Willmore Surfaces in $S^n$ *

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Abstract. A surface $x: M \rightarrow S^n$ is called a Willmore surface if it is a critical surface of the Willmore functional. It is well known that any minimal surface is a Willmore surface and that many nonminimal Willmore surfaces exists. In this paper, we establish an integral inequality for compact Willmore surfaces in $S^n$ and obtain a new characterization of the Veronese surface in $S^4$ as a Willmore surface. Our result reduces to a well-known result in the case of minimal surfaces.


Key words: Willmore surface, minimal surface, pinching, Veronese surface.

1. Introduction

Let $x: M \rightarrow S^n$ be a surface in an $n$-dimensional unit sphere space $S^n$. If $h^a_{ij}$ denotes the second fundamental form of $M$, $S$ denotes the square of the length of the second fundamental form, $H$ denotes the mean curvature vector, and $H$ denotes the mean curvature of $M$, then we have

$$S = \sum_\alpha \sum_{i,j} (h^a_{ij})^2, \quad H = \sum_\alpha H^a e_\alpha, \quad H^a = \frac{1}{2} \sum_k h^a_{kk}, \quad H = |H|,$$

where $e_\alpha$ ($3 \leq \alpha \leq n$) are orthonormal vector fields of $M$ in $S^n$.

We define the following nonnegative function on $M$:

$$\rho^2 = S - 2H^2,$$  \hfill (1.1)

which vanishes exactly at the umbilic points of $M$.

The Willmore functional is the following non-negative functional (see [2, 4] or [21])

$$W(x) = \int_M \rho^2 \, dv = \int_M (S - 2H^2) \, dv,$$  \hfill (1.2)

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it was shown in [4] (also see [17, 19]) that this functional is an invariant under conformal transformations of $S^n$. The Willmore conjecture says that $W(x) \geq 4\pi^2$ holds for all immersed tori $x: M \to S^3$. The conjecture has been proved in some conformal classes by Li and Yau [12] and Montiel and Ros [14]. The conjecture is also known to be true for flat tori (see [5]) and tori whose images under stereographic projection are surfaces of revolution in $R^3$ (see [8, 10]). It is a natural idea to approach the Willmore conjecture by studying the critical surfaces of the Willmore functional $W(x)$. A surface in $S^n$ is called a Willmore surface if it is a critical surface of the above Willmore functional.

Let $M$ be a surface in $S^n$, it was proved by Bryant in the case $n = 3$ (see [2]) and by Weiner [18] in the general case $n \geq 3$, that $M$ is a Willmore surface if and only if

$$\Delta^\perp H^\alpha + \sum_{\beta, i, j} h^\alpha_{ij} h^\beta_{ij} H^\beta - 2H^2 H^\alpha = 0, \quad 3 \leq \alpha \leq 2 + p.$$  \hspace{1cm} (1.3)

**Remark 1.1.** From (1.3), it is obvious that all minimal surfaces in $S^n$ are Willmore surfaces (see [18]). In [5], Pinkall constructed many compact nonminimal flat Willmore surfaces in $S^3$. In [3], Castro and Urbano constructed many compact nonminimal Willmore surfaces in $S^4$ and Ejiri [7] constructed a compact nonminimal flat Willmore surfaces in $S^5$.

In order to state our main result, we first recall the following example:

**EXAMPLE 1** (see [6]). Veronese surface. Let $(x, y, z)$ be the canonical coordinate system in $R^3$ and $u = (u_1, u_2, u_3, u_4, u_5)$ the canonical coordinate system in $R^5$. We consider the mapping defined by

$$u_1 = \frac{1}{\sqrt{3}}yz, \quad u_2 = \frac{1}{\sqrt{3}}xz, \quad u_3 = \frac{1}{\sqrt{3}}xy,$$

$$u_4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2),$$

where $x^2 + y^2 + z^2 = 3$. This defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points $(x, y, z)$ and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4$. This real projective plane imbedded in $S^4$ is called the Veronese surface. We know that the Veronese surface is a minimal surface in $S^4$ (see [6]), thus it is a Willmore surface. We also note that $\rho^2$ of the Veronese surface satisfies

$$\rho^2 = \frac{4}{3}.$$  \hspace{1cm} (1.4)

