NOTE ON BRENDLE-EICHMAIR'S PAPER
“ISOPERIMETRIC AND WEINGARTEN SURFACES IN THE SCHWARCHILD MANIFOLD”

HAIZHONG LI, YONG WEI, AND CHANGWEI XIONG

ABSTRACT. In this short note, we show that the assumption “convex” in Theorem 7 of Brendle-Eichmair’s paper [4] is unnecessary.

1. Introduction

For \( n \geq 3 \), let \( \lambda : [0, \bar{r}) \to \mathbb{R} \) be a smooth positive function which satisfies the following conditions (see [4]):

(H1) \( \lambda'(0) = 0 \) and \( \lambda''(0) > 0 \).

(H2) \( \lambda'(r) > 0 \) for all \( r \in (0, \bar{r}) \).

(H3) The function

\[
2 \frac{\lambda''(r)}{\lambda(r)} - (n - 2) \frac{1 - \lambda'(r)^2}{\lambda(r)^2}
\]

is non-decreasing for \( r \in (0, \bar{r}) \).

(H4) \( \frac{\lambda''(r)}{\lambda(r)} + \frac{1 - \lambda'(r)^2}{\lambda(r)^2} > 0 \) for all \( r \in (0, \bar{r}) \).

Now we consider the manifold \( M = S^{n-1} \times [0, \bar{r}) \) equipped with a Riemannian metric \( \bar{g} = dr \otimes dr + \lambda(r)^2 g_{S^{n-1}} \). Let \( \Sigma \) be a closed embedded star-shaped hypersurface in \( (M, \bar{g}) \), where star-shaped means that the unit outward normal \( \nu \) satisfies \( \langle \partial_r, \nu \rangle \geq 0 \). Denote by \( \sigma_p \) the \( p \)-th elementary symmetric polynomial of the principal curvatures. In fact, for this manifold \( (M, \bar{g}) \) Brendle and Eichmair proved the following theorem

**Theorem 1** (Theorem 7 of [4]). Let \( \Sigma \) be a closed embedded hypersurface in the manifold \( (M, \bar{g}) \) that is star-shaped and convex. If \( \sigma_p \) is constant, then \( \Sigma \) is a slice \( S^{n-1} \times \{ r \} \) for some \( r \in (0, \bar{r}) \).

In this note, we show that the assumption “ convex” in Theorem [4] is unnecessary. That is we have

**Theorem 2.** Let \( \Sigma \) be a closed, embedded and star-shaped hypersurface in the manifold \( (M, \bar{g}) \). If \( \sigma_p \) is constant, then \( \Sigma \) is a slice \( S^{n-1} \times \{ r \} \) for some \( r \in (0, \bar{r}) \).
Note that the conditions (H1)-(H4) are all satisfied on the deSitter-Schwarzschild manifolds (see [4]). So we have the following Corollary

**Corollary 3.** Let \( \Sigma \) be a closed, embedded and star-shaped hypersurface in the deSitter-Schwarzschild manifold \((M, \bar{g})\). If \( \sigma_p \) is constant, then \( \Sigma \) is a slice \( S^{n-1} \times \{r\} \) for some \( r \in (0, \bar{r}) \).

2. **Proof of Theorem 2**

In this section, by observing the existence of an elliptic point on \( \Sigma \) and some basic facts about the function \( \sigma_p \), we can remove the assumption “convex”.

Let \( X = \lambda(r)\partial_r \). It is easy to see that \( X \) is a conformal vector field satisfying \( \nabla X = \lambda' \bar{g} \). Following the argument as Lemma 5.3 in [1], we have

**Lemma 4.** Let \( \psi : \Sigma \to (M, \bar{g}) \) be a closed hypersurface. Then there exists an elliptic point \( x \) on \( \Sigma \), i.e., all the principal curvatures are positive at \( x \).

**Proof.** Let \( h = \pi_I \circ \psi : \Sigma \to I \) be the height function on \( \Sigma \), where \( \pi_I \) is the projection \( \pi_I(r, \theta) = r \). At any point \( x \in \Sigma \), we have

\[
\nabla h = (\nabla r)^\top = (\partial_r)^\top. \tag{1}
\]

Let \( \{e_1, \cdots, e_{n-1}\} \) be a local orthonormal frame on \( \Sigma \), and assume that the second fundamental form \( h_{ij} = \langle \nabla_{e_i} \nu, e_j \rangle \) is diagonal with eigenvalues \( \kappa_1, \cdots, \kappa_{n-1} \). Then

\[
\nabla_{e_i} \nabla h
= \nabla_{e_i} \left( \frac{1}{\lambda(h)} \lambda(h) \partial_r^\top \right)
= -\frac{\lambda'}{\lambda} \nabla_{e_i} h \partial_r^\top + \frac{1}{\lambda} \nabla_{e_i} (\lambda \partial_r^\top). \tag{2}
\]

