STABILITY OF HYPERSURFACES WITH CONSTANT \( r \)-TH ANISOTROPIC MEAN CURVATURE

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Abstract. Given a positive function \( F \) on \( S^n \) which satisfies a convexity condition, we define the \( r \)-th anisotropic mean curvature function \( H^F_r \) for hypersurfaces in \( \mathbb{R}^{n+1} \) which is a generalization of the usual \( r \)-th mean curvature function. Let \( X: M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional closed hypersurface with \( H^F_{r+1} \) constant, for some \( r \) with \( 0 \leq r \leq n-1 \), which is a critical point for a variational problem. We show that \( X(M) \) is stable if and only if \( X(M) \) is the Wulff shape.

§1. Introduction

Let \( F: S^n \to \mathbb{R}^+ \) be a smooth function which satisfies the following convexity condition:

\[
(D^2F + F1)_x > 0, \quad \forall x \in S^n,
\]

where \( S^n \) denotes the standard unit sphere in \( \mathbb{R}^{n+1} \), \( D^2F \) denotes the intrinsic Hessian of \( F \) on \( S^n \) and 1 denotes the identity on \( T_xS^n \), \( > 0 \) means that the matrix is positive definite. We consider the map

\[
\phi: S^n \to \mathbb{R}^{n+1},
\]

\[
x \mapsto F(x)x + (\text{grad}_{S^n} F)_x,
\]

its image \( W_F = \phi(S^n) \) is a smooth, convex hypersurface in \( \mathbb{R}^{n+1} \) called the Wulff shape of \( F \) (see [3], [7], [8], [10], [13], [17], [18]). We note when \( F \equiv 1 \), \( W_F \) is just the sphere \( S^n \).

Now let \( X: M \to \mathbb{R}^{n+1} \) be a smooth immersion of a closed, orientable hypersurface. Let \( \nu: M \to S^n \) denotes its Gauss map, that is, \( \nu \) is the unit inner normal vector of \( M \).

Let \( A_F = D^2F + F1 \), \( S_F = -d(\phi \circ \nu) = -A_F \circ d\nu \). \( S_F \) is called the \( F \)-Weingarten operator, and the eigenvalues of \( S_F \) are called anisotropic principal curvatures. Let \( \sigma_r \) be

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the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$:

\[
\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \leq r \leq n).
\]

We set $\sigma_0 = 1$. The $r$-th anisotropic mean curvature $H^F_r$ is defined by $H^F_r = \sigma_r / C^n_r$, also see Reilly [15].

For each $r$, $0 \leq r \leq n-1$, we set

\[
(1.4) \quad \mathcal{A}_{r,F} = \int_M F(\nu) \sigma_r dA_X.
\]

The algebraic $(n+1)$-volume enclosed by $M$ is given by

\[
(1.5) \quad V = \frac{1}{n+1} \int_M \langle X, \nu \rangle dA_X.
\]

We consider those hypersurfaces which are critical points of $\mathcal{A}_{r,F}$ restricted to those hypersurfaces enclosing a fixed volume $V$. By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

\[
(1.6) \quad \mathcal{F}_{r,F;\Lambda} = \mathcal{A}_{r,F} + \Lambda V(X),
\]

where $\Lambda$ is a constant. We will show the Euler-Lagrange equation of $\mathcal{F}_{r,F;\Lambda}$ is:

\[
(1.7) \quad (r + 1)\sigma_{r+1} - \Lambda = 0.
\]

So the critical points are just hypersurfaces with $H^F_{r+1} =$ const.

If $F \equiv 1$, then the function $\mathcal{A}_{r,F}$ is just the functional $\mathcal{A}_r = \int_M S_r dA_X$ which was studied by Alencar, do Carmo and Rosenberg in [1], where $H_r = S_r / C^n_r$ is the usual $r$-th mean curvature. For such a variational problem, they call a critical immersion $X$ of the functional $\mathcal{A}_r$ (that is, a hypersurface with $H_{r+1} =$ constant) stable if and only if the second variation of $\mathcal{A}_r$ is non-negative for all variations of $X$ preserving the enclosed $(n+1)$-volume $V$. They proved:

**Theorem 1.1.** ([1]) Suppose $0 \leq r \leq n-1$. Let $X: M \to \mathbb{R}^{n+1}$ be a closed hypersurface with $H_{r+1} =$ constant. Then $X$ is stable if and only if $X(M)$ is a round sphere.

