Hall type algebras associated to triangulated categories

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1. Ringel-Hall type algebras associated to abelian categories
2. Hall type algebras associated to triangulated categories
3. An improvement of Peng-Xiao's theorem

Outline

Ringel-Hall type algebras I

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ for an abelian category $\mathcal{A}$ with finite conditions:

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where $[X]$ isomorphism class. The multiplication:

$$[X]*[Y] = \sum_{[L]} g_{XY}^{L}[L],$$

where the Hall number
$g_{XY}^{L} =| \{(f,g) \mid 0 \to X \xrightarrow{f} L \xrightarrow{g} Y \to 0\}/\mathrm{Aut}X \times \mathrm{Aut}Y | \in \mathbb{Z}$.

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New types (potential):

- **Constructible functions on Artin stacks** are used by Kapranov- Vasserot, Joyce.
- **Cluster multiplication formulas**, as by Caldero-Chapoton-Keller, Geiss-Leclerc-Schroeer, Hubery, Xiao-Xu, Palu.

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- Elliptic Lie algebras of type $D_4^{(1,1)}$ and $E_{6,7,8}^{(1,1)}$ by Peng and Lin (2004).

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- Loop algebras of Kac-Moody algebras via Hall algebras of weighted projective lines by Schiffmann (2004).
- A characterization of Drinfeld new presentation of quantum loop algebras via Hall algebras of weighted projective lines by Dou-Jiang-Xiao (2010).
- Counting stability conditions and wall-crossing via holomorphic functions to Hall Lie algebras of stack functions by Joyce (2007).

Ringel-Hall type algebras IV

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$$\begin{array}{ccccc} X & \longrightarrow & L & \longrightarrow & Y \\ \| & & \downarrow & & \downarrow \\ X & \longrightarrow & M & \longrightarrow & L' \\ & & \downarrow & & \downarrow \\ & & Z & = & Z \end{array}$$

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Question: How to define the Hall algebra $\mathcal{H}(\mathcal{C})$ associated to a triangulated category $\mathcal{C}$?
Let k be a field with $|k| = q$ and $\mathcal{C}$ be a k-additive finitary, Krull-Schmidt triangulated category.

- $\mathcal{C}$ is (left) homologically finite if for $X, Y \in \mathcal{C}, \mathrm{Hom}(X[i], Y) = 0$ for $i >> 0$.
  Example: the derived category of an abelian category
- $\mathcal{C}$ is n-periodic if the translation functor $T = [1]$ such that $T^n \cong 1$.
  Example: 2-periodic orbit categories of $\mathcal{D}^b(H)$ for hereditary category H as triangulated categories of singularities in homological mirror symmetry.

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Theorem (Toën, Xiao-Xu)

Let $\mathcal{C}$ be a (left) homologically finite triangulated category and $\mathcal{H}(\mathcal{C})$ be the $\mathbb{Q}$-space with the basis $\{[X] \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by

$$[X] * [Y] = \sum_{[L]} F_{XY}^{L} [L],$$

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $[0]$.

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$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $[0]$. Here

$$F_{XY}^{L} = \frac{|(X, L)_Y|}{|\mathrm{Aut}X|} \cdot \Big(\prod_{i>0} \frac{|Hom\,(X[i], L)|^{(-1)^i}}{|Hom\,(X[i], X)|^{(-1)^i}}\Big) \in \mathbb{Q}.$$

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Theorem (Chen-Xu)

Let $\mathcal{C}$ be an odd periodic triangulated category and $\mathcal{H}(\mathcal{C})$ be the $\mathbb{Q}(v)$-space with the basis $\{[X] \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by

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$$[X] * [Y] = \sum_{[L]} F_{XY}^{L} [L],$$

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$$F_{XY}^{L} = \frac{|(X, L)_Y|}{|\mathrm{Aut} X|} \cdot \Big(\prod_{i=1}^{t} \frac{|\mathit{Hom}\,(X[i], L)|^{(-1)^i}}{|\mathit{Hom}\,(X[i], X)|^{(-1)^i}}\Big)^{\frac{1}{2}} \in \mathbb{Q}[v, v^{-1}].$$

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- $\mathcal{D}_3 := \mathcal{D}_3(\mathcal{A})$: the 3-periodic derived category;
- Objects in $\mathcal{D}_3$:

$$\dot{X}: \quad \cdots \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_1 \xrightarrow{f_1} \cdots$$

- Indecomposable objects in $\mathcal{D}_3(\mathcal{A})$: $X[i]$ for $X \in ind\mathcal{A}$ and $i = 0, 1, 2$.

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- Indecomposable objects in $\mathcal{D}_3(\mathcal{A})$: $X[i]$ for $X \in ind\mathcal{A}$ and $i = 0, 1, 2$.
- There is a Galois covering (dense) functor $F : \mathcal{D} \to \mathcal{D}_3$.