Note that \( X = \lambda \partial_r \) is a conformal vector field, we have

\[
\nabla_{e_i} (\lambda \partial_r^\top) = \nabla_{e_i} (\lambda \partial_r - \langle \lambda \partial_r, \nu \rangle \nu)
= (\nabla_{e_i} (\lambda \partial_r - \langle \lambda \partial_r, \nu \rangle \nu))^\top
= \lambda' e_i - \langle \lambda \partial_r, \nu \rangle \kappa_i e_i. \tag{3}
\]

Substituting (3) into (2) gives that

\[
\nabla_{e_i} \nabla h = -\frac{\lambda'}{\lambda} (\nabla_{e_i} h) \partial_r^\top + \frac{1}{\lambda} (\lambda' - \langle \lambda \partial_r, \nu \rangle \kappa_i) e_i. \tag{4}
\]

Now we consider the maximum point \( x \) of \( h \). We have \( \nabla h = 0, \nu = \partial_r \) and \( \nabla^2 h \leq 0 \) at \( x \). Then from (4), we get

\[
\kappa_i \geq \frac{\lambda'}{\lambda} > 0, \quad i = 1, \cdots, n - 1,
\]

i.e., \( x \) is an elliptic point of \( \Sigma \). \qed
Remark 5. If we assume that the closed embedded hypersurface $\Sigma$ in $M$ satisfies $\langle \partial_r, \nu \rangle > 0$, then $\Sigma$ can be parametrized by a graph on $\mathbb{S}^{n-1}$ (see [5]):

$$\Sigma = \{(r(\theta), \theta) : \theta \in \mathbb{S}^{n-1}\}.$$ 

Define a function $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}$ by $\varphi(\theta) = \Phi(r(\theta))$, where $\Phi(r)$ is a positive function satisfying $\Phi' = 1/\lambda$. Let $\varphi_i, \varphi_{ij}$ be covariant derivatives of $\varphi$ with respect to $g_{\mathbb{S}^{n-1}}$. Define $v = \sqrt{1 + |\nabla \varphi|_{g_{\mathbb{S}^{n-1}}}^2}$. Then the same calculation as in Proposition 5 in [5] gives that the second fundamental form of $\Sigma$ has the expression

$$h_{ij} = \frac{\lambda'}{v^2} g_{ij} - \frac{\lambda}{v^2} \varphi_{ij},$$

where $g_{ij}$ is the induced metric on $\Sigma$ from $(M, \bar{g})$. At the maximum point $x$ of $\varphi$, we have $\varphi_i = 0, \varphi_{ij} \leq 0$. Then we have $h_{ij} \geq \frac{\lambda'}{v^2} g_{ij}$, i.e., $x$ is an elliptic point of $\Sigma$. Note that the maximum point $x$ of $\varphi$ is also a maximum point of $r$.

Recall that for $1 \leq k \leq n - 1$ the convex cone $\Gamma^+_k \subset \mathbb{R}^{n-1}$ is defined by

$$\Gamma^+_k = \{ \bar{\kappa} \in \mathbb{R}^{n-1} | \sigma_j(\bar{\kappa}) > 0 \text{ for } j = 1, \ldots, k \},$$

or equivalently

$$\Gamma^+_k = \text{component of } \{ \sigma_k > 0 \} \text{ containing the positive cone}.\text{.}$$

It is clearly that $\Gamma^+_k$ is a cone with vertex at the origin and $\Gamma^+_k \subset \Gamma^+_j$ for $j \leq k$. We write $\sigma_0 = 1$, $\sigma_k = 0$ for $k > n - 1$, and denote $\sigma_k(\bar{\kappa}) = \sigma_k(\bar{\kappa}|_{\kappa_i=0})$, i.e., $\sigma_k(\bar{\kappa})$ is the $k$-the elementary symmetric polynomial of $(\kappa_1, \ldots, \kappa_i-1, \kappa_{i+1}, \ldots, \kappa_{n-1})$. Then we have the following classical result (see, e.g., [10] Lemma 2.3).

Lemma 6. If $\bar{\kappa} \in \Gamma^+_k$, then $\sigma_{k-1;j}(\bar{\kappa}) > 0$ for each $1 \leq i \leq n - 1$ and

$$\sigma_{j-1} \geq \frac{j}{n-j} \left( \frac{n-1}{j} \right)^{1/j} \sigma_j^{(j-1)/j}, \text{ for } 1 \leq j \leq k. \quad (5)$$

The following Lemma shows that on connected closed hypersurface in $(M, \bar{g})$, the positiveness of $\sigma_p$ implies that the principal curvatures $\bar{\kappa} \in \Gamma^+_p$.