Analogously, we call a critical immersion $X$ of the functional $\mathcal{A}_{r,F}$ stable if and only if the second variation of $\mathcal{A}_{r,F}$ (or equivalently of $\mathcal{F}_{r,F;\Lambda}$) is non-negative for all variations of $X$ preserving the enclosed $(n+1)$-volume $V$.

In [13], Palmer proved the following theorem (also see Winklmann [18]):

**Theorem 1.2.** ([13]) Let $X: M \to \mathbb{R}^{n+1}$ be a closed hypersurface with $H^F_1 =$constant. Then $X$ is stable if and only if, up to translations and homotheties, $X(M)$ is the Wulff shape.

In this paper, we prove the following theorem:
Theorem 1.3. Suppose $0 \leq r \leq n - 1$. Let $X : M \to \mathbb{R}^{n+1}$ be a closed hypersurface with $H^{F+1}_{r+1} = \text{constant}$. Then, $X$ is stable if and only if, up to translations and homotheties, $X(M)$ is the Wulff shape.

Remark 1.4. In the case $F \equiv 1$, Theorem 1.3 becomes Theorem 1.1. Theorem 1.3 gives an affirmative answer to the problem proposed in [8].

§2. Preliminaries

Let $X : M \to R^{n+1}$ be a smooth closed, oriented hypersurface with Gauss map $\nu : M \to S^n$, that is, $\nu$ is the unit inner normal vector field. Let $X_t$ be a variation of $X$, and $\nu_t : M \to S^n$ be the Gauss map of $X_t$. We define

$$\psi = \langle \frac{dX_t}{dt}, \nu_t \rangle, \quad \xi = (\frac{dX_t}{dt})^\top,$$

where $\top$ represents the tangent component and $\psi, \xi$ are dependent of $t$. The corresponding first variation of the unit normal vector is given by (see [10], [13], [18])

$$\nu'_t = - \text{grad} \psi + d\nu_t(\xi),$$

the first variation of the volume element is (see [2], [4] or [9])

$$\partial_t dA_{X_t} = (\text{div} \xi - nH\psi)dA_{X_t},$$

and the first variation of the volume $V$ is

$$V'(t) = \int_M \psi dA_{X_t},$$

where $\text{grad}$, $\text{div}$, $H$ represents the gradients, the divergence, the mean curvature with respect to $X_t$ respectively.

Let $\{E_1, \cdots, E_n\}$ be a local orthogonal frame on $S^n$, let $e_i = e_i(t) = E_i \circ \nu_t$, where $i = 1, \cdots, n$ and $\nu_t$ is the Gauss map of $X_t$, then $\{e_1, \cdots, e_n\}$ is a local orthogonal frame of $X_t : M \to \mathbb{R}^{n+1}$.

The structure equation of $x : S^n \to \mathbb{R}^{n+1}$ is:

$$\begin{cases} 
    \frac{dx}{dt} = \sum_i \theta_i E_i \\
    \frac{dE_i}{dt} = \sum_j \theta_{ij} E_j - \theta_i x \\
    \frac{d\theta_i}{dt} = \sum_j \theta_{ij} \wedge \theta_j \\
    \frac{d\theta_{ij}}{dt} - \sum_k \theta_{ik} \wedge \theta_{kj} = -\frac{1}{2} \sum_{kl} \bar{R}_{ijkl} \theta_k \wedge \theta_l = -\theta_i \wedge \theta_j
\end{cases}$$

where $\theta_{ij} + \theta_{ji} = 0$ and $\bar{R}_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$. 

The structure equation of \( X_t \) is (see \([11], [12]\)):

\[
\begin{align*}
\text{d}X_t &= \sum_i \omega_i e_i \\
\text{d}\nu_t &= -\sum_{ij} h_{ij} \omega_j e_i \\
\text{d}e_i &= \sum_j \omega_{ij} e_j + \sum_j h_{ij} \nu_t \\
\text{d}\omega_i &= \sum_j \omega_{ij} \wedge \omega_j \\
\text{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} &= -\frac{1}{2} \sum_{kl} R_{ijkl} \theta_k \wedge \theta_l
\end{align*}
\]

where \( \omega_{ij} + \omega_{ji} = 0, R_{ijkl} + R_{ijlk} = 0, \) and \( R_{ijkl} \) are the components of the Riemannian curvature tensor of \( X_t(M) \) with respect to the induced metric \( dX_t \cdot dX_t \). Here we have omitted the variable \( t \) for some geometric quantities.