In the theory of minimal surfaces in $S^n$, the following integral inequality is well known:
THEOREM 1 ([1] or [13]). Let $M$ be a compact minimal surface with Gauss curvature $K$ in an $n$-dimensional unit sphere $S^n$. Then we have

$$\int_M (1 - K)(3K - 1) \, dv \leq 0. \quad (1.5)$$

In particular, if

$$\frac{1}{3} \leq K \leq 1, \quad (1.6)$$

then either $K = 1$ and $M$ is totally geodesic or $K = 1/3$, $n = 4$ and $M$ is the Veronese surface given by Example 1.

For minimal surfaces in $S^n$, the Gauss equation reads $2K = 2 - S$, thus Theorem 1 is equivalent to:

THEOREM 2. Let $M$ be a compact minimal surface in an $n$-dimensional unit sphere $S^n$. Then we have

$$\int_M S (2 - \frac{4}{3}S) \, dv \leq 0. \quad (1.5')$$

In particular, if

$$0 \leq S \leq \frac{4}{3}, \quad (1.6')$$

then either $S = 0$ and $M$ is totally geodesic, or $S = 4/3$, $n = 4$ and $M$ is the Veronese surface given by Example 1.

In this paper we prove the following integral inequality for compact Willmore surfaces in $S^n$.

THEOREM 3. Let $M$ be a compact Willmore surface in an $n$-dimensional unit sphere $S^n$. Then we have

$$\int_M \rho^2 \left(2 - \frac{4}{3}\rho^2\right) \, dv \leq 0. \quad (1.7)$$

In particular, if

$$0 \leq \rho^2 \leq \frac{4}{3}, \quad (1.8)$$

then either $\rho^2 = 0$ and $M$ is totally umbilic, or $\rho^2 = 4/3$. In the latter case, $n = 4$ and $M$ is the Veronese surface given by Example 1.

Remark 1.2. In the case of minimal surfaces, Theorem 3 reduces to Theorem 2.
Remark 1.3. In [11], we proved an integral inequality for compact \((n-1)\)-dimensional Willmore hypersurfaces in \(S^n\). In the case \(n = 3\), our result is:

THEOREM 4 (see [11, theorem 3]). Let \(M\) be a compact Willmore surface in \(S^3\). Then we have

\[
\int_M \rho^2 (2 - \rho^2) \, dv \leq 0.
\] (1.9)

In particular, if

\[
0 \leq \rho^2 \leq 2,
\] (1.10)

then either \(\rho^2 = 0\) and \(M\) is totally umbilic, or \(\rho^2 = 2\) and

\[
M = S^1 \left( \frac{1}{\sqrt{3}} \right) \times S^1 \left( \frac{1}{\sqrt{3}} \right).
\]

Remark 1.4. We would like to mention the following recent paper: Zhen Guo, H. Li and C. P. Wang, The second variational formula for Willmore submanifolds in \(S^n\), Res. in Math. 40 (2001), 205–225. In this paper, authors studied the stability of \(m\)-dimensional Willmore submanifolds in a sphere and constructed some examples of Willmore submanifolds. In particular they proved that minimal submanifolds are not necessary to be Willmore submanifolds for \(m \geq 3\), but all \(m\)-dimensional minimal Einstein submanifolds in a sphere are Willmore submanifolds.

2. Preliminaries

Let \(x: M \to S^n\) be a surface in an \(n\)-dimensional unit sphere. We choose an orthonormal basis \(e_1, \ldots, e_n\) of \(S^n\) such that \(\{e_1, e_2\}\) are tangent to \(x(M)\) and \(\{e_3, \ldots, e_n\}\) is a local frame in the normal bundle. Let \(\{\omega_1, \omega_2\}\) be the dual forms of \(\{e_1, e_2\}\).