Examples from Hereditary categories II

$\mathcal{H}(\mathcal{D})$ has the set of generators $\{u_{[X[i]]} \mid i \in \mathbb{Z}, X \in \mathcal{A}\}$ with the following generating relations:

1. $u_{[X[m]]} \cdot u_{[Y[m]]} = \sum_{[L]} F_{XY}^{L} \cdot u_{[L[m]]}$;
2. $u_{[X]}^{[m+1]} \cdot u_{[Y]}^{[m]} = \sum_{[K],[C]} F_{X,Y[1]}^{K[m+1] \oplus C[m]} \cdot u_{[K[m+1]]} \cdot u_{[C[m]]}$;
3. $u_{[X[n]]} \cdot u_{[Y[m]]} = q^{(-1)^{n-m} <Y,X>} u_{[Y[m]]} \cdot u_{[Y[n]]}$ if $n > m+1$.

Examples from Hereditary categories III

Now $\mathcal{A} = \mathrm{mod} kQ$ for a finite quiver without oriented cycles and a finite field k. $\mathcal{H}(\mathcal{D}_3)$ has the set of generators $\{u_{X[i]} \mid i = 0, 1, 2, X \in \mathcal{A}\}$ with the following generating relations:

1. for $n = 0, 1, 2$, $u_{[X[n]]} \cdot u_{[Y[n]]} = \sum_{[L]} F_{XY}^{L} \cdot u_{[L[n]]}$;
2. for $n = 0, 1$,
$$u_{[X[n]]} \cdot u_{[Y[n+1]]} = \sum_{[K],[C]} F_{X,Y[1]}^{K[1]\oplus C} \cdot u_{[K[n+1]]} \cdot u_{[C[n]]};$$
3. $u_{[X[2]]} \cdot u_{[Y]} = \sum_{[K],[C]} F_{X,Y[1]}^{K[1]\oplus C} \cdot u_{[K]} \cdot u_{[C[2]]}$.

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Hall Lie algebras of triangulated categories I

still open: How to define Hall algebras for 2-periodic triangulated categories? The partial answer is given by Hall Lie algebras.

Theorem (Peng-Xiao)

Let $\mathcal{C}$ be 2-periodic over a field k with $|k| = q$ and $G(\mathcal{C})$ be the Grothendieck group. For any $M \in \mathcal{C}$, h_M its canonical image in $G(\mathcal{C})$. Set

$$L(\mathcal{C}) = \sum_{[M];M\in \mathrm{ind}\mathcal{C}} \mathbb{Z}\overline{h}_M / \bigoplus \bigoplus_{[X];X\in \mathrm{ind}\mathcal{C}} \mathbb{Z}[X]$$

where $d(M) = \dim_k(\mathrm{End}M/\mathrm{radEnd}M)$ and $\overline{h}_M = h_M/d(M)$. Then $L(\mathcal{C})/(q-1)L(\mathcal{C})$ is a Lie algebra over $\mathbb{Z}/(q-1)$.

Note: there is a geometric construction of Lie algebras from 2-periodic derived categories by Xiao-Xu-Zhang.

Hall Lie algebras of triangulated categories II

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$$[X] * [Y]([L]) = F_{XY}^{L}.$$

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Write

$$[X] * [Y]([L]) = F_{XY}^{L}.$$

The multiplication is not associative, i.e., it does not hold

$$(([X] * [Y]) * [Z] - [X] * ([Y] * [Z]))([M]) = 0$$

for any $M \in \mathcal{C}$.

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Partial answer:

- Inspired by theorems by Toën, Xiao-Xu and Chen-Xu, F_{XY}^{L} should be chosen in $\mathbb{Q}$ or $\mathbb{Q}(v)$. Then, use the approach to prove the associativity.

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- Inspired by Lusztig's construction of finite dimensional quotients of enveloping algebras, the multiplication should not be intrinsic.

An improvement of Peng-Xiao's theorem II

For any objects U, V and W in $\mathcal{C}$, define

1. $F_{UV}^{W} := \frac{|(U,W)_V|}{|\mathrm{Aut}U|}$, if $U \not\cong W \oplus V[1]$,
2. $\overline{F}_{UV}^{W} := \frac{|(W,V)_{U[1]}|}{|\mathrm{Aut}V|}$, if $V \not\cong W \oplus U[1]$,
3. $F_{W\oplus V[1],V}^{W} := |\mathrm{Hom}(W, V[1])| \cdot \frac{|(W\oplus V[1],W)_V|}{|\mathrm{Aut}(W\oplus V[1])|} = \frac{1}{|\mathrm{Aut}V|}$,
4. $\overline{F}_{U,W\oplus U[1]}^{W} := |\mathrm{Hom}(U[1], W)| \cdot \frac{|(W,W\oplus U[1])_{U[1]}|}{|\mathrm{Aut}(W\oplus U[1])|} = \frac{1}{|\mathrm{Aut}U|}$.

An improvement of Peng-Xiao's theorem III

For any $X \in \mathcal{C}$, we introduce a new variant θ_X associated to X. Define a new multiplication between indecomposable objects by the following rules. For $X, Y \in \mathcal{C}$,

1. $\theta_X \theta_Y = \theta_Y \theta_X$,
2. $\theta_X - \theta_{X[1]} = \overline{h}_X$,
3. if $X \ncong Y[1]$, $[X] * [Y] := \sum_{[L]} F_{YX}^{L}[L]$,
4. $[X] * [X[1]] := \sum_{[L]} F_{X[1]X}^{L} + \theta_{X[1]}$,
5. $[Y] * \theta_X := -\frac{\dim_k \mathrm{Hom}(X[1],Y)}{d(X)}$,
6. $\theta_X * [Y] := -\frac{\dim_k \mathrm{Hom}(Y,X)}{d(X)}$.