Lemma 7. Let $\Sigma$ be a connected, closed hypersurface in $(M, \bar{g})$. If $\sigma_p > 0$ on $\Sigma$, then we have $\sigma_j > 0$ on $\Sigma$ for each $1 \leq j \leq p - 1$.

Proof. We believe that the proof of this Lemma can be found in literature, for example, see the proof of Proposition 3.2 in [3]. For convenience of the readers, we include the proof here. Lemma 6 implies that there exists an elliptic point $x$ on $\Sigma$. By continuity there exists an open neighborhood $U$ around $x$ such that the principal curvatures are positive in $U$. Hence $\sigma_k$ are positive in $U$ for each $1 \leq k \leq n - 1$. Denote by $\mathcal{G}_j$ the connected component of the set $\{ x \in \Sigma : \sigma_j|_x > 0 \}$ containing $U$.

Claim 8. For each $j$, we have $\mathcal{G}_{j+1} \subset \mathcal{G}_j$.  

Proof of the Claim. For each $k$, define the open set $$V_k = \bigcap_{j=1}^{k} G_j.$$ It suffices to show that $V_k = G_k$. Since $\sigma_j > 0$ in $V_k$ for $1 \leq j \leq k$, Lemma 6 implies that at each point of this open set $V_k$ the inequalities (5) hold. By continuity (3) also hold at the boundary of $V_k$. If a point $y$ of the boundary of $V_k$ belongs to $G_k$, then (5) implies $y \in G_j$ for each $j \leq k$ and therefore belongs to $V_k$. This shows that the boundary of $V_k$ is contained in the boundary of $G_k$. Since by definition $V_k \subset G_k$ and they are both open sets, $G_k$ is connected, we have $V_k = G_k$. This completes the proof of the Claim. \hfill \Box

Now we continue the proof of Lemma 7. We will show that $G_{p-1}$ is closed. Pick a point $y$ at the boundary of $G_{p-1}$. By continuity $\sigma_{p-1} \geq 0$ at $y$. Then Claim 8 implies that $\sigma_j \geq 0$ at $y$ for each $1 \leq j \leq p-1$. If $\sigma_{p-1} = 0$ at $y$, by hypothesis $\sigma_p > 0$ and using Lemma 6, we have

$$0 = \sigma_{p-1} \geq \frac{p}{n-p} \left( \frac{n-1}{p} \right)^{1/p} \sigma_p^{(p-1)/p} > 0,$$

which is a contradiction. This implies $\sigma_{p-1} \neq 0$ at $y$, and $y$ belongs to the interior of $G_{p-1}$. Therefore $G_{p-1}$ is closed. Since it is also open, and then $G_{p-1} = \Sigma$ by the connectedness of $\Sigma$. Then Claim 8 shows that $G_j = \Sigma$ for each $1 \leq j \leq p-1$, this implies $\sigma_j > 0$ for $1 \leq j \leq p-1$ on $\Sigma$ and completes the proof of Lemma 7. \hfill \Box

Now we can prove Theorem 2. As in [4], it suffices to prove the Heintze-Karcher-type inequality and Minkowski-type inequality.

If $\sigma_p$ is a constant on $\Sigma$, then Lemma 3 implies $\sigma_p = \text{const} > 0$. Denote by $\vec{\kappa} = (\kappa_1, \ldots, \kappa_{n-1})$ the principal curvatures of $\Sigma$. Then Lemma 7 implies $\vec{\kappa} \in \Gamma^+_1$ on $\Sigma$. Thus $\vec{\kappa} \in \Gamma^+_1$ and $\Sigma$ is mean convex. So the Heintze-Karcher-type inequality

$$(n-1) \int_{\Sigma} \frac{\lambda'}{\Sigma} \geq \int_{\Sigma} \langle X, \nu \rangle$$

can be obtained as in [2].