We use the following convention on the ranges of indices:

\[
1 \leq i, j, k, \ldots \leq 2; \quad 3 \leq \alpha, \beta, \gamma, \ldots \leq n.
\]

Then we have the structure equations

\[
dx = \sum_i \omega_i e_i,
\] (2.1)

\[
de_i = \sum_j \omega_{ij} e_j + \sum_{\alpha, j} h^\alpha_{ij} \omega_j e_\alpha - \omega_i x,
\] (2.2)

\[
de_\alpha = -\sum_{i, j} h^\alpha_{ij} \omega_j e_i + \sum_\beta \omega_\alpha e_\beta, \quad h^\alpha_{ij} = h^\alpha_{ji}.
\] (2.3)
The Gauss equations and Ricci equations are

\[ R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha} (h_{ij}^\alpha h_{kl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \quad (2.4) \]

\[ R_{ik} = \delta_{ik} + 2 \sum_{\alpha} H^\alpha h_{ik}^\alpha - \sum_{\alpha, j} h_{ij}^\alpha h_{jk}^\alpha, \quad (2.5) \]

\[ 2K = 2 + 4H^2 - S, \quad (2.6) \]

\[ R_{\beta\alpha ikl} = \sum_i (h_{\beta 1}^\alpha h_{ikl}^\alpha - h_{\beta 2}^\alpha h_{ikl}^\alpha), \quad (2.7) \]

where \( K \) is the Gauss curvature of \( M \) and \( S = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2 \) is the square of the norm of the second fundamental form; \( H = \sum_{\alpha} H^\alpha e_\alpha = (1/2) \sum_{\alpha} (\sum_k h_{ik}^\alpha) e_\alpha \) is the mean curvature vector and \( H = |H| \) is the mean curvature of \( M \).

By Gauss equation (2.6), (1.2) becomes

\[ W(x) = 2 \int_M (H^2 - K + 1) \, dv. \quad (2.8) \]

It was shown in [4] and [19] that this functional is an invariant under conformal transformations of \( S^n \).

We have the following Codazzi equations and Ricci identities:

\[ h_{ijk}^\alpha - h_{ikj}^\alpha = 0, \quad (2.9) \]

\[ h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{mj}^\alpha R_{mikl} + \sum_m h_{im}^\alpha R_{mjkl} + \sum_\beta h_{ij}^\beta R_{\beta kkl}, \quad (2.10) \]

where \( h_{ijk}^\alpha \) and \( h_{ijkl}^\alpha \) are defined by

\[ \sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{ikj}^\alpha \omega_k + \sum_k h_{ijk}^\alpha \omega_k + \sum_\beta h_{ij}^\beta \omega_{\beta k}, \quad (2.11) \]

\[ \sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_l h_{jik}^\alpha \omega_l + \sum_l h_{ijkl}^\alpha \omega_l + \sum_\beta h_{ijl}^\beta \omega_{\beta k} + \sum_\beta h_{ijl}^\beta \omega_{\beta k}. \quad (2.12) \]

As \( M \) is a two-dimensional surface, we have from (2.6) and (1.1)

\[ 2K = 2 + 4H^2 - S = 2 + 2H^2 - \rho^2, \quad (2.13) \]

\[ R_{ijkl} = K (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad R_{ik} = K \delta_{ik}. \quad (2.14) \]
LEMMA 2.1. Let \( x: M \to S^n \) be a surface. Then we can write, for \( 3 \leq \alpha \leq n \),

\[
\frac{h_{11}^\alpha - h_{22}^\alpha}{2} = \lambda \cos \theta_\alpha, \quad \sum_\alpha \cos^2 \theta_\alpha = 1, \quad \lambda \geq 0, \tag{2.15}
\]

\[
h_{12}^\alpha = \mu \cos \phi_\alpha, \quad \sum_\alpha \cos^2 \phi_\alpha = 1, \quad \mu \geq 0, \tag{2.16}
\]

\[
\rho^2 = 2(\lambda^2 + \mu^2). \tag{2.17}
\]

Proof. By choosing

\[
\lambda = \frac{1}{2} \sqrt{\sum_\alpha (h_{11}^\alpha - h_{22}^\alpha)^2}, \quad \mu = \sqrt{\sum_\alpha (h_{12}^\alpha)^2}, \tag{2.18}
\]

we get (2.15) and (2.16). (2.17) comes from (1.1) and (2.18). \( \square \)

LEMMA 2.2. Let \( x: M \to S^n \) be a surface. Then we have

\[
\frac{1}{2} \Delta \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 \geq |\nabla h|^2 + \sum_{\alpha,i,j} (h_{ij}^\alpha h_{kki}^\alpha) - 4|\nabla^\perp H|^2 + 2(1 + H^2) \rho^2 - \frac{3}{2} \rho^4, \tag{2.19}
\]

where

\[
|\nabla h|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 \quad \text{and} \quad |\nabla^\perp H|^2 = \sum (H_{ij}^\alpha)^2.
\]

Proof. Using (2.7), (2.9), (2.10), (2.13) and (2.14), we have the following calculations:

\[
\frac{1}{2} \Delta \sum_{\alpha,i,j} (h_{ij}^\alpha)^2
\]

\[
= \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 + \sum_{\alpha,i,j} h_{ij}^\alpha h_{ij}^\alpha
\]

\[
= \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 + \sum_{\alpha,i,j} h_{ij}^\alpha h_{kki}^\alpha + \sum_{\alpha,i,j} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mj}) +
\]

\[
+ \sum_{\alpha,i,j} h_{ij}^\alpha h_{\beta i}^\alpha R_{\beta ijk}
\]

\[
= \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 + \sum_{\alpha,i,j} h_{ij}^\alpha h_{kki}^\alpha + 2K \rho^2 + \sum_{\alpha,\beta} \left[ \sum_i (h_{ij}^\alpha h_{i2}^\alpha - h_{ij}^\alpha h_{i1}^\alpha) \right] R_{\beta i12}
\]

\[
= \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 + \sum_{\alpha,i,j} h_{ij}^\alpha h_{kki}^\alpha + 2K \rho^2 - \sum_{\alpha,\beta} (R_{\beta i12})^2. \tag{2.20}
\]
From (2.7), (2.15) and (2.16)

\[
R_{\beta \alpha \alpha 12} = \sum_{i=1}^{2} (h_{1i}^\beta h_{i1}^\alpha - h_{2i}^\beta h_{i1}^\alpha)
\]

\[
= (h_{11}^\beta - h_{22}^\beta)h_{11}^\alpha - (h_{11}^\alpha - h_{22}^\alpha)h_{12}^\beta
\]

\[
= 2\lambda \mu (\cos \theta_\beta \cos \phi_\alpha - \cos \theta_\alpha \cos \phi_\beta).
\]

Putting (2.21) into (2.20) and using (2.13), we have

\[
\frac{1}{2} \Delta \sum_{\alpha, i, j} (h_{ij}^\alpha)^2
\]

\[
= \sum_{\alpha, i, j, k} (h_{ijk}^\alpha)^2 + \sum_{\alpha, i, j} h_{ij}^\alpha h_{kij}^\alpha + 2K \rho^2
\]

\[- 4\lambda^2 \mu^2 \sum_{\alpha, \beta} (\cos \theta_\beta \cos \phi_\alpha - \cos \theta_\alpha \cos \phi_\beta)^2
\]

\[
= \sum_{\alpha, i, j, k} (h_{ijk}^\alpha)^2 + \sum_{\alpha, i, j} h_{ij}^\alpha h_{kij}^\alpha + 2K \rho^2
\]