From \( \text{d}e_i = \text{d}(E_i \circ \nu_t) = \nu_t^* \text{d}E_i = \sum_j \nu_t^* \theta_{ij} e_j - \nu_t^* \theta_i \nu_t \), we get

\[
\begin{align*}
\omega_{ij} &= \nu_t^* \theta_{ij} \\
\nu_t^* \theta_i &= -\sum_j h_{ij} \omega_j,
\end{align*}
\]

where \( \omega_{ij} + \omega_{ji} = 0, h_{ij} = h_{ji} \).

Let \( F: S^n \to \mathbb{R}^+ \) be a smooth function, we denote the coefficients of covariant differential of \( F \), \( \text{grad}_{S^n} F \) with respect to \( \{E_i\}_{i=1,\ldots,n} \) by \( F_i, F_{ij} \) respectively.

From \( (2.7) \), \( \text{d}(F(\nu_t)) = \nu_t^* \text{d}F = \nu_t^* (\sum_i F_i \theta_i) = -\sum_{ij} (F_i \circ \nu_t) h_{ij} \omega_j \), thus

\[
\text{grad}(F(\nu_t)) = -\sum_{ij} (F_i \circ \nu_t) h_{ij} e_j = \text{d}\nu_t(\text{grad}_{S^n} F).
\]

Through a direct calculation, we easily get

\[
\text{d}\phi = (D^2 F + F1) \circ \text{d}x = \sum_{ij} A_{ij} \theta_i E_j,
\]

where \( A_{ij} \) is the coefficient of \( A_F \), that is, \( A_{ij} = F_{ij} + F \delta_{ij} \).

Taking exterior differential of \( (2.9) \) and using \( (2.5) \) we get

\[
A_{ijk} = A_{jik} = A_{ikj},
\]

where \( A_{ijk} \) denotes coefficient of the covariant differential of \( A_F \) on \( S^n \).

We define \( (A_{ij} \circ \nu_t)_k \) by

\[
\text{d}(A_{ij} \circ \nu_t) + \sum (A_{kj} \circ \nu_t) \omega_{ki} + \sum (A_{ik} \circ \nu_t) \omega_{kj} = \sum (A_{ij} \circ \nu_t)_k \omega_k.
\]

By a direct calculation using \( (2.7) \) and \( (2.11) \), we have

\[
(A_{ij} \circ \nu_t)_k = -\sum_l h_{kl} A_{ijl} \circ \nu_t.
\]

We define \( L_{ij} \) by

\[
\left(\frac{\text{d}e_i}{\text{d}t}\right)^\top = -\sum_j L_{ij} e_j,
\]
where $\top$ denote the tangent component, then $L_{ij} = -L_{ji}$ and we have (see [2], [4] or [9])

\[(2.14) \quad h'_{ij} = \psi_{ij} + \sum_k \{ h_{ijk} \xi_k + \psi h_{ik} h_{jk} + h_{ik} L_{kj} + h_{jk} L_{ki} \}.
\]

Let $s_{ij} = \sum_k A_{ik} h_{kj}$, then from (2.7) and (2.9), we have

\[(2.15) \quad d(\phi \circ \nu) = \nu^* d\phi = -\sum_{ij} s_{ij} \omega_j e_i.
\]

We define $S_F$ by

\[(2.16) \quad S_F = -d(\phi \circ \nu) = -A_F \circ d\nu, \quad \text{then we have} \ S_F(e_j) = \sum_i s_{ij} e_i. \quad \text{We call} \ S_F \ \text{to be F-Weingarten operator.}
\]

Taking exterior differential of (2.15) and using (2.6) we get

\[d((A_{ij} E_i \otimes E_j) \circ \nu) = \sum_{ij} \{ A'_{ij} e_i \otimes e_j + A_{ij} \left( \frac{de_i}{dt} \right) \top \otimes e_j + A_{ij} e_i \otimes \left( \frac{de_j}{dt} \right) \top \}. \quad (2.20)
\]