An improvement of Peng-Xiao's theorem IV

Theorem

With the above notation, in $\mathbb{Z}[\frac{1}{q}]/(q-1)$, *we have*

$$[([Y]\cdot[X])\cdot[Z]-[Y]\cdot([X]\cdot[Z])](M)=0.$$

for X,Y,Z *and* $M\neq 0$ *in* $\mathrm{ind}\mathcal{C}$.

We call this property **the almost associativity of the multiplication for indecomposable objects**. The multiplication induces Lie brackets in Peng-Xiao's theorem.

Corollary

The almost associativity $\Rightarrow$ *Peng-Xiao's theorem*

The idea of the proof of the almost associativity I

Consider the key expression

$$S = \sum_{[L], L\in\mathcal{C}} F_{XY}^{L} F_{ZL}^{M} - \sum_{[L'], L'\in\mathcal{C}} F_{ZX}^{L'} F_{L'Y}^{M}.$$

By definition, it is

$$\sum_{[L]} \frac{|(X,L)_Y|}{|\mathrm{Aut}X|} \cdot \frac{|(M,L)_{Z[1]}|}{|\mathrm{Aut}L|} - \sum_{[L']} \frac{|(L',X)_{Z[1]}|}{|\mathrm{Aut}X|} \cdot \frac{|(L',M)_Y|}{|\mathrm{Aut}L'|}.$$

Set

$$\mathrm{Hom}(M\oplus X, L)_{L'[1]}^{Y,Z[1]} := \{ \begin{pmatrix} m \\ f \end{pmatrix} \in \mathrm{Hom}(M\oplus X, L) \mid$$

$$\mathrm{Cone}(f) \simeq Y, \mathrm{Cone}(m) \simeq Z[1] \text{ and } \mathrm{Cone}\begin{pmatrix} m \\ f \end{pmatrix} \simeq L'[1]\}$$

The idea of the proof of the almost associativity II

$$S = \sum_{[L],[L']} \frac{1}{|\mathrm{Aut} X|} \cdot \delta_{L, M\oplus Z[1]} \cdot \frac{|\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\mathrm{Aut} L|}$$

$$- \sum_{[L],[L']} \frac{1}{|\mathrm{Aut} X|} \cdot \delta'_{L', M\oplus Y[1]} \cdot \frac{|\mathrm{Hom}(L', M \oplus X)_{L}^{Y, Z[1]}|}{|\mathrm{Aut} L'|}$$

where $\delta_{L, M\oplus Z[1]} = |\mathrm{Hom}(Z[1], M)|$ for $L \cong M \oplus Z[1]$ and 1 otherwise, $\delta_{L', M\oplus Y[1]} = |\mathrm{Hom}(M, Y[1])|$ for $L' \cong M \oplus Y[1]$ and 1 otherwise.

The idea of the proof of the almost associativity III

The equivalence between the octahedral axiom and the pushout $\Rightarrow$

$$\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}/\mathrm{Aut}L = \mathrm{Hom}(L', M \oplus X)_{L}^{Y,Z[1]}/\mathrm{Aut}L'$$
$$:= V(L', L; M \oplus X)_{Y,Z[1]}.$$

Lemma

$$\frac{|(M,L)_{Z[1]}|}{|\mathrm{Aut}L|} = \sum_{\alpha \in V(Z,L;M)} \frac{|\mathrm{End}L_1(\alpha)|}{|n(\alpha)\mathrm{Hom}(Z[1],L)||\mathrm{Aut}L_1(\alpha)|}$$

and

$$\frac{|(Z,M)_L|}{|\mathrm{Aut}Z|} = \sum_{\alpha \in V(Z,L;M)} \frac{|\mathrm{End}Z_1(\alpha)|}{|\mathrm{Hom}(Z[1],L)n(\alpha)||\mathrm{Aut}Z_1(\alpha)|}$$

The idea of the proof of the almost associativity IV

Except two exceptional cases, the sum S is

$$\sum_{[L],[L']}\Big(\sum_{\bar{\alpha}\in\overline{V}(L',L;M\oplus X)_{Y,Z[1]}}\Big(\frac{1}{q^{s(\alpha)}}-\frac{1}{q^{t(\alpha)}}\Big)\cdot\frac{1}{|G_\alpha|}\Big).$$

Here, G_α is the stable subgroup of $\mathrm{Aut}X$ and $\overline{V}(L',L;M\oplus X)_{Y,Z[1]}=V(L',L;M\oplus X)_{Y,Z[1]}/\mathrm{Aut}X.$

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Except two exceptional cases, G_α is a vector space $\Rightarrow S=0$ over $\mathbb{Z}[\frac{1}{q}]/(q-1)$.