THE MONGE-AMPÈRE EQUATION WITH INFINITE BOUNDARY VALUE

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1. Introduction

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( \psi \) a positive function defined on \( \Omega \times \mathbb{R} \times \mathbb{R}^n \). In this paper we study the Dirichlet problem for the Monge-Ampère equation

\[
\det D^2 u = \psi(x, u, Du) > 0 \quad \text{in} \quad \Omega,
\]

with the infinite boundary value condition

\[
u = +\infty \quad \text{on} \quad \partial \Omega.
\]

We will look for strictly convex solutions in \( C^\infty(\Omega) \); it is necessary to assume the underlying domain \( \Omega \) to be convex for such solutions to exist.

This problem was first considered by Cheng and Yau ([5], [6]) for \( \psi(x, u) = e^{Ku} f(x) \) in bounded convex domains and for \( \psi(u) = e^{2u} \) in unbounded domains. More recently, Matero [11] treated the case \( \psi = \psi(x, u) \) for bounded strictly convex domains, generalizing a result of Keller [8] and Osserman [12] for the Laplace operator; his results were further extended by Salani [13] to some Hessian equations. (See also [9] where problem (1.1)-(1.2) were studied for \( \psi(x, u) = e^{u} f(x) \) and \( \psi(x, u) = u^p f(x) \).) For the complex Monge-Ampère equation with \( \psi(z, u) = e^{Ku} f(z) \) the corresponding problem was also treated in [5] in connection with the problem of finding complete Kähler-Einstein metrics on pseudoconvex domains. In this article we will consider more general cases, including allowing unbounded and non-strictly convex domains when \( \psi = \psi(x, u) \). Our main results are stated as follows.

**Theorem 1.1.** Let \( \Omega \) be a bounded strictly convex domain and \( \psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \), \( \psi > 0 \). Suppose

\[
M(z^+)^p \leq \psi(x, z, p), \quad \forall \ (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n
\]

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where \( p > n \), \( M > 0 \), \( z^+ = \max\{z, 0\} \), and
\[
\psi(x, z, p) \leq \Psi(z)(1 + |p|^n), \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n
\]
where \( \Psi \) is a smooth positive function and
\[
\sup_{z \leq 0} e^{-\varepsilon z} \psi(x, z) < +\infty
\]
for some \( \varepsilon > 0 \). Then there exists a strictly convex solution \( u \in C^\infty(\Omega) \) to (1.1)-(1.2). Moreover, there exist functions \( \underline{h}, \bar{h} \in C(\mathbb{R}^+) \) with \( \underline{h}(r), \bar{h}(r) \to \infty \) as \( r \to 0 \), such that
\[
\underline{h}(d(x)) \leq u(x) \leq \bar{h}(d(x)), \quad \forall x \in \Omega
\]
where \( d \) is the distance function to \( \partial \Omega \).

When \( \psi \) does not depend on \( Du \), Theorem 1.1 holds under weaker conditions. In particular, \( \Omega \) is not necessarily bounded or strictly convex. More precisely, we have

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n \) be a convex domain which does not contain a straight line and \( \psi \in C^\infty(\bar{\Omega} \times \mathbb{R}) \), \( \psi > 0 \). Suppose \( \psi \) satisfies (1.3) for some \( p > n \) and
\[
\sup_{x \in \Omega, z \leq 0} e^{-\varepsilon z} \psi(x, z) < +\infty
\]
for some \( \varepsilon > 0 \). Then (1.1)-(1.2) has a strictly convex solution \( u \in C^\infty(\Omega) \) which also satisfies (1.6). In addition, when \( \Omega \) is bounded, assumption (1.7) can be weakened to allow \( \varepsilon = 0 \).

**Remark 1.3.** Suppose \( \psi_z \geq 0 \) and there exists a convex supersolution \( \bar{u} \in C^2(\Omega) \) satisfying
\[
\det D^2 \bar{u} \leq \psi(x, \bar{u}, D\bar{u}) \quad \text{in } \Omega,
\]
\[
\bar{u} = \infty \quad \text{on } \partial \Omega.
\]
Theorems 1.1 and 1.2 then remain valid, with \( \bar{u}(x) \) in place of the function \( \bar{h}(d(x)) \) in (1.6), without assumption (1.3).

The following non-existence results complement Theorems 1.1 and 1.2 and indicate that the growth conditions in Theorems 1.1 and 1.2 are nearly optimal.

**Theorem 1.4.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^n \). If
\[
0 \leq \psi(x, z, p) \leq M (1 + (z^+)^p)(1 + |p|^q), \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n
\]
for some \( p, q \geq 0, p + q \leq n \), then there exists no convex solution to (1.1)-(1.2).
Theorem 1.5. Let $\Omega$ be a convex domain in $\mathbb{R}^n$. If
\begin{equation}
\psi(x, z, p) \geq M(1 + |p|^n)^{\alpha}, \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n
\end{equation}
where $\alpha > 1$ and $M > 0$, and $\Omega$ contains a ball of radius $a > (M(\alpha - 1))^{-\frac{1}{n}}$, then there exists no convex solution to (1.1)-(1.2).

Note that $\Omega$ is not assumed to be bounded in Theorem 1.5.

Theorem 1.6. Assume $\Omega$ is an unbounded convex domain which contains a straight line. If $\psi > 0$ satisfies (1.3) where $p > n$ then there is no convex solution to (1.1)-(1.2) in $C^2(\Omega)$.

The article is organized as follows. We start with some comparison principle and uniqueness results in section 2. In section 3 we construct some radially symmetric functions which will be used as barriers in proving our theorems. Section 3 also contains the proofs of Theorems 1.4-1.6, while Theorems 1.1 and 1.2 are proved in sections 4 and 5.

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2. THE COMPARISON PRINCIPLE AND UNIQUENESS

Throughout this section $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded convex domain and $u, v \in C^2(\Omega)$ are convex functions satisfying
\begin{equation}
\det D^2u \geq \psi(x, u, Du), \quad \det D^2v \leq \phi(x, v, Dv) \quad \text{in } \Omega
\end{equation}
respectively, where $\psi, \phi \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and
\begin{equation}
\psi(x, z, p) \geq \phi(x, z, p) \geq 0, \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.
\end{equation}
For later references we first recall the following comparison principle which will be used repeatedly.

Lemma 2.1. Assume $u, v \in C(\Omega)$ and $u \leq v$ on $\partial \Omega$. If either $\psi_z(x, z, p) > 0$ or $\phi_z(x, z, p) > 0$ for any $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, then $u \leq v$ in $\Omega$.

Proof. Assume that
\[
u(y) = \max_{\Omega} (u - v) > 0
\]
for some $y \in \Omega$. Then $\det D^2 u(y) \leq \det D^2 v(y)$ as the Hessian $D^2(v - u)$ is positive semidefinite at $y$. On the other hand, we have

$$\det D^2 v(y) \leq \psi(y, u(y), Du(y)) \leq \det D^2 u(y)$$

since $u(y) > v(y)$ and $Du(y) = Dv(y)$. This contradiction shows $u \leq v$ in $\Omega$. □

We have the following comparison principle and uniqueness for solutions of problem (1.1)-(1.2).

**Theorem 2.2.** Assume $u = +\infty$, $v = +\infty$ on $\partial \Omega$ and $v$ is strictly convex in $\Omega$. Suppose $\Omega$ contains the origin in $\mathbb{R}^n$ and $\psi$ satisfies

$$x \cdot D_x \psi(x, z, p) \leq 0, \quad p \cdot D_p \psi(x, z, p) \geq 0, \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{2.3}$$

If, in addition, either

$$\psi(x, \lambda z^+, p) \geq \lambda^p \psi(x, z, p), \quad \forall \lambda \geq 1, \quad (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{2.4}$$

where $p > n$, or there exists $\varepsilon > 0$ such that

$$\psi_z(x, z, p) \geq \varepsilon \psi(x, z, p), \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{2.5}$$

then $u \leq v$ in $\Omega$. In particular, problem (1.1)-(1.2) admits at most one strictly convex solution in $C^2(\Omega)$.

**Remark 2.3.** Assumption (2.4) implies $\psi = 0$ where $z \leq 0$. Thus any strictly convex solution of (1.1)-(1.2) must be positive when (2.4) is satisfied. Note also that $\psi_z > 0$ wherever $\psi > 0$ if either (2.4) or (2.5) holds.

**Remark 2.4.** If $\psi_z(x, z, p) \geq 0$, then $(z^+)^p \psi(x, z, p)$ and $e^{\varepsilon z} \psi(x, z, p)$ satisfy (2.4) and (2.5), respectively.

**Proof of Theorem 2.2.** Let $u \in C^2(\Omega)$ be a convex solution of (1.1)-(1.2). Consider for $0 < \lambda \leq 1$,

$$u_\lambda(x) := \lambda^\alpha u(\lambda x) - a, \quad x \in \Omega_\lambda$$

where $\Omega_\lambda = \{x \in \mathbb{R}^n : \lambda x \in \Omega\}$ and

$$a = 0, \quad \alpha = \frac{2n}{p-n}, \quad \text{if (2.4) holds},$$

$$\alpha = 0, \quad a = \frac{\varepsilon}{\lambda - (2+\alpha) n}, \quad \text{if (2.5) holds}.$$
We calculate

\[
\begin{align*}
\det D^2 u_\lambda(x) &= \lambda^{(2+\alpha)n} \det D^2 u(\lambda x) \\
&= \lambda^{(2+\alpha)n} \psi(\lambda x, u(\lambda x), Du(\lambda x)) \\
&= \lambda^{(2+\alpha)n} \psi(\lambda x, \lambda^{-\alpha}(u_\lambda(x) + a), \lambda^{-(1+\alpha)} Du_\lambda(x)) \\
&\geq \lambda^{(2+\alpha)n} \psi(x, \lambda^{-\alpha}(u_\lambda(x) + a), Du_\lambda(x)) \\
&\geq \psi(x, \lambda^{-\alpha}(u_\lambda(x) + a), Du_\lambda(x))
\end{align*}
\]

by assumption (2.3), when either (2.4) or (2.5) holds.

Now note that \( \Omega \subset \Omega_\lambda \) and \( v - u_\lambda = +\infty \) on \( \partial \Omega \) for all \( 0 < \lambda < 1 \). We claim that \( v \geq u_\lambda \) on \( \Omega \) for all \( 0 < \lambda < 1 \). Indeed, assume that

\[
u_\lambda(y) - v(y) = \max_{\Omega}(u_\lambda - v) > 0
\]

for some \( y \in \Omega \). Then

\[
\psi(y, u_\lambda(y), Du_\lambda(y)) \geq \psi(y, v(y), Dv(y)) \geq \phi(y, v(y), Dv(y)) \geq \det D^2 v > 0
\]

It follows that \( \psi_\lambda(y, u_\lambda(y), Du_\lambda(y)) > 0 \); see Remark 2.3. Consequently, we obtain a contradiction as in the proof of Lemma 2.1. This proves our claim, that is, \( v \geq u_\lambda \) on \( \Omega \) for all \( 0 < \lambda < 1 \). Letting \( \lambda \to 1 \) we obtain \( v \geq u \). \( \square \)

3. Barriers

The main purpose of this section is to construct some radially symmetric strictly convex functions that will be used as barriers in proving our main results. Using these barriers we present proofs of Theorems 1.4-1.6 at the end of this section.

Let \( u(x) = u(|x|) \) be a radially symmetric function. A straightforward calculation shows that

\[
\det D^2 u = \left(\frac{u'}{r}\right)^{n-1} u'', \quad r = |x|,
\]

Thus equation (1.1) takes the form

\[
(u')^{n-1} u'' = r^{n-1} \psi(x, u, u')
\]

for radially symmetric functions.

**Lemma 3.1.** Let \( \eta \in C^1(\mathbb{R}) \) satisfy \( \eta(z) > 0, \eta'(z) \geq 0 \) for all \( z \in \mathbb{R} \). Then, for any \( a > 0 \), there exists a strictly convex radially symmetric function \( v \in C^2(B_a(0)) \)
satisfying
\[ \det D^2 v \geq e^v \eta(v)(1 + |Dv|^n) \text{ in } B_a(0) \]
\[ v = +\infty \text{ on } \partial B_a(0). \]