By use of (2.13), we get from (2.19) and (2.20)

\[\frac{d(A_{ij})}{dt} = A'_{ij}(t) = \sum_k \{ -A_{ijk} \psi_k - \sum_l A_{ijk} h_{kl} \xi_l + A_{ik} L_{kj} + A_{jk} L_{ki} \}. \quad (2.21)
\]

By (2.12), (2.14), (2.21) and the fact $L_{ij} = -L_{ji}$, through a direct calculation, we get the following lemma:
Lemma 2.1. \[ \frac{ds_{ij}}{dt} = s'_{ij}(t) = \sum_k \{(A_{ik} \psi_k)_{,j} + s_{ijk} \xi_k + \psi s_{ikj} + s_{kj} L_{ki} + s_{ik} L_{kj}\} . \]

As \( M \) is a closed oriented hypersurface, one can find a point where all the principal curvatures with respect to \( \nu \) are positive. By the positiveness of \( A_F \), all the anisotropic principal curvatures are positive at this point. Using the results of Gårding (\cite{5}), we have the following lemma:

Lemma 2.2. Let \( X : M \rightarrow \mathbb{R}^{n+1} \) be a closed, oriented hypersurface. Assume \( H^F_{r+1} > 0 \) holds on every point of \( M \), then \( H^F_k > 0 \) holds on every point of \( M \) for every \( k = 1, \ldots, r \).

Using the characteristic polynomial of \( S_F \), \( \sigma_r \) is defined by
\[
\det(t I - S_F) = \sum_{r=0}^n (-1)^r \sigma_r t^{n-r}.
\]
So, we have
\[
\sigma_r = \frac{1}{r!} \sum_{i_1, \ldots, i_r; j_1, \ldots, j_r} \delta_{i_1 \ldots i_r}^{j_1 \ldots j_r} s_{i_1 j_1} \cdots s_{i_r j_r},
\]
where \( \delta_{i_1 \ldots i_r}^{j_1 \ldots j_r} \) is the usual generalized Kronecker symbol, i.e., \( \delta_{i_1 \ldots i_r}^{j_1 \ldots j_r} = +1 \) (resp. -1) if \( i_1 \cdots i_r \) are distinct and \( (j_1 \cdots j_r) \) is an even (resp. odd) permutation of \( (i_1 \cdots i_r) \) and in other cases it equals zero.

We introduce two important operators \( P_r \) and \( T_r \) by
\[
P_r = \sigma_r I - \sigma_{r-1} S_F + \cdots + (-1)^r S_F^r, \quad r = 0, 1, \ldots, n,
\]
\[
T_r = P_r A_F, \quad r = 0, 1, \ldots, n - 1.
\]
Obviously \( P_n = 0 \) and we have
\[
P_r = \sigma_r I - P_{r-1} S_F = \sigma_r I + T_{r-1} d\nu, \quad r = 1, \ldots, n.
\]

From the symmetry of \( A_F \) and \( d\nu, S_F A_F \) and \( d\nu \circ S_F \) are symmetric, so \( T_r = P_r A_F \) and \( d\nu \circ P_r \) are also symmetric for each \( r \).

Lemma 2.3. The matrix of \( P_r \) is given by:
\[
(P_r)_{ij} = \frac{1}{r!} \sum_{i_1, \ldots, i_r; j_1, \ldots, j_r} \delta_{i_1 \ldots i_r}^{j_1 \ldots j_r} s_{i_1 j_1} \cdots s_{i_r j_r}.
\]

Proof. We prove Lemma 2.3 inductively. For \( r = 0 \), it is easy to check that (2.27) is true. Assume (2.27) is true for \( r = k \), then from (2.26),
\[
(P_{k+1})_{ij} = \frac{1}{(k+1)!} \sum (P_k)_{il} s_{lj} - \sum_{l}(P_k)_{il} s_{lj} = \frac{1}{(k+1)!} \sum \delta_{i_1 \ldots i_{k+1}}^{j_1 \ldots j_{k+1}} s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}}.
\]
Lemma 3.1. We only prove (ii), the others are easily obtained from (2.23), (2.26) and (2.27).

Proof. We only prove (ii), the others are easily obtained from (2.23), (2.26) and (2.27).

Noting (j, j_r) is symmetric in s_{i_j,1} \cdots s_{i_j, r} by (2.10) and (j, j_r) is skew symmetric in \delta_{i_1 \cdots i_r}^{j_1 \cdots j_r}, we have

\[ \sum_j (P_r)_{jj} = \frac{1}{(r-1)!} \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r} \delta_{i_1 \cdots i_r}^{j_1 \cdots j_r} s_{i_1 j_1} \cdots s_{i_r j_r} = 0. \]

Remark 2.5. When F = 1, Lemma 2.4 was a well-known result (for example, see Barbosa-Colares [2], Reilly [14], or Rosenberg [16]).

Since \( P_{r-1} S_F \) is symmetric and \( L_{ij} \) is anti-symmetric, we have

\[ \sum_{i,j,k} (P_{r-1})_{ji} (s_{kj}L_{ki} + s_{ik}L_{kj}) = 0. \] (2.28)

From (2.10), (2.26) and (i) of Lemma 2.4 we get

\[ (\sigma_r)_k = \sum_j (\sigma_r)_{jk} = \sum_j (P_r)_{jk} + \sum_j [(P_{r-1})_{ji} s_{ik}] = \sum_j (P_{r-1})_{ji} s_{ijk}. \] (2.29)

§3. First and Second Variation Formulas of \( F_{r,F;A} \)

Define the operator \( L_r : C^\infty(M) \to C^\infty(M) \) as following:

\[ L_r f = \sum_{i,j} [(T_r)_{ij} f_j]_i. \] (3.1)

Lemma 3.1. \( \frac{d\sigma_r}{dt} = \sigma'_r(t) = L_{r-1} \psi + \psi(T_{r-1} \circ dv_t, dv_t) + \langle \text{grad} \sigma_r, \xi \rangle. \)

Proof. Using (2.23), (2.28), (2.29), Lemma 2.4, Lemma 2.5 and (i) of Lemma 2.4 we have

\[ \sigma'_r = \frac{1}{(r-1)!} \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r} \delta_{i_1 \cdots i_r}^{j_1 \cdots j_r} s_{i_1 j_1} \cdots s_{i_r j_r} s_{i_{r-1} j_{r-1}} s_{i_r j_r} \]

\[ = \sum_{i,j,k} (P_{r-1})_{ji} s_{ik} \]

\[ = \sum_{i,j,k} (P_{r-1})_{ji} [(A_{ik} \psi_k) + \psi s_{ik} h_{kj} + s_{ij} \xi_k + s_{kj} L_{ki} + s_{ik} L_{kj}] \]

\[ = \sum_{i,j,k} (P_{r-1})_{ji} A_{ik} \psi_k + \psi \sum_{i,j,k} (P_{r-1})_{ji} A_{ik} h_{kj} + \sum_k (\sigma_r)_{ki} \xi_k \]

\[ = \sum_{i,j,k} [(T_{r-1})_{jk} \psi_k] + \psi \sum_{i,j,k} (T_{r-1})_{ji} h_{kj} + \sum_k (\sigma_r)_{ki} \xi_k \]

\[ = L_{r-1} \psi + \psi(T_{r-1} \circ dv_t, dv_t) + \langle \text{grad} \sigma_r, \xi \rangle. \]
Lemma 3.2. For each $0 \leq r \leq n$, we have

\begin{equation}
\text{div}(P_r(\text{grad}_S F) \circ \nu_t) + F(\nu_t) \text{tr}(P_r \circ d\nu_t) = -(r + 1)\sigma_{r+1},
\end{equation}

and

\begin{equation}
\text{div}(P_rX^\top) + \langle X, \nu_t \rangle \text{tr}(P_r \circ d\nu_t) = (n-r)\sigma_r.
\end{equation}

\begin{proof}
From (2.16), (2.15) and Lemma 2.4, we have

\begin{equation}
\text{div}(P_r(\text{grad}_S F) \circ \nu_t) = \sum_{ij}((P_r)_{ji} \circ \nu_t, e_i)_j = -\sum_{ij}((P_r)_{ji} \circ \nu_t + F(\nu_t) \sum_{ij}((P_r)_{ji} h_{ij}) = -\text{tr}(P_rS_F) - F(\nu_t) \text{tr}(P_r \circ d\nu_t) = -(r + 1)\sigma_{r+1} - F(\nu_t) \text{tr}(P_r \circ d\nu_t),
\end{equation}

\begin{equation}
\text{div}(P_rX^\top) = \sum_{ij}((P_r)_{ji}(X, e_i)_j = \sum_{ij}((P_r)_{ji} \delta_{ij} + \sum_{ij}((P_r)_{ji} h_{ij}) \langle X, \nu_t \rangle = \text{tr}(P_r) - \text{tr}(P_r \circ d\nu_t)(X, \nu_t) = (n-r)\sigma_r - \text{tr}(P_r \circ d\nu_t)(X, \nu_t).
\end{equation}

Thus, the conclusion follows.
\end{proof}

Theorem 3.3. (First variational formula of $\mathcal{A}_{r,F}$)

\begin{equation}
\mathcal{A}_{r,F}'(t) = -(r + 1) \int_M \psi \sigma_{r+1} dA_X.
\end{equation}

\begin{proof}
We have $(F(\nu_t))' = (\text{grad}_S F, \nu'_t)$, so by use of Lemma 3.1, Lemma 3.2, (2.2), (2.3), (2.8), (2.20) and Stokes formula, we have

\begin{equation}
\mathcal{A}_{r,F}'(t) = \int_M (F(\nu_t)\sigma_r' + (F(\nu_t))' \sigma_r) dA_X + F(\nu_t) \sigma_r \partial_t dA_X,
\end{equation}

\begin{equation}
= \int_M \{F(\nu_t) \text{div}(T_{r-1} \text{grad} \psi) + F(\nu_t) \langle T_{r-1} \circ d\nu_t, d\nu_t \rangle + F(\nu_t)(\text{grad} \sigma_r, \xi) + \langle \sigma_r(\text{grad}_S F) \circ \nu_t, -\text{grad} \psi + d\nu_t(\xi) \rangle + F(\nu_t)\sigma_r(-nH\psi + \text{div} \xi) \} dA_X,
\end{equation}

\begin{equation}
= \int_M \{-\text{grad}(F(\nu_t), T_{r-1} \text{grad} \psi) + F(\nu_t) \langle T_{r-1} \circ d\nu_t, d\nu_t \rangle + \langle F(\nu_t) \text{grad} \sigma_r, \xi \rangle + \psi \text{div}(\sigma_r(\text{grad}_S F) \circ \nu_t) + \langle \sigma_r, \text{grad}(F(\nu_t)) \rangle, \xi \rangle - nH\psi F(\nu_t) \sigma_r + F(\nu_t)\sigma_r \text{div} \xi \} dA_X,
\end{equation}

\begin{equation}
= \int_M \{-\langle T_{r-1} \text{grad} F(\nu_t), \text{grad} \psi \rangle + F(\nu_t) \langle T_{r-1} \circ d\nu_t, d\nu_t \rangle - \langle F(\nu_t)(\text{grad} F(\nu_t)) \rangle \text{div}(\text{grad}_S F) \circ \nu_t \}
\end{equation}

\begin{equation}
= \int_M \{\text{div}(\sigma_r(\text{grad}_S F) \circ \nu_t) + \text{div}(T_{r-1} \text{grad} F(\nu_t)) + F(\nu_t)(\text{grad} F(\nu_t)) \}
\end{equation}

\begin{equation}
= \int_M \{\text{div}[(\sigma_r + T_{r-1} \circ d\nu_t)(\text{grad}_S F) \circ \nu_t] + F(\nu_t) \text{tr}(T_{r-1} \circ d\nu_t + \sigma_r I) \circ d\nu_t] \} dA_X,
\end{equation}

\begin{equation}
= \int_M \{\text{div}(P_r(\text{grad}_S F) \circ \nu_t) + F(\nu_t) \text{tr}(P_r \circ d\nu_t) \} dA_X,
\end{equation}

\begin{equation}
= -(r + 1) \int_M \psi \sigma_{r+1} dA_X.
\end{equation}

\end{proof}
Remark 3.4. When $F = 1$, Lemma 4.1 and Theorem 3.3 were proved by R. Reilly \[14\] (also see \[2\], \[4\]).