\[- 4\lambda^2 \mu^2 \left[ 2 \sum_{\alpha, \beta} \cos^2 \theta_\beta \cos^2 \phi_\alpha - 2 \sum_{\alpha, \beta} \cos \theta_\beta \cos \theta_\alpha \cos \phi_\beta \cos \phi_\alpha \right]
\]

\[
= |\nabla h|^2 + \sum_{\alpha, i, j, k} (h_{ij}^\alpha h_{kij}^\alpha) - \sum_{\alpha, i, j, k} h_{ij}^\alpha h_{kij}^\alpha + 2K \rho^2
\]

\[- 4\lambda^2 \mu^2 \left[ 2 - 2 \left( \sum_{\alpha} \cos \theta_\alpha \cos \phi_\alpha \right)^2 \right]
\]

\[
\geq |\nabla h|^2 + \sum_{\alpha, i, j, k} (h_{ij}^\alpha h_{kij}^\alpha) - 4 \sum_{\alpha, i} (H_{ij}^\alpha)^2 + 2(1 + H^2) \rho^2 - \frac{1}{2} \rho^4,
\]

(2.22)

and, for the inequality, we used

\[
8\lambda^2 \mu^2 \leq 2(\lambda^2 + \mu^2)^2 = \frac{1}{2} \rho^4.
\]

The following Euler–Lagrange equation for the Willmore functional was derived by Weiner in [18].

**Lemma 2.3.** Let \( x: M \to S^n \) be a surface in an \( n \)-dimensional unit sphere \( S^n \). Then \( M \) is a Willmore surface if and only if

\[
\Delta^{\perp} H^\alpha + \sum_{\beta, i, j} h_{ij}^\alpha h_{ij}^\beta H^\beta - 2H^2 H^\alpha = 0, \quad 3 \leq \alpha \leq n.
\]

(2.23)
We also need the following lemma to prove our Theorem 3:

**LEMMA 2.4.** Let \( M \) be a surface in \( S^n \), then we have

\[
|\nabla h|^2 \geq 3|\nabla^\perp H|^2,
\]

where

\[
|\nabla h|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2, \quad |\nabla^\perp H|^2 = \sum_{\alpha,i} (H_i^\alpha)^2, \quad H_i^\alpha = \nabla_i H^\alpha.
\]

**Proof.** We construct the following symmetric tracefree tensor:

\[
F_{ijk}^\alpha = h_{ijk}^\alpha - \frac{1}{2}(H_i^\alpha \delta_{jk} + H_j^\alpha \delta_{ik} + H_k^\alpha \delta_{ij}).
\]

(2.25)

Then we can easily compute

\[
|F|^2 = \sum_{\alpha,i,j,k} (F_{ijk}^\alpha)^2 = |\nabla h|^2 - 3|\nabla^\perp H|^2
\]

and we get \( |\nabla h|^2 \geq 3|\nabla^\perp H|^2 \), which proves Lemma 2.4.

**Remark 2.1.** The analogue of Lemma 2.4 for hypersurfaces in \( S^n \) can be found in [1, 9, 11].

Now we define the following tracefree tensor

\[
\tilde{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij},
\]

(2.26)

then Lemma 2.3 becomes

**LEMMA 2.5.** Let \( x: M \to S^n \) be a surface in an \( n \)-dimensional unit sphere \( S^n \). Then \( M \) is a Willmore surface if and only if

\[
\Delta^\perp H^\alpha + \sum_{\beta,i,j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta H^\beta = 0, \quad 3 \leq \alpha \leq n.
\]

(2.27)

**LEMMA 2.6.** Let \( x: M \to S^n \) be a Willmore surface, then

\[
\int_M |\nabla^\perp H|^2 = \int_M \sum_{\alpha,i} (H_i^\alpha)^2 = \int_M \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta} H^\alpha H^\beta,
\]

(2.28)

where

\[
\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta.
\]