On the other hand, we can prove

**Proposition 9** (Minkowski-type inequality). For $1 \leq p \leq n-1$, suppose that $\Sigma$ is star-shaped and $\sigma_p > 0$. Then

$$p \int_{\Sigma} \langle X, \nu \rangle \sigma_p \geq (n-p) \int_{\Sigma} \lambda' \sigma_{p-1}$$

Proof. Let $\xi = X - \langle X, \nu \rangle \nu$ and $T^{(p)}_{ij} = \frac{\partial \sigma_p}{\partial h_{ij}}$. Then

$$\nabla_i \xi_j = \nabla_i X_j - \langle X, \nu \rangle h_{ij} = \lambda' \tilde{g}_{ij} - \langle X, \nu \rangle h_{ij}$$
Therefore
\[
\sum_{i,j=1}^{n-1} \nabla_i (\xi_j T^{(p)}_{ij}) = \lambda' \sum_{i=1}^{n-1} T^{(p)}_{ii} - \sum_{i,j=1}^{n-1} T^{(p)}_{ij} \langle X, \nu \rangle h_{ij} + \sum_{i,j=1}^{n-1} \xi_j \nabla_i T^{(p)}_{ij}
\]
\[
= \lambda' (n-p) \sigma_{p-1} - p \sigma_p \langle X, \nu \rangle + \sum_{i,j=1}^{n-1} \xi_j \nabla_i T^{(p)}_{ij} \quad (8)
\]

Next as the proof of Proposition 8 in [4], we can get
\[
\sum_{i,j=1}^{n-1} \xi_j \nabla_i T^{(p)}_{ij} = -\frac{n-p}{n-2} \sum_{j=1}^{n-1} \sigma_{p-2,j}(\vec{\kappa}) \xi_j \text{Ric}(e_j, \nu)
\]
By direct calculation, we have
\[
\text{Ric}(e_j, \nu) = - (n-2) \left( \frac{\lambda''(r)}{\lambda(r)} + \frac{1 - \lambda'(r)^2}{\lambda(r)^2} \right) \frac{\xi_j}{\lambda} \langle \partial_r, \nu \rangle.
\]
Thus, using the assumption “star-shaped” \( \langle \partial_r, \nu \rangle \geq 0 \) and the condition \((H4)\), we have \( \xi_j \text{Ric}(e_j, \nu) \leq 0 \) for each \( 1 \leq j \leq n-1 \). On the other hand, from Lemma 7 and Lemma 6, \( \vec{\kappa} \in \Gamma^+_{p-1} \) on \( \Sigma \) and \( \sigma_{p-2,j}(\vec{\kappa}) > 0 \) for each \( 1 \leq j \leq n-1 \). Therefore we have
\[
\sum_{i,j=1}^{n-1} \xi_j \nabla_i T^{(p)}_{ij} \geq 0.
\]

Putting (9) into (8) and integrating on \( \Sigma \), we get the Proposition 8. \( \square \)

Once obtaining the Heintze-Karcher-type inequality (6) and the Minkowski-type inequality (7), we can go through the remaining proof as in [4], which completes the proof of Theorem [2]

**Appendix A. Further remark**

Finally we give a remark about the generalization of Theorem [2]. For \( n \geq 3 \), let \( (N, g_N) \) be a compact Einstein manifold of dimension \( n-1 \) satisfying \( \text{Ric}_N = (n-2)Bg_N \) for some constant \( B \). Moreover, let \( \lambda: [0, \bar{r}) \to \mathbb{R} \) be a smooth positive function which satisfies the following conditions:

\((H1)’\) \( \lambda’(0) = 0 \) and \( \lambda''(0) > 0 \).

\((H2)’\) \( \lambda'(r) > 0 \) for all \( r \in (0, \bar{r}) \).

\((H3)’\) The function
\[
2 \frac{\lambda''(r)}{\lambda(r)} - (n-2) \frac{B - \lambda'(r)^2}{\lambda(r)^2}
\]
is non-decreasing for \( r \in (0, \bar{r}) \).

\((H4)’\) \( \frac{\lambda''(r)}{\lambda(r)} + \frac{B - \lambda'(r)^2}{\lambda(r)^2} > 0 \) for all \( r \in (0, \bar{r}) \).
Let manifold $M = N \times [0, r)$ with a Riemannian metric $\bar{g} = dr \otimes dr + \lambda(r)^2 g_N$. By use of the similar arguments as proof of Theorem 2, we can obtain the following generalization of Theorem 2.

**Theorem 10.** Let $\Sigma$ be a closed, embedded and star-shaped hypersurface in the manifold $(M, \bar{g})$. If $\sigma_p$ is constant, then $\Sigma$ is a slice $N \times \{r\}$ for some $r \in (0, \bar{r})$.

**References**


Department of mathematical sciences, and Mathematical Sciences Center, Tsinghua University, 100084, Beijing, P. R. China

E-mail address: hli@math.tsinghua.edu.cn

Department of mathematical sciences, Tsinghua University, 100084, Beijing, P. R. China

E-mail address: wei-y09@mails.tsinghua.edu.cn

Department of mathematical sciences, Tsinghua University, 100084, Beijing, P. R. China

E-mail address: xiongcw10@mails.tsinghua.edu.cn