From (1.6), (2.4) and (3.4), we get

Proposition 3.5. (the first variational formula). For all variations of $X$ preserving $V$, we have

$$
(3.5) \quad \mathcal{A}_r'(t) = \mathcal{F}_{r,F;\Lambda}(t) = - \int_M \psi \{(r + 1)\sigma_{r+1} - \Lambda\} dA_X.
$$

Hence we obtain the Euler-Lagrange equation for such a variation

$$
(3.6) \quad (r + 1)\sigma_{r+1} - \Lambda = 0.
$$

Theorem 3.6. (the second variational formula). Let $X: M \to \mathbb{R}^{n+1}$ be an $n$-dimensional closed hypersurface, which satisfies (3.6), then for all variations of $X$ preserving $V$, the second variational formula of $\mathcal{A}$ at $t = 0$ is given by

$$
(3.7) \quad \mathcal{A}_r''(0) = \mathcal{F}_{r,F;\Lambda}(0) = -(r + 1) \int_M \psi \{L_r \psi + \psi \langle T_r \circ d\nu, d\nu \rangle \} dA_X,
$$

where $\psi$ satisfies

$$
(3.8) \quad \int_M \psi dA_X = 0.
$$

Proof. Differentiating (3.5), we get (3.7) by use of (3.6). \[\square\]

We call $X: M \to \mathbb{R}^{n+1}$ to be a stable critical point of $\mathcal{A}_{r,F}$ for all variations of $X$ preserving $V$, if it satisfies (3.6) and $\mathcal{A}_r''(0) \geq 0$ for all $\psi$ with condition (3.8).

§4. PROOF OF THEOREM 1.3

Firstly, we prove that if $X(M)$ is, up to translations and homotheties, the Wulff shape, then $X$ is stable.

From $d\phi = (D^2F + F1) \circ dx$, $d\phi$ is perpendicular to $x$. So $\nu = -x$ is the unit inner normal vector. We have

$$
(4.1) \quad d\phi = -A_F \circ d\nu = \sum_{ijk} A_{jk} h_{ki} \omega_i e_j.
$$

On the other hand,

$$
(4.2) \quad d\phi = \sum_i \omega_i e_i,
$$

so we have

$$
(4.3) \quad s_{ij} = \sum_k A_{ik} h_{kj} = \delta_{ij}.
$$
From this, we easily get $\sigma_r = C^r_n$ and $\sigma_{r+1} = C^{r+1}_n$, thus the Wulff shape satisfies (3.6) with $\Lambda = (r + 1)C^{r+1}_n$. Through a direct calculation, we easily know for Wulff shape, 

\[
\mathcal{A}^{\prime\prime}_r(0) = -(r + 1)C^{r-1}_n \int_M [\text{div}(A_F \text{grad} \psi) + \psi(A_F \circ d\nu, d\nu)]dA_X,
\]

and $\psi$ satisfies 

\[
\int_M \psi dA_X = 0.
\]

From Palmer [13] (also see Winklmann [18]), we know $\mathcal{A}^{\prime\prime}_r(0) \geq 0$, that is, the Wulff shape is stable.

Next, we prove that if $X$ is stable, then up to translations and homotheties, $X(M)$ is the Wulff shape. We recall the following lemmas:

**Lemma 4.1.** ([7], [8]) For each $r = 0, 1, \ldots, n - 1$, the following integral formulas of Minkowski type hold:

\[
\int_M (H^F_r F(\nu) + H^F_{r+1} \langle X, \nu \rangle) dA_X = 0, \quad r = 0, 1, \ldots, n - 1.
\]

**Lemma 4.2.** ([7], [8], [13]) If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{const} \neq 0$, then up to translations and homotheties, $X(M)$ is the Wulff shape.