(2.29)
Proof. By use of (2.27), we have
\[
|\nabla^\perp H|^2 = \sum_{\alpha,i} (H_{\alpha}^i)^2
\]
\[
= \sum_{\alpha,i} (H_{\alpha}^i H_{\alpha}^i) - \sum_{\alpha} H_{\alpha}^2 \Delta^\perp H_{\alpha}^\alpha
\]
\[
= \sum_{\alpha,i} (H_{\alpha}^i H_{\alpha}^i) + \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta} H_{\alpha}^\alpha H_{\beta}^\beta. \tag{2.30}
\]

We get (2.28) by integrating (2.30) over \( M \).

We note that the \((n - 2) \times (n - 2)\)-matrix \((\tilde{\sigma}_{\alpha\beta})\) is symmetric, then it can be assumed to be diagonal for a suitable choice of \( e_3, \ldots, e_n \). We set
\[
\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha\delta} \tilde{\sigma}_{\delta\beta}. \tag{2.31}
\]

By use of (2.8), (2.26) and (2.29), we have
\[
\rho^2 = \sum_{\alpha} \tilde{\sigma}_{\alpha}. \tag{2.32}
\]

3. Proof of Theorem 3

In this section, we give the proof of Theorem 3. Integrating (2.19) over \( M \), by using Lemmas 2.4 and 2.6 we have
\[
0 \geq \int_M \left[ (|\nabla h|^2 - 3|\nabla^\perp H|^2) - |\nabla^\perp H|^2 + 2(1 + H^2)\rho^2 - \frac{3}{2}\rho^4 \right]
\]
\[
\geq \int_M \left[ -|\nabla^\perp H|^2 + 2(1 + H^2)\rho^2 - \frac{3}{2}\rho^4 \right]
\]
\[
= \int_M (2 - \frac{3}{2}\rho^2) \rho^2 + \int_M H^2 \rho^2 + \int_M \left( H^2 \rho^2 - \sum_{\alpha,\beta} H_{\alpha}^\alpha H_{\beta}^\beta \tilde{\sigma}_{\alpha\beta} \right)
\]
\[
\geq \int_M (2 - \frac{3}{2}\rho^2) \rho^2 + \int_M \left( H^2 \rho^2 - \sum_{\alpha,\beta} H_{\alpha}^\alpha H_{\beta}^\beta \tilde{\sigma}_{\alpha\beta} \right). \tag{3.1}
\]

From (2.31) and (2.32), we get
\[
H^2 \rho^2 = \left( \sum_{\alpha} (H_{\alpha}^\alpha)^2 \right) \left( \sum_{\beta} \tilde{\sigma}_{\beta} \right) \geq \sum_{\alpha} (H_{\alpha}^\alpha)^2 \tilde{\sigma}_{\alpha} = \sum_{\alpha,\beta} H_{\alpha}^\alpha H_{\beta}^\beta \tilde{\sigma}_{\alpha\beta}. \tag{3.2}
\]
Putting (3.2) into (3.1), we reach the following integral inequality:

$$\int_M \rho^2 \left(2 - \frac{3}{2} \rho^2\right) \leq 0.$$  

(3.3)

Therefore we have proved the integral inequality (1.7) in Theorem 3.

If (1.8) holds, then from (3.3) we conclude that either $\rho^2 \equiv 0$ or $\rho^2 \equiv 4/3$. In the first case, we know that $S \equiv 2H^2$, i.e. $M$ is totally umbilic; in the latter case, i.e., $\rho^2 \equiv 4/3$, (3.1) becomes an equality, thus we have

$$\int_M H^2 \rho^2 = 0.$$  

(3.4)

Formula (3.4) implies that $H = 0$, thus $x: M \to S^n$ is a minimal surface with $S = 4/3$.

From Theorem 2 we can conclude that $n = 4$ and $x: M \to S^4$ is a Veronese surface, which is given by Example 1. We complete the proof of Theorem 3.

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References


