The 2-Calabi-Yau property and the multiplication formulas for cluster algebras

Fan Xu

April 16, 2010

Outline

Quivers

$$Q = (Q_0, Q_1) \quad \text{quiver } (= \text{oriented graph})$$

$$Q_0 = \text{set of vertices } = \{1, \ldots, n\} \qquad \text{(finite)}$$

$$Q_1 = \text{set of arrows} \qquad \text{(finite)}$$

$$s : Q_1 \to Q_0, \quad s(i \xrightarrow{\alpha} j) = i \qquad \text{“start”}$$

$$t : Q_1 \to Q_0, \quad t(i \xrightarrow{\alpha} j) = j \qquad \text{“terminal”}$$

Path algebra

k = algebraically closed field

kQ = path algebra of Q
basis: { paths in Q }
multiplication:

$$cc' = \begin{cases} c \cdot c' & \text{if } s(c') = t(c) \\ 0 & \text{otherwise} \end{cases}$$

The elements of kQ are formal sums

$$\sum_{c \text{ path}} a_c \, c, \quad a_c \in k.$$

Examples of path algebras

1. $Q = 1 \circlearrowleft \alpha$ paths: $e_1, \alpha, \alpha^2, \alpha^3, \ldots$
   $kQ \cong k[\alpha]$ polynomial algebra

Examples of path algebras

1. $Q = 1 \circlearrowleft \alpha$ paths: $e_1, \alpha, \alpha^2, \alpha^3, \ldots$
   $kQ \cong k[\alpha]$ polynomial algebra
2. $Q = 1 \to 2$ paths: $e_1, e_2, 1 \to 2$
   $kQ \cong$ lower triangular 2×2 matrices

$$\begin{bmatrix} a_{e_1} & 0 \\ a_{1\to 2} & a_{e_2} \end{bmatrix}$$

For any finite dimensional k-algebra A there is a quotient B of a path algebra such that the module categories of A and B are equivalent.

Representations of Q

- A *representation* $\mathbf{V}$ of Q is a family of

k-vector spaces	$(V_i)_{i \in Q_0}$
and k-linear maps	$(f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)})_{\alpha \in Q_1}$.

- Example:

$$Q = 1 \longrightarrow 2 \longleftarrow 3 \qquad \mathbf{V} = k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} k$$

- The *dimension* of $\mathbf{V}$ is $\mathbf{d} = (d_1, \ldots, d_n)$, where $d_i = \dim(V_i)$.

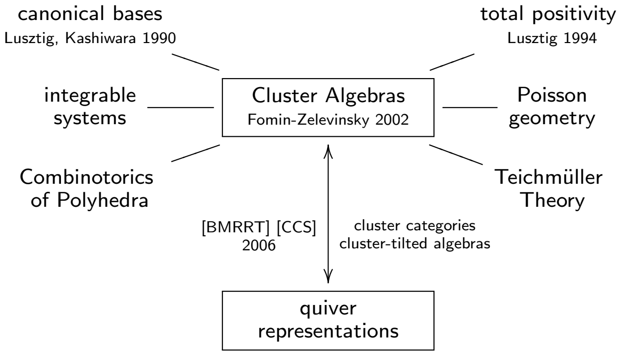
Category of representations

- A *morphism* $g = (g_i)_{i \in Q_0} : \mathbf{V} \to \mathbf{V}'$ is a collection of k-linear maps $g_i : V_i \to V_i'$ such that

$$\begin{CD} V_{s(\alpha)} @>{f_\alpha}>> V_{t(\alpha)} \\ @V{g_{s(\alpha)}}VV @VV{g_{t(\alpha)}}V \\ V'_{s(\alpha)} @>{f'_\alpha}>> V'_{t(\alpha)} \end{CD}$$

  commutes, for each arrow $\alpha \in Q_1$.
- Let $\mathrm{Rep}(kQ)$ be the category of representations of Q.

Cluster Algebras



Construction of the set of generators via mutation

Q quiver, $n = |Q_0|$, $\mathcal{A}(Q)$ cluster algebra

- commutative subalgebra inside $\mathbb{Q}(x_1, \ldots, x_n)$
- generators: cluster variables, grouped set of n cluster variables: cluster
- initial cluster: $\mathbf{x} = \{x_1, \ldots, x_n\}$
- construct the other cluster variables by an iterative process called *mutations*:

$$\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$$

$$x'_k = \left(\prod_{\alpha : s(\alpha) = k} x_{t(\alpha)} + \prod_{\alpha : t(\alpha) = k} x_{s(\alpha)} \right) \frac{1}{x_k}$$

Some results (Fomin-Zelevinsky)

Thm: (Laurent Phenomenon) Let $\{x_1, \ldots, x_n\}$ be a cluster. Each cluster variable u is a Laurent polynomial in $x_1, \ldots, x_n$, i.e.

$$u = \frac{\text{polynomial}}{x_1^{d_1} \ldots x_n^{d_n}} \qquad d_i \geq 0$$

Thm: (Finite type classification) The number of cluster variables is finite if and only if Q is a quiver of Dynkin type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ or G_2.

Categorification

Q without oriented cycles, $\mathcal{A}(Q)$ cluster algebra.

Categorification

Q without oriented cycles, $\mathcal{A}(Q)$ cluster algebra.

- Buan-Marsh-Reineke-Reiten-Todorov (2006) and Caldero-Chapoton-S. (2006) introduced a triangulated category $\mathcal{C}_Q = \mathcal{D}^b(Q)/S \circ [-2]$ where S is the Serre functor, the *cluster category*, s.t.

Categorification

Q without oriented cycles, $\mathcal{A}(Q)$ cluster algebra.

- Buan-Marsh-Reineke-Reiten-Todorov (2006) and Caldero-Chapoton-S. (2006) introduced a triangulated category $\mathcal{C}_Q = \mathcal{D}^b(Q)/S \circ [-2]$ where S is the Serre functor, the *cluster category*, s.t.

cluster category		cluster algebra
indec. (rigid) objects	$\overset{1:1}{\longleftrightarrow}$	cluster variables
tilting objects	$\overset{1:1}{\longleftrightarrow}$	clusters
exchange triangles	$\leftrightsquigarrow$	mutations, relations

Outline

2-Calabi-Yau categories

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$,

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau if there is a bifunctorial isomorphism $\mathrm{Ext}^1(X, Y) \cong \mathrm{DExt}^1(Y, X)$ where $D = \mathrm{Hom}_k(-, k)$ and $\mathrm{Ext}^1(X, Y) = \mathcal{C}(X, Y[1])$.

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau if there is a bifunctorial isomorphism $\mathrm{Ext}^1(X,Y) \cong \mathrm{DExt}^1(Y,X)$ where $D = \mathrm{Hom}_k(-,k)$ and $\mathrm{Ext}^1(X,Y) = \mathcal{C}(X,Y[1])$. Examples:

1. The cluster category $\mathcal{C}_H$ for a hereditary category H (BMRRT);

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau if there is a bifunctorial isomorphism $\mathrm{Ext}^1(X,Y) \cong \mