An upper bound for the permanent of (0, 1)-matrices

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Abstract

A novel upper bound for the permanent of (0, 1)-matrices is obtained in this paper, by using an unbiased estimator of permanent [Random Structures Algorithms 5 (1994) 349]. It is a refinement of Minc’s very famous result, and apparently tighter than the current best general bound in some cases.

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1. Introduction

Let \( A = [a_{ij}] \) be an \( n \times n \) matrix with 0-1 entries, which is called a \((0, 1)\)-matrix for brevity. Its permanent is defined as

\[
\text{Per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)},
\]

where the sum goes over every permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \). \( \text{Per}(A) \) looks similar to the determinant of matrices. However, it is much harder to be computed. Valiant [6] proves that evaluating the permanent of a \((0, 1)\)-matrix is a \#P-complete
problem. The following are the two most well known upper bounds for the permanent of \((0, 1)\)-matrices.

**Theorem 1.1** \([3]\). Let \(A = [a_{ij}]\) be a \((0, 1)\)-matrix of order \(n\). Its row sums are defined by \(r_i = \sum_{j=1}^{n} a_{ij}, i = 1, 2, \ldots, n\). Then

\[
\text{Per}(A) \leq \prod_{i=1}^{n} \left( r_i + \frac{1}{2} \right).
\]

**Theorem 1.2** \([1]\). Let \(A = [a_{ij}]\) be a \((0, 1)\)-matrix with row sums \(r_1, r_2, \ldots, r_n\). Then

\[
\text{Per}(A) \leq \prod_{i=1}^{n} \left( r_i! \right)^{1/r_i}.
\]

Theorem 1.2 is the best upper bound known for the permanent of \((0, 1)\)-matrices. It was conjectured by Minc in 1963 \([3]\). The bound given by Theorem 1.2 is tighter than that of Theorem 1.1. A novel upper bound is obtained in this paper. Our tool is an unbiased estimator for the permanent of \((0, 1)\)-matrices given by Rasmussen \([5]\). The new upper bound is a refinement of the result of Theorem 1.1, the very famous Minc bound, and sharper than that of Theorem 1.2 in some special cases.

2. Rasmussen’s estimator (RAS)

Let \(A(i, j)\) be the \((n-1) \times (n-1)\) matrix obtained by deleting the \(i\)th row and the \(j\)th column from the matrix \(A\); and \(A(i, : )\) be the \(i\)th row of matrix \(A\). For any set \(S\), let \(|S|\) be the number of its elements. Algorithm 2.1 gives Rasmussen’s unbiased estimator for permanent \([5]\).

**Algorithm 2.1 (RAS)**

**Input:** \(A\)—an \(n \times n\) \((0, 1)\)-matrix.

**Output:** \(X_A\)—the estimation of \(\text{Per}(A)\).

**step0:** Let \(p_i = 0\) for \(i = 1, \ldots, n\);

**step1:** For \(i = 1\) to \(n\)

- If \(|A(1, :)| = 0\), goto step2;
- Choose \(a_{1j}\) from the nonzero elements of \(A(1, :)\) uniformly at random;
- Let \(p_i = |A(1, :)|\);
- \(A = A(1, j)\);

**End**;

**step2:** \(X_A = p_1 \times \cdots \times p_n\).
Through one stochastic experiment of Algorithm 2.1, one obtains either a permutation \( \sigma \) of \( \{1, 2, \ldots, n\} \) such that \( a_{i, \sigma(i)} = 1 \) for all \( i = 1, 2, \ldots, n \), or a permutation \( \sigma' \) of a subset of \( \{1, 2, \ldots, n\} \) such that \( (\sigma'(1), \ldots, \sigma'(j)) \), \( j < n \). We call the permutation obtained in this way a “random path”. \( X_A \) given by Algorithm 2.1 is called random path value. It defines a random variable. A random path \( \sigma \) is said to be feasible if \( X_A(\sigma) \neq 0 \). Note that permutations \( \sigma \) which satisfy \( \prod_{i=1}^{n} a_{i, \sigma(i)} = 1 \) are one-to-one correspondent to feasible paths.

**Theorem 2.1.** Let \( X_A \) be the random variable given by Algorithm 2.1. Then
\[
E[X_A] = \text{Per}(A).
\]

**Proof.** For a feasible path \( \sigma = (j_1, j_2, \ldots, j_n) \), one can get
\[
P[\sigma = (j_1, j_2, \ldots, j_n)] = P[\sigma(1) = j_1, \ldots, \sigma(n) = j_n]
= P[\sigma(1) = j_1] \cdot P[\sigma(2) = j_2 | \sigma(1) = j_1] \cdots
= \frac{1}{p_1} \cdot \frac{1}{p_2} \cdots \frac{1}{p_n}
= \frac{1}{X_A(\sigma)},
\]
where \( P[\sigma] \) represents the probability that the random path \( \sigma \) is chosen in the process of Algorithm 2.1. Denote all feasible paths of matrix \( A \) as \( \{\sigma_1, \ldots, \sigma_N\} \) where \( N = \text{Per}(A) \). Hence we have
\[
E[X_A] = \sum_{i=1}^{N} \frac{1}{X_A(\sigma_i)} \cdot X_A(\sigma_i) = N = \text{Per}(A). \quad \square
\]

**3. Main results**

Let \( \lceil x \rceil \) denote the smallest integer such that \( \lceil x \rceil \geq x \), and \( \lfloor x \rfloor \) denote the largest integer such that \( \lfloor x \rfloor \leq x \). The main result of this paper is the following.

**Theorem 3.1.** Let matrix \( A = [a_{ij}] \) and its row sums \( r_i, \quad i = 1, 2, \ldots, n \) be given as in Theorem 1.1. Then
\[
\text{Per}(A)^2 \leq \prod_{i=1}^{n} a_i (r_i - a_i + 1),
\]
where \( a_i = \min \left\{ \lceil \frac{r_i+1}{2} \rceil, \lfloor \frac{i}{2} \rfloor \right\} \).

**Proof.** Matrix \( B = [b_{ij}] \) is defined such that \( b_{ij} = a_{n-i+1, j} \). Hence \( \text{Per}(A) = \text{Per}(B) \). By the estimator in Algorithm 2.1, every feasible path \( \sigma = \{j_1, \ldots, j_n\} \)
of matrix $A$ has a dual path $\sigma' = \{j_n, \ldots, j_1\}$ of matrix $B$. This clearly gives a one-to-one correspondence between $\sigma$ and $\sigma'$.

Denote $S_i = \{j \mid a_{ij} = 1, \ 1 \leq j \leq n\}$, $p_i = |S_i \cup \bigcup_{i=1}^n \{j_i\}|$ and $p_i' = |S_i \setminus \bigcup_{i=1}^n \{j_i\}|$. Then one gets

$$X_A(\sigma) = \prod_{i=1}^n p_i, \quad X_B(\sigma') = \prod_{i=1}^n p_i'.$$

Note that $p_i + p_i' = r_i + 1$, so we have

$$X_B(\sigma') = \prod_{i=1}^n (r_i - p_i + 1),$$

and hence

$$X_A(\sigma)X_B(\sigma') = \prod_{i=1}^n p_i(r_i - p_i + 1) \leq \prod_{i=1}^n a_i(r_i - a_i + 1),$$

where $a_i = \min \{\lceil \frac{r_i + 1}{2} \rceil, \ n - i + 1\}$. Rearrange the rows of matrix $A$ in the order of $\{1, n, 2, n - 1, \ldots, i, n - i + 1, \ldots\}$, the corresponding $a_i$ can be rewritten as $\min \{\lceil \frac{r_i + 1}{2} \rceil, \ \lceil \frac{i}{2} \rceil\}$. Denote $\text{Per}(A) = N$, we have

$$\text{Per}(A)^2 = N^2 \leq \frac{n}{X_A(\sigma_1) + \cdots + X_A(\sigma_N)} \cdot \frac{n}{X_B(\sigma_1') + \cdots + X_B(\sigma_N')} \leq \prod_{i=1}^n a_i(r_i - a_i + 1).$$

\[\square\]

**Theorem 3.2.** Let matrix $A = [a_{ij}]$ and its row sums $r_i, \ i = 1, 2, \ldots, n$ be given as in Theorem 1.1, and $a_i$ be as in Theorem 3.1. Then

$$\prod_{i=1}^n a_i(r_i - a_i + 1) \leq \left(\prod_{j=1}^n \frac{r_j + 1}{2}\right)^2.$$

**Proof.** Assume $r_i \in \mathbb{N}, 1 \leq i \leq n$. For any $1 \leq i \leq n$, it is easy to show that

$$a_i(r_i - a_i + 1) \leq \left(\frac{r_i + 1}{2}\right)^2,$$

where $a_i = \min \{\lceil \frac{r_i + 1}{2} \rceil, \ \lceil \frac{i}{2} \rceil\}$. Hence

$$\prod_{i=1}^n a_i(r_i - a_i + 1) \leq \left(\prod_{j=1}^n \frac{r_j + 1}{2}\right)^2. \ \ [\square]$$
Theorem 3.2 shows that the upper bound given by Theorem 3.1 is always tighter than that of Theorem 1.1. Hence Theorem 3.1 gives a refinement of Minc’s result.

Numerical experiments show that the bound given by Theorem 3.1 is apparently tighter than that of Theorem 1.2 for a large proportion of matrices when \( n \) is relatively small, though this proportion decreases as \( n \) grows. The following example shows that the result of Theorem 3.1 is tighter than that of Theorem 1.2 for some special classes of problems.

Example 3.1. Consider matrices consisting of \( k (0 < k < n) \) full rows followed by \( n - k \) rows with single or two 1’s. Note the fact that

(i) \( \prod_{j=1}^{n} a_j (r_j - a_i + 1) \left( \frac{1}{2} \right)^{\frac{1}{r_j}} = r_j! \),

(ii) \( \prod_{i=1}^{k} \left( n - \left\lceil \frac{i}{2} \right\rceil + 1 \right) \left( \prod_{i=1}^{n} \left( n - \left\lceil \frac{i}{2} \right\rceil + 1 \right) \right)^{\frac{1}{n}} = (n!)^{\frac{2}{n}} \).

Hence the upper bound given by Theorem 3.1 is sharper than that of Theorem 1.2.

Acknowledgments

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References