**Proof.** Consider the initial value problem
\[ \varphi' = \left( \exp \{ r^n e^{\varphi} \eta(\varphi) \} - 1 \right)^{\frac{1}{n}}, \quad r > 0 \]
\[ \varphi(0) = 0. \]
Let \([0, R)\) be the maximal interval on which the solution to (3.4) exists. We claim that \(R\) is finite. Indeed, by (3.4) we have
\[ \varphi'(r) \geq r \left( e^{\varphi} \eta(\varphi) \right)^{\frac{1}{n}} \geq r \left( e^{\varphi(r)} \eta(0) \right)^{\frac{1}{n}}, \quad 0 < r < R \]
since \(\eta' \geq 0\), \(\varphi' \geq 0\) and \(\varphi(0) = 0\). It follows that
\[ n \geq n \left( 1 - e^{-\varphi(\rho)} \right) \geq \int_0^\rho \varphi'(r) e^{-\varphi(r)} \, dr \geq (\eta(0))^\frac{1}{n} \int_0^\rho r \, dr = \frac{1}{2} (\eta(0))^\frac{1}{n} \rho^2 \]
for any \(\rho < R\). This proves that \(R < \infty\). Moreover, by the theory of ordinary differential equations we see that \(\varphi \in C^2[0, R)\) and \(\varphi(R) = +\infty\) as \(\varphi\) is strictly increasing. Rewriting (3.4) in the form
\[ \log \left( 1 + (\varphi')^n \right) = r^n e^\varphi \eta(\varphi) \]
we obtain by differentiation
\[ \frac{n(\varphi')^{n-1}\varphi''}{1 + (\varphi')^n} = (r^n e^\varphi \eta(\varphi))' \geq nr^{n-1} e^\varphi \eta(\varphi), \quad 0 < r < R. \]
In particular, \(\varphi''(r) > 0\) for \(0 < r < R\).

For given \(a > 0\), let \(v\) be defined by
\[ v(x) := \varphi(\lambda |x|) - 2n(-\log \lambda)^+, \quad x \in B_a(0) \]
where \(\lambda = \frac{R}{a}\). Note that \(\varphi'(0) = 0\). We see that \(v \in C^2(B_a(0))\) and is strictly convex since \(\varphi \in C^2[0, R)\) and \(\varphi'' > 0\). By (3.1) and (3.5) we obtain in \(B_a(0)\)
\[ \det D^2 v(x) = \lambda^{2n} \frac{\varphi'(\lambda |x|)^{n-1}\varphi''(\lambda |x|)}{(\lambda |x|)^{n-1}} \]
\[ \geq \lambda^{2n} e^{\varphi(\lambda |x|)} \eta(\varphi(\lambda |x|))(1 + (\varphi'(\lambda |x|))^n) \]
\[ \geq \lambda^{2n} e^{v(x) + 2n(-\log \lambda)^+} \eta(v(x) + 2n(-\log \lambda)^+) (1 + \lambda^{-n}|Dv(x)|^n) \]
\[ \geq e^{v(x)} \eta(v(x))(1 + |Dv(x)|^n). \]
In the last inequality we used the fact that \(\eta\) is nondecreasing. This completes the proof of Lemma 3.1. \(\square\)
Remark 3.2. In sequel we will denote the function $v \in C^2(B_a(0))$ in Lemma 3.1 by $v^{a,\eta}$. We will also write $v^{a,\eta}(x) = v^{a,\eta}(|x|)$ as it is radially symmetric.

By Lemma 2.1 we have

**Lemma 3.3.** Let $u \in C^2(\Omega)$ be a strictly convex solution of (1.1)-(1.2) where $\Omega$ is a bounded convex domain contained in a ball $B_a(x_0)$. Suppose $\psi(x, z, p) \leq e^z \eta(z)(1 + |p|^n)$, $\forall (x, z, p) \in \Omega$, $\eta \in C^1(R)$ satisfies $\eta(z) > 0$ and $\eta'(z) \geq 0$. Then $u(x) \geq v^{a,\eta}(x)$, $\forall x \in \Omega$.

**Proof.** We may assume $x_0 = 0$. For any $r > a$, note that $u - v^{r,\eta} = +\infty$ on $\partial \Omega$. From Lemma 2.1 follows that $u \geq v^{r,\eta}$. Letting $r \to a$ we obtain $u \geq v^{a,\eta}$. □

We next construct a function on $B_a(0)$ that will serve as a upper barrier when $\psi$ satisfies (1.3) with $p > n$. A straightforward calculation shows that when $p > n$ the function

$$w(x) := (1 - |x|^2)^{\frac{n+1}{n-p}}$$

is strictly convex and satisfies the inequality

$$\det D^2 w \leq C(n, p)w^p \text{ in } B_1(0)$$

where $C(n, p) = p[2(n + 1)/(p - n)]^{n+1}$. By rescaling, we have

**Lemma 3.4.** Let $a, M > 0$ and $p > n$ and define $w^{a,M} \in C^\infty(B_a(0))$ by

$$w^{a,M}(x) := \lambda w\left(\frac{x}{a}\right), \quad x \in B_a(0),$$

where

$$\lambda = \left(\frac{C(n, p)}{a^{2n}M}\right)^{\frac{1}{p-n}}.$$

Then

$$\det D^2 w^{a,M} \leq M(w^{a,M})^p \text{ in } B_a(0).$$

**Proof.** One calculate directly

$$\det D^2 w^{a,M}(x) = \frac{\lambda^n}{a^{2n}} \det D^2 w\left(\frac{x}{a}\right) \leq \frac{\lambda^n}{a^{2n}C(n, p)} \left(w\left(\frac{x}{a}\right)^p = M(w^{a,M}(x))^p. \right.$$

This completes the proof. □

¿From Lemma 3.4 and Lemma 2.1 we derive the following comparison lemma.

**Lemma 3.5.** Let $u \in C^2(\Omega)$ be a strictly convex solution of (1.1). Suppose $\psi$ satisfies (1.3) with $p > n$ and $\Omega$ contains a ball $B_a(x_0)$. Then $u(x) \leq w^{a,M}(x - x_0)$ for all $x \in \Omega$. 
The second inequality in (1.6) now follows from Lemma 3.5. More precisely,

**Corollary 3.6.** Let \( u \in C^2(\Omega) \) be a strictly convex solution of (1.1) where \( \Omega \) is a convex (not necessarily bounded) domain in \( \mathbb{R}^n \). Suppose \( \psi \) satisfies (1.3) with \( p > n \) and \( M > 0 \). Then

\[
\forall \ x \in \Omega \quad u(x) \leq \bar{h}(d(x)),
\]

where \( d \) is the distance function to \( \partial \Omega \) and \( \bar{h} \in C^\infty(\mathbb{R}^+) \) is given by

\[
(3.6) \quad \bar{h}(r) := w^{r,M}(0), \quad r > 0.
\]

We next construct subsolutions which are defined on the whole space \( \mathbb{R}^n \) to (1.1) when \( \psi \) satisfies (1.9) with \( p + q \leq n \).

