]
\[
= C^{r+1}_n \{ \alpha [n H^F_1 H^F_{r+1} - (n - r - 1) H^F_{r+2}] - (r+1)(H^F_{r+1})^2 \}.
\]
Therefore, we obtain from Lemma 4.1 (recall \(H^F_{r+1}\) is constant and \(\int_M \psi \ dA_X = 0\))
\[
\frac{1}{r+1} \mathcal{F}_r''(0)
\]
\[
= - \int_M \psi I_r[\psi] \ dA_X
\]
\[
= - \int_M \psi C^{r+1}_n \{ \alpha [n H^F_1 H^F_{r+1} - (n - r - 1) H^F_{r+2}] - (r+1)(H^F_{r+1})^2 \} \ dA_X
\]
\[
= -\alpha C^{r+1}_n \int_M [\alpha F(\nu) + H^F_{r+1} \langle X, \nu \rangle][n H^F_1 H^F_{r+1} - (n - r - 1) H^F_{r+2}] \ dA_X
\]
\[
= -\alpha^2 C^{r+1}_n \int_M F(\nu)[n H^F_1 H^F_{r+1} - (n - r - 1) H^F_{r+2}] \ dA_X
\]
\[
- \alpha C^{r+1}_n H^F_{r+1} \int_M \langle X, \nu \rangle [n H^F_1 H^F_{r+1} - (n - r - 1) H^F_{r+2}] \ dA_X
\]
\[
= -\alpha^2 C^{r+1}_n \int_M F(\nu)[n H^F_1 H^F_{r+1} - (n - r - 1) H^F_{r+2}] \ dA_X
\]
\[
+ \alpha C^{r+1}_n H^F_{r+1} \int_M F(\nu)[n H^F_1 - (n - r - 1) H^F_{r+1}] \ dA_X
\]
\[
= -\alpha^2 (n - r - 1) C^{r+1}_n \int_M F(\nu)(H^F_1 H^F_{r+1} - H^F_{r+2}) \ dA_X
\]
\[
- \frac{\alpha (r+1) C^{r+1}_n (H^F_{r+1})^2}{\int_M F(\nu) \ dA_X}
\]
\[
\times \left\{ \int_M F(\nu) H^F_1 \ dA_X \int_M F(\nu) H^F_{r+1} \ dA_X - \left( \int_M F(\nu) \ dA_X \right)^2 \right\},
\]
where we used $\alpha = \int_M F(\nu) H_r^F dA_X / \int_M F(\nu) dA_X$ in the last equality of the above formula.

As $H_{r+1}^F$ is a constant, it must be positive by the compactness of $M$. Thus, by Lemma 2.2, $H_1^F, \ldots, H_r^F$ are all positive. So, from [6] or [20], we have:

(i) for each $0 \leq r < n - 1$,
\begin{equation}
H_1^F H_{r+1}^F - H_{r+2}^F \geq 0,
\end{equation}
with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$, and
(ii) for each $1 \leq r \leq n - 1$,
\begin{equation}
\int_M F(\nu) H_1^F dA_X \int_M F(\nu) H_{r+1}^F dA_X - \left( \int_M F(\nu) dA_X \right)^2 
\geq 0,
\end{equation}
with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.

From (4.12) and (4.13), we easily obtain that, for each $0 \leq r \leq n - 1$,
\[ A''_r(0) \leq 0, \]
with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$. Thus, from Lemma 4.2, up to translations and homotheties, $X(M)$ is the Wulff shape. We complete the proof of Theorem 1.3.

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