From Lemma 4.1 and (3.8), we can choose $\psi = \alpha F(\nu) + H^F_{r+1} \langle X, \nu \rangle$ as the test function, where $\alpha = \int_M F(\nu) H^F_r dA_X / \int_M F(\nu) dA_X$. For every smooth function $f : M \to \mathbb{R}$, and each $r$, we define:

\[
I_r[f] = L_r f + f(T_r \circ d\nu, d\nu),
\]

Then, we have from (3.7)

\[
\mathcal{A}^{\prime\prime}_r(0) = -(r + 1) \int_M \psi I_r[\psi] dA_X.
\]

**Lemma 4.3.** For each $0 \leq r \leq n - 1$, we have

\[
I_r[F \circ \nu] = -\langle \text{grad} \sigma_{r+1}, (\text{grad}_{S^n} F) \circ \nu \rangle + \sigma_{1} \sigma_{r+1} - (r + 2) \sigma_{r+2},
\]

and

\[
I_r[\langle X, \nu \rangle] = -\langle \text{grad} \sigma_{r+1}, X^T \rangle - (r + 1) \sigma_{r+1}.
\]

**Proof.** From (2.8) and (2.26), we have

\[
I_r[F \circ \nu] = \text{div}\{T_r \text{grad}(F(\nu))\} + F(\nu)\langle T_r \circ d\nu, d\nu \rangle = \text{div}(T_r \circ d\nu(\text{grad}_{S^n} F) \circ \nu) + F(\nu)\langle T_r \circ d\nu, d\nu \rangle = \text{div}(P_{r+1}(\text{grad}_{S^n} F) \circ \nu) + F(\nu) \text{tr}(P_{r+1}d\nu) - \langle \text{grad} \sigma_{r+1}, (\text{grad}_{S^n} F) \circ \nu \rangle - \sigma_{r+1}\{\text{div}(P_0(\text{grad}_{S^n} F) \circ \nu) + F(\nu) \text{tr}(P_0d\nu)\},
\]

\[
I_r[\langle X, \nu \rangle] = \text{div}(T_r(\text{grad}_{S^n} F) \circ \nu) + \text{tr}(P_r d\nu) = \text{div}(P_{r+1}^0(\text{grad}_{S^n} F) \circ \nu) + F(\nu) \text{tr}(P_{r+1}^0d\nu) - \langle \text{grad} \sigma_{r+1}, (\text{grad}_{S^n} F) \circ \nu \rangle - \sigma_{r+1}\{\text{div}(P_0^0(\text{grad}_{S^n} F) \circ \nu) + F(\nu) \text{tr}(P_0^0d\nu)\}.
\]
Therefore we obtain from Lemma 4.1 (recall □
So the conclusions follow from Lemma 3.2.

As $H^F_{r+1}$ is a constant, from (4.9) and (4.10), we have

$$ I_r[\psi] = \alpha I_r[F \circ \nu] + H^F_{r+1} I_r[(X, \nu)] $$

(4.11)

$$ = \alpha(\sigma_1 \sigma_{r+2} - (r+2)\sigma_{r+2}) - (r+1)H^F_{r+1}\sigma_{r+1} $n

$$ = C^{r+1}_n\{\alpha[nH^F_1 H^F_{r+1} - (n - r - 1)H^F_{r+2} - (r+1)(H^F_{r+1})^2]. $n

Therefore we obtain from Lemma 4.1 (recall $H^F_{r+1}$ is constant and $\int_M \psi dA_X = 0$)

$$ \frac{1}{r+1}\mathcal{A}_r''(0) $$

$$ = -\int_M \psi I_r[\psi] dA_X $$

$$ = -\int_M \psi C^{r+1}_n\{\alpha[nH^F_1 H^F_{r+1} - (n - r - 1)H^F_{r+2} - (r+1)(H^F_{r+1})^2] dA_X $$

(4.12)

$$ = \alpha H^F_1 H^F_{r+1} - H^F_{r+2} \geq 0, $$

with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$, and

(ii) for each $1 \leq r \leq n - 1,$

$$ \int_M F(\nu) H^F_{r+1} dA_X \int_M F(\nu) H^F_{r+2} dA_X - (\int_M F(\nu) dA_X)^2 \geq 0, $$

(4.13)

with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$.

From (4.12) and (4.13), we easily obtain that, for each $0 \leq r \leq n - 1,$

$$ \mathcal{A}_r''(0) \leq 0, $$
with the equality holds if and only if \( \lambda_1 = \cdots = \lambda_n \). Thus, from Lemma 4.2, up to translations and homotheties, \( X(M) \) is the Wulff shape. We complete the proof of Theorem 1.3.

REFERENCES


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