**Lemma 3.7.** Assume \( p, q \geq 0, \ p + q \leq n \) and \( M > 0 \). Then there exists a strictly convex radially symmetric positive function \( \tilde{u} \in C^\infty(\mathbb{R}^n) \) satisfying

\[
(3.7) \quad \det D^2\tilde{u}(x) \geq M\left(1 + (\tilde{u}(x))^p\right)\left(1 + |D\tilde{u}(x)|^q\right), \quad \forall \ x \in \mathbb{R}^n.
\]

**Proof.** Without loss of generality we may assume \( M = 1 \). Let us consider separately three different cases: i) \( q = 0 \); ii) \( q = n \); and iii) \( 0 < q < n \).

**Case i) \( q = 0 \).** Consider the initial value problem

\[
(3.8) \quad \begin{cases}
\varphi' = r\left(1 + \varphi^p\right)^{\frac{2}{n}}, & r > 0 \\
\varphi(0) = 1.
\end{cases}
\]

It is easy to see that when \( p \leq n \) there exists a unique smooth solution \( \varphi \) to (3.8) which is defined for all \( r \geq 0 \) and strictly increasing. Indeed, suppose \( \varphi \) is defined on \([0, R)\). For any \( \rho < R \), by (3.8) we have

\[
\rho^2 = 2\int_0^\rho r dr \\
= 2\int_0^\rho \frac{\varphi'(r)}{(1 + (\varphi(r))^p)\frac{n}{p}} dr \\
\geq \int_0^\rho \frac{\varphi'(r)}{(\varphi(r))^\frac{n}{p}} dr \\
= \begin{cases}
\log \varphi(\rho), & p = n; \\
\frac{n}{n-p}(\varphi(\rho))^\frac{n-p}{n}, & p < n.
\end{cases}
\]

It follows that \( \lim_{\rho \to \infty} \varphi(\rho) = +\infty \) if and only if \( R = +\infty \). We rewrite (3.8) in the form

\[
(\varphi')^n = r^n(1 + \varphi^p)
\]
and take derivatives of both sides to obtain
\[(\varphi')^{n-1}\varphi'' \geq r^{n-1}(1 + \varphi^p).\]

By (3.1) we see that the function \(\tilde{u}(x) := \varphi(|x|)\) is strictly convex and
\[\det D^2 \tilde{u} \geq (1 + \tilde{u}^p) \text{ in } \mathbb{R}^n.\]

**Case ii) \(q = n\).** In this case we note that \(p = 0\). Let \(\varphi \in C^\infty(\mathbb{R}^+)\) be the function given by
\[\varphi(r) := \int_0^r (e^{r^p} - 1)^{\frac{1}{n}} dr, \quad r \geq 0.\]
Then
\[\varphi'(r) = (e^{r^p} - 1)^{\frac{1}{n}} > 0, \quad r > 0.\]
Moreover \(\varphi\) is strictly convex and \(\varphi(0) = \varphi'(0) = 0\). Rewriting (3.9) in the form
\[\log (1 + (\varphi')^n) = r^n\]
and taking derivatives, we obtain
\[(\varphi')^{n-1}\varphi'' = r^{n-1}(1 + (\varphi')^n).\]
Consequently, the function \(\tilde{u}(x) := 1 + \varphi(|x|), \ x \in \mathbb{R}^n\), which is smooth and strictly convex, satisfies
\[\det D^2 \tilde{u} = (1 + |D\tilde{u}|^n) \text{ in } \mathbb{R}^n.\]

**Case iii) \(0 < q < n\).** Let \(\varphi\) be the solution defined in some interval \([0, R]\) of the initial value problem
\[\varphi' = \left((1 + r^n(1 + \varphi^p))^{\frac{n}{n-q}} - 1\right)^{\frac{1}{n}}, \quad r > 0\]
\[\varphi(0) = 1.\]
Then \(\varphi'(0) > 0\) and \(\varphi'(r) > 0\) for \(r > 0\). Moreover,
\[\varphi' \leq (1 + r^n(1 + \varphi^p))^{\frac{1}{n-q}} \leq (1 + \varphi^p)^{\frac{1}{n-q}}(1 + r^n)^{\frac{1}{n-q}} \leq 2\varphi^{\frac{p}{n-q}}(1 + r^n)^{\frac{1}{n-q}}.\]
Since \(p + q \leq n\), we see that \(\varphi\) is defined for all \(r \geq 0\). Rewriting (3.10) in the form
\[\left(1 + (\varphi')^n\right)^{\frac{n-q}{n}} = 1 + r^n(1 + \varphi^p)\]
we obtain by differentiation
\[(n - q)(\varphi')^{n-1}\varphi'' = r^{n-1}(1 + \varphi^p)(1 + (\varphi')^n)^{\frac{n}{n}}\]
\[\geq \frac{r^{n-1}}{2}(1 + \varphi^p)(1 + (\varphi')^q).\]
Consequently the function $\tilde{u}(x) := c\varphi(|x|)$, where $c$ is a constant, is smooth, strictly convex and satisfies (3.7) when $c$ is large enough. □

We conclude this section with proofs of Theorems 1.4-1.6.

**Proof of Theorem 1.4.** Let $u \in C^2(\Omega)$ be a convex solution of (1.1)-(1.2) where $\Omega$ is bounded and $\psi$ satisfies (1.9). Let $\tilde{u} \in C^\infty(\mathbb{R}^n)$ satisfy (3.7) in Lemma 3.7. Note that $u - C\tilde{u} = \infty$ on $\partial\Omega$ for any $C > 0$. Since $\tilde{u} > 0$, we can choose $C > 1$ such that $u(y) - C\tilde{u}(y) = \min_{\Omega}(u - C\tilde{u}) < 0$ for some $y \in \Omega$.

By (1.9) we have

$$\det D^2 u(y) \leq M(1 + (u^+(y))^p)(1 + |Du(y)|^q)$$

$$\leq M(1 + (C\tilde{u}(y))^p)(1 + C^q|D\tilde{u}(y)|^q)$$

$$\leq C^n M(1 + (\tilde{u}(y))^p)(1 + |D\tilde{u}(y)|^q)$$

$$\leq C^n \det D^2 \tilde{u}(y),$$

since $C > 1$, $p + q \leq n$ and $Du(y) = D\tilde{u}(y)$, contradicting the fact that $D^2(u - C\tilde{u})(y)$ is a positive semidefinite matrix. The proof is complete. □

**Proof of Theorem 1.6.** We follow an idea of Cheng and Yau [6]. Assume $\Omega$ contains the line $L : x_1 = 0, \ldots, x_{n-1} = 0$.

Since $\Omega$ is convex, it contains a solid cylinder $\{ x := (x', x_n) \in \mathbb{R}^n : |x'| < \delta \}$ for some $\delta > 0$, where $x' = (x_1, \ldots, x_{n-1})$. For any $\lambda > 0$, let $E_\lambda$ be the ellipsoid

$$\frac{|x'|^2}{\delta^2} + \frac{x_n^2}{(\delta\lambda)^2} \leq 1$$

and consider the function

$$w_\lambda(x) := \lambda^n w^{\delta,M}(x', \lambda^{-1}x_n), \ x \in E_\lambda$$

where $\alpha = \frac{2}{n-p}$ and $w^{\delta,M}$ is as in Lemma 3.4. We have

$$\det D^2 w_\lambda(x) = \lambda^{n\alpha - 2} \det D^2 w^{\delta,M}(x', \lambda^{-1}x_n)$$

$$\leq M(\lambda^n w^{\delta,M}(x', \lambda^{-1}x_n))^p$$

$$= M(w_\lambda(x))^p, \ \forall x \in E_\lambda.$$
Assume now that \( u \in C^2(\Omega) \) is a convex solution of (1.1)-(1.2) in \( \Omega \), where \( \psi \) satisfies (1.3). Since \( w_\lambda = +\infty \) on \( \partial E_\lambda \subset \Omega \), we have by Lemma 2.1

\[
w_\lambda \geq u \text{ in } E_\lambda
\]

Note that \( \alpha < 0 \). Letting \( \lambda \to \infty \), we see that \( u = 0 \) on \( L \). It follows that \( u_{x_n}x_n = 0 \) on \( L \), contradicting the fact that \( \det D^2u > 0 \) everywhere in \( \Omega \). \( \Box \)

Finally, Theorem 1.5 follows from the comparison principle (Lemma 2.1) and the following

**Lemma 3.8.** Let \( \alpha > 1 \) and \( a > 0 \). There exists a strictly convex radially symmetric function \( \bar{u} \in C^2(B_a(0)) \) satisfying

\[
det D^2\bar{u} = \frac{1}{a^n(\alpha - 1)}(1 + |D\bar{u}|^n)^\alpha \text{ in } B_a(0),
\]

\[
\frac{\partial \bar{u}}{\partial \nu} = +\infty \text{ on } \partial B_a(0)
\]

where \( \nu \) is the unit normal to \( \partial B_a(0) \). Moreover, if \( \alpha > \frac{n+1}{n} \) then \( \bar{u} \in C^0(\overline{B_a(0)}) \).

**Proof.** Let \( \beta > 0 \) and consider the function \( \varphi \) defined by

\[
\varphi(r) := \int_0^r \left((1 - r^n)^{-\beta} - 1\right)^{\frac{1}{n}} dr, \quad 0 < r < 1,
\]

Then

\[
1 + (\varphi')^n = \frac{1}{(1 - r^n)^\beta}
\]

and

\[
(\varphi')^{n-1}\varphi'' = \beta r^{n-1}(1 + (\varphi')^n)^{\frac{\beta+1}{\beta}}.
\]

We see that \( \varphi(0) = \varphi'(0) = 0, \varphi''(r) > 0 \) for all \( 0 \leq r < 1 \) and \( \lim_{r\to 1} \varphi'(r) = \infty \). Note also that if \( \beta < n \),

\[
\varphi(r) \leq \int_0^r (1 - r^n)^{-\beta} dr \leq \int_0^r (1 - r)^{-\beta} dr \leq \frac{n}{n - \beta}, \quad \forall r < 1.
\]

Taking \( \beta = \frac{1}{\alpha - 1} \), we obtain the desired function \( \bar{u}(x) := a\varphi(a^{-1}|x|), x \in B_a(0) \). \( \Box \)
4. Proof of Theorem 1.1

By assumption (1.5) we may find a positive nondecreasing function \( \eta \in C^\infty(\mathbb{R}^n) \) satisfying
\[
e^{e_2} \eta(z) \geq \max_{y \leq z} \Psi(y), \quad \forall z \in \mathbb{R}.
\]
For simplicity we will assume throughout this section that \( \varepsilon = 1 \) as this may be achieved by rescaling.

We first assume \( \Omega \) to be smooth. For each integer \( k \geq 1 \), consider the Dirichlet problem
\[
det D^2 u = \psi(x, u, Du) > 0 \text{ in } \Omega,
\]
\[u = k \text{ on } \partial \Omega.
\]
Since \( \Omega \) is bounded, we may choose \( r > 0 \) sufficiently large such that \( \Omega \subset B_r(0) \) and \( v^{r, \eta} \leq 1 \) on \( \partial \Omega \). It then follows from Lemma 2.1 that \( v^{r, \eta} \leq u \leq k \) in \( \Omega \) and, therefore,
\[|u| \leq C_k \]
for any convex solution \( u \) of (4.2), where \( C_k \) is a constant depending on \( k \). By a result of Lions [10] (see also [4]), there exists for each \( k \) a strictly convex function \( u_k \in C^2(\Omega) \) satisfying
\[
det D^2 u_k \geq \Psi(C_k)(1 + |Du_k|^n) \text{ in } \Omega,
\]
\[u_k = k \text{ on } \partial \Omega.
\]
Note that \( u_k \) is a subsolution of \( (4.2) \). By a theorem of Caffarelli-Nirenberg-Spruck [4] there exists a strictly convex solution \( u_k \in C^\infty(\overline{\Omega}) \) of (4.2) satisfying \( u_k \geq u_k \) in \( \overline{\Omega} \) for each \( k \geq 1 \). Moreover, \( u_k \) satisfies the a priori estimate
\[
\|u_k\|_{C^2, \alpha}(\overline{\Omega}) \leq C(k), \quad k \geq 1
\]
where \( C(k) > 0 \) depends on \( k \). We next need to derive a priori interior estimates which are independent of \( k \).

**Proposition 4.1.** For an arbitrary compact subset \( K \) of \( \Omega \), there exists constant \( C \) independent of \( k \) such that
\[
\|u_k\|_{C^2, \alpha(K)} \leq C, \quad \forall k \geq 1.
\]

The proof of this estimate is based on the following lemma and some well known results in the theory of Monge-Ampère and more general fully nonlinear elliptic equations.
Lemma 4.2. There exists $a > 0$ depending only on $\Omega$ and an decreasing sequence $a_k \to a$ ($k \to \infty$) such that

\begin{equation}
(4.6) \quad v^{a_k,\eta}(a - d(x)) \leq u_k(x) \leq \bar{h}(d(x)), \quad \forall \, x \in \Omega, \, k \geq 1,
\end{equation}

where $d$ is the distance function to the boundary of $\Omega$. (See (3.6) and Remark 3.2 for the definitions of $\bar{h}$ and $v^{a_k,\eta}$, respectively.)

Proof. The second inequality follows from Corollary 3.6. Next, let $a > 0$ be the smallest number such that for any point $\bar{x} \in \partial \Omega$ there is a ball $B_a(x_0)$ of radius $a$ with $\Omega \subset B_a(x_0)$ and $\overline{\Omega} \cap \partial B_a(x_0) = \{\bar{x}\}$; such a number exists as $\Omega$ is bounded and strictly convex. Choose $a_1 > a_2 > \cdots > a_k > \cdots, a_k \to a$ ($k \to \infty$), such that $v^{a_k,\eta}(a) = k$ for each $k \geq 1$. For $x \in \Omega$, choose $\bar{x} \in \partial \Omega$ and a ball $B_a(x_0)$ such that $d(x) = |x - \bar{x}|, \Omega \subset B_a(x_0)$ and $\overline{\Omega} \cap \partial B_a(x_0) = \{\bar{x}\}$. Since $v^{a_k,\eta}(x - x_0) \leq u_k(x)$ for all $x \in \partial \Omega$, by Lemma 2.1

\begin{equation}
v^{a_k,\eta}(x - x_0) \leq u_k(x), \quad \forall \, x \in \Omega.
\end{equation}

This proves the first inequality in (4.6). \hfill \square

For convenience let us now introduce some notation. Let $h, v_k$ denote the functions defined in $\Omega$ by

\begin{equation*}
h(x) := \bar{h}(d(x)), \quad v_k(x) := v^{a_k,\eta}(a - d(x)), \quad x \in \Omega
\end{equation*}

respectively. For $l > 0$ and $k \geq 1$ write

\begin{align*}
H_l & := \{x \in \Omega : h(x) < l\}, \\
U_{k,l} & := \{x \in \Omega : u_k(x) < l\}, \\
V_{k,l} & := \{x \in \Omega : v_k(x) < l\}.
\end{align*}

By (4.6) we have $H_l \subset U_{k,l} \subset V_{k,l}$ for each $k \geq 1$.

Proof of Proposition 4.1. Let $K$ be a compact subset of $\Omega$. We may choose $l > 0$ and then $k_0$ sufficiently large so that $K \subset H_{\frac{l}{2}}$ and $\overline{\Omega \setminus U_{k_0,4l}} \subset \Omega$. From (4.6) we see that

\begin{equation}
(4.7) \quad |u_k| \leq C_0 \text{ in } \overline{U}_{k,2l}, \quad \forall \, k \geq k_0
\end{equation}
where $C_0$ is independent of $k$. Moreover, by the strict convexity of $u_k$,
\[
\max_{\mathcal{U}_{k,2l}} |Du_k| = \max_{\partial \mathcal{U}_{k,2l}} |Du_k| 
\]
\[
\leq \max_{x \in \partial \mathcal{V}_{k_0,4l}} \frac{u_k(x) - 2l}{\text{dist}(\mathcal{U}_{k,2l}, \partial \mathcal{V}_{k_0,4l})} 
\]
\[
\leq \max_{x \in \partial \mathcal{V}_{k_0,4l}} \frac{h(x) - 2l}{\text{dist}(\mathcal{V}_{k,2l}, \partial \mathcal{V}_{k_0,4l})} 
\]
\[
\leq \max_{x \in \partial \mathcal{V}_{k_0,4l}} \frac{h(x) - 2l}{\text{dist}(\mathcal{V}_{k_0,2l}, \partial \mathcal{V}_{k_0,4l})} 
\]
\[
= C_1 
\]
for all $k \geq k_0$, where the last two inequalities follow from the facts that $u_k < h$ and $U_{k,2l} \subset V_{k,2l} \subset V_{k_0,2l}$ since $v_k \geq v_{k_0}$ for $k \geq k_0$.

Next, applying Pogorelov’s interior estimates (cf. [7]) we obtain
\[
|D^2 u_k(x)| \leq \frac{C_2}{\text{dist}(x, \partial \mathcal{U}_{k,2l})}, \quad \forall \ x \in \mathcal{U}_{k,2l}, k \geq k_0 
\]
where $C_2$ depends on $C_0, C_1$ and the $C^2$ norm of $\psi$, as well as $\min \psi$, in $\bar{\Omega} \times \{|z| \leq C_0\} \times \{|p| \leq C_1\}$, but is independent of $k$. Note that, since $H_l \subset H_{2l} \subset U_{k,2l}$,
\[
\text{dist}(H_l, \partial U_{k,2l}) \geq \text{dist}(H_l, \partial H_{2l}). 
\]
It follows from (4.9) that
\[
\|D^2 u_k\|_{C^0(\overline{\Omega})} \leq \frac{C_2}{\text{dist}(H_l, \partial H_{2l})}. 
\]
Finally, by the Evans-Krylov theorem (cf. [3]) we have
\[
\|D^2 u_k\|_{C^0(\overline{\mathcal{U}_{k,2l}})} \leq C_3, \quad \forall \ k \geq k_0 
\]
where $C_3$ is independent of $k$. Now (4.5) follows from (4.7), (4.8), (4.10) and (4.11), combining with (4.4) for $k \leq k_0$. \hfill \Box

By Proposition 4.1, there exists a subsequence $\{u_{k_j}\}$ and $u \in C^{2,\alpha}(\Omega)$ such that
\[
\lim_{j \to \infty} \|u_{k_j} - u\|_{C^{2,\alpha}(K)} = 0
\]
for any compact subset $K$ of $\Omega$. We see that $u$ is strictly convex and solves (1.1).

From (4.6) we obtain
\[
\tilde{h}(d(x)) := v^{\alpha-n}(a - d(x)) \leq u(x) \leq \tilde{h}(d(x)), \quad \forall \ x \in \Omega.
\]
Consequently, $u = +\infty$ on $\partial \Omega$. This completes the proof of Theorem 1.1 when $\Omega$ is smooth.
Suppose now that $\Omega$ is not smooth. We choose a sequence of smooth strictly convex domains
\[ \Omega_1 \subseteq \cdots \subseteq \Omega_k \subseteq \cdots \subseteq \Omega \]
such that
\[ \Omega = \bigcup_{k=1}^{\infty} \Omega_k. \]
For each $k \geq 1$, let $u_k \in C^\infty(\Omega_k)$ be a strictly convex solution of the problem
\[
\det D^2 u = \psi(x, u, Du) \text{ in } \Omega_k, \\
u = \infty \text{ on } \partial \Omega_k.
\]
We have
\[
v^{\alpha, \eta}(a - d(x)) \leq u_k(x) \leq \overline{h}(d_k(x)), \quad \forall \, x \in \Omega_k
\]
where $a$ is as in (4.12) and $d_k$ is the distance function to $\partial \Omega_k$. Using this in place of Lemma 4.2 we can derive the estimate (4.5) as before, and therefore obtain a subsequence which converges to a solution $u \in C^{2,\alpha}(\Omega)$ of (1.1)-(1.2) satisfying (4.12). That $u \in C^\infty(\Omega)$ follows from the elliptic regularity theory. The proof of Theorem 1.1 is complete.

**Remark 4.3.** We remark that as an alternative approach one may first prove the existence of a convex weak solution and then apply the strict convexity and regularity theorems of Caffarelli [1], [2] to prove Theorem 1.1.

### 5. Proof of Theorem 1.2

The proof of Theorem 1.2 follows that of Theorem 1.1 in the previous section except that we have to reconstruct lower barriers when $\Omega$ is not unbounded or strictly convex. To this end we consider the equation
\[
\det D^2 u = F(u) \text{ in } \Gamma^+ := \{ x \in \mathbb{R}^n : x_i > 0 \}.
\]
where $F$ is a positive nondecreasing function. When $F(u) = e^{2u}$, Cheng and Yau [6] observed that $u(x) := -\log(x_1 \ldots x_n)$ is a strictly convex solution of (5.1) in $\Gamma^+$. Inspired by this we look for solutions to (5.1) of the form
\[ u(x) = \varphi(a - \log(x_1 \ldots x_n)), \quad x = (x_1, \ldots, x_n) \in \Gamma^+ \]
for some function $\varphi$, where $a$ is a constant. We calculate
\[ u_{x_i} = \frac{-\varphi'}{x_i}, \quad u_{x_i x_j} = \frac{1}{x_i x_j} (\varphi'' + \varphi' \delta_{ij}). \]
It follows that
\[ \det D^2 u = \frac{1}{(x_1 \ldots x_n)^2}(\varphi')^{n-1}(n\varphi'' + \varphi'). \]

Equation (5.1) thus reduces to
\[ (\varphi')^{n-1}(n\varphi'' + \varphi') = e^{2(a-t)}F(\varphi). \]

**Lemma 5.1.** Let \( a > 0 \) and \( F \in C^\infty(\mathbb{R}) \) satisfying \( F > 0, F' \geq 0 \) and
\[ F(z) \geq M(z^+)^p \forall z \in \mathbb{R} \]
where \( p > n \). Then there exists a strictly increasing function \( \varphi \in C^\infty(\mathbb{R}^+) \) with
\[ (\varphi')^{n-1}(n\varphi'' + \varphi') \geq e^{2(a-t)}F(\varphi(t)), \forall t \geq 0 \]
and
\[ \lim_{t \to +\infty} \varphi(t) = +\infty. \]

**Proof.** We will construct \( \varphi \) from \( F \). For convenience we write \( f := A(e^{2a}F)^{\frac{1}{n}} \) where \( A \) is an undetermined constant, and define
\[ g(z) := \int_0^z \frac{dz}{f(z)}. \]
We see that \( g \) is a strictly increasing function defined for all \( z \in \mathbb{R} \). Let \( g^{-1} \) denote the inverse function of \( g \) and define
\[ \varphi(t) := g^{-1}(B - e^{-\beta t}), \]
where \( \beta \) is a constant to be determined and
\[ B := \int_0^\infty \frac{dz}{f(z)} < \infty \]
by assumption (5.3). It is clear that \( \varphi \) satisfies (5.5). We calculate
\[ \varphi'(t) = \frac{\beta e^{-\beta t}}{g'(\varphi(t))} = \beta e^{-\beta t}f(\varphi(t)) > 0 \forall t \in \mathbb{R} \]
and
\[ \varphi''(t) = \beta e^{-\beta t}(f'(\varphi(t))\varphi'(t) - \beta f(\varphi(t))) \]
\[ = \beta^2 e^{-\beta t}f(\varphi(t))(e^{-\beta t}f'(\varphi(t)) - 1) \]
\[ \geq -\beta^2 e^{-\beta t}f(\varphi(t)) \]
since \( f'(\varphi(t)) \geq 0 \). It follows that
\[ (\varphi')^{n-1}(n\varphi'' + \varphi') \geq \beta^n(1 - n\beta)e^{-n\beta t}(f(\varphi(t)))^n. \]
Taking \( \beta < \frac{1}{n} \) and \( A = \beta^{-1}(1 - n\beta)^{-1/n} \) we obtain (5.4). \( \square \)
From the above proof one sees that a slight modification will yield the following.

**Lemma 5.2.** Let \( a > 0 \) and \( F(z) = e^{\varepsilon z} \eta(z) \) where \( \varepsilon > 0 \) and \( \eta \in C^\infty(\mathbb{R}) \) is a positive nondecreasing function. Then there exists a strictly increasing function \( \varphi \in C^\infty(\mathbb{R}^+) \) satisfying (5.4) for all \( t \in \mathbb{R} \) and (5.5). Moreover, \( \varphi \) is a convex function.

**Proof.** As in the proof of Lemma 5.1 we define \( \varphi \) by (5.7). Note that in this case we still have \( B := \int_0^\infty dz/f(z) < +\infty \). Write \( s = \varphi(t) \). By (5.7) we have

\[
e^{-\beta t} = B - g(s) = \int_s^\infty \frac{dz}{f(z)} \geq \frac{1}{A(e^{2a}\eta(s))^{1/n}} \int_s^\infty e^{-\varepsilon z/n} dz = \frac{n}{\varepsilon f(s)}
\]

since \( \eta \) is nondecreasing. Next,

\[
f'(s) = \frac{A^n e^{2a} F'(s)}{n(f(s))^{n-1}} = \frac{A^n e^{2a} \varepsilon \eta(s) + \eta'(s)}{n(f(s))^{n-1}} \geq \frac{\varepsilon A^n e^{2a} F(s)}{n(f(s))^{n-1}} = \frac{\varepsilon f(s)}{n}
\]

since \( \eta' \geq 0 \). Consequently, \( \varphi''(t) \geq 0 \) by (5.8). Finally, taking \( \beta = A^{-1} = \frac{2}{n} \) we have

\[
(\varphi')^{n-1}(n\varphi'' + \varphi') \geq (\varphi'(t))^n = \beta^n e^{-n\beta t}(f(\varphi(t)))^n = e^{2(a-t)} F(\varphi(t))
\]

for all \( t \in \mathbb{R} \). \( \square \)

**Remark 5.3.** Let \( \varphi \) be the unique solution of (5.2) satisfying the initial data

\[
(5.10) \quad \varphi(0) = \varphi(0), \quad \varphi'(0) = \varphi'(0).
\]

We have \( \varphi(t) \leq \varphi(t) \) for all \( t > 0 \) where \( \varphi(t) \) is defined. Writing equation (5.2) in the form

\[
(5.11) \quad (e^t(\varphi'))^n = e^{2a-t} F(\varphi),
\]

we see that \( 0 < \varphi'(t) \leq \varphi'(t) \) for all \( t > 0 \) where \( \varphi(t) \) is defined. By the extension theorem we see that \( \varphi \) is defined for all \( t > 0 \). However, \( \varphi \) may be bounded above on the whole \( \mathbb{R}^+ \). So we can not replace \( \varphi \) by \( \varphi \) in the construction below.

We are now ready to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** As in the last part of the proof of Theorem 1.1 we choose a sequence of bounded smooth strictly convex domains \( \Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_k \subseteq \cdots \subseteq \Omega \) such that \( \Omega = \bigcup \Omega_k \) and consider for each \( k = 1, 2, \ldots \)

\[
(5.12) \quad \det D^2 u = \psi(x, u) \text{ in } \Omega_k, \quad u = k \text{ on } \partial \Omega_k.
\]
Let $u_k \in C^\infty(\Omega_k)$ be a strictly convex solution of (5.12); the existence of $u_k$ follows from [4]. By assumption (1.3) and Corollary 3.6 we have
\begin{equation}
(5.13)\quad u_k(x) \leq \bar{h}(d_k(x)), \quad \forall x \in \Omega_k
\end{equation}
where $d_k$ is the distance function to $\partial \Omega_k$. We need an \textit{a priori} lower bound for $u_k$, which is derived below (Lemma 5.4). With the aid of such estimates, the rest of proof proceeds as that of Theorem 1.1. \hfill \Box

\begin{lemma}
There exists an increasing sequence of functions $h_k \in C(\mathbb{R}^+)$ such that
\begin{equation}
(5.14)\quad \lim_{k \to \infty} \lim_{r \to 0} h_k(r) = +\infty
\end{equation}
and
\begin{equation}
(5.15)\quad u_k(x) \geq h_k(d(x)) \quad \forall x \in \Omega_k,
\end{equation}
for all $k$ sufficiently large, where $d(x) = \text{dist}(x, \partial \Omega)$.
\end{lemma}

\begin{proof}
By assumption (1.7) we may find a function $\eta \in C^\infty(\mathbb{R})$ with $\eta > 0$, $\eta' \geq 0$ and $F(z) := \exp(\varepsilon \eta(z)) \geq \psi(x, z)$ for all $(x, z) \in \overline{\Omega} \times \mathbb{R}$, where $\varepsilon \geq 0$ as in Theorem 1.2. We consider two different cases.

\textbf{Case i)} $\varepsilon > 0$. We apply Lemma 5.2 with $a = 0$ to obtain $\varphi \in C(\mathbb{R})$ satisfying (5.4) and (5.5). By the assumption that $\Omega$ does not contain any straight lines we may assume $\Omega \subset \Gamma^+ = \{ x \in \mathbb{R}^n : x_i > 0 \}$. For a fixed point $x_0 \in \Omega$ let $\bar{x}$ be a point on $\partial \Omega$ such that $d(x_0) = \text{dist}(x_0, \bar{x})$. We may assume $\bar{x}$ lies on the hyperplane $x_1 = 0$. For each integer $k \geq 1$ let
\begin{equation}
u_k(x) := \varphi \left( - \log ((x_1 + b_k) \cdots (x_n + b_k)) \right), \quad x \in \Gamma^+
\end{equation}
where $b_k$ satisfies $\varphi(-n \log b_k) = k$. Then $\nu_k \in C^\infty(\Gamma^+)$ is strictly convex and
\begin{equation}
det D^2 \nu_k(x) \geq F(\nu_k(x)) \geq \psi(x, \nu_k(x)), \quad x \in \Omega.
\end{equation}
Note that $\nu_k \leq u_k$ on $\partial \Omega_k$. By Lemma 2.1 we obtain
\begin{equation}
u_k \leq u_k \quad \text{in } \Omega.
\end{equation}
In particular, $\nu_k(x_0) \leq u_k(x_0)$ if $k$ is sufficiently large and $x_0 \in \Omega_k$. The function
\begin{equation}
h_k(r) := \min_{|x-\bar{x}|=r, x \in \Gamma^+} \nu_k(x)
\end{equation}
then has the desired properties.

\textbf{Case ii)} $\varepsilon = 0$ and $\Omega$ is bounded. We may assume that $\Omega \subset Q := \{ x \in \mathbb{R}^n : 0 < x_i < \rho, 1 \leq i \leq n \} \subset \mathbb{R}^n$
and $\bar{x} = (0, \frac{a}{n}, \ldots, \frac{a}{n})$, where $\rho$ is the diameter of $\Omega$. Applying Lemma 5.1 to $F$ with $a = a_k := n \log(\rho + b_k)$, where $b_k > 0$ is to be determined, we obtain $\varphi_k \in C^\infty(\mathbb{R}^+)$ satisfying (5.4) for $t \geq 0$ and (5.5). Let

$$u_k(x) := \varphi_k(a_k - \log((x_1 + b_k) \cdots (x_n + b_k))), \quad x \in Q$$

and choose a decreasing sequence $b_k$ such that $\varphi_k(a_k - n \log b_k) \leq k$ for all $k$ sufficiently large. We now can proceed as in the previous case.

□

This completes the proof of Theorem 1.2. Finally, it is clear from the proof that with minor modifications one can prove Theorems 1.1 and 1.2 with assumption (1.8) in place of (1.3) when $\psi_z \geq 0$. (See Remark 1.3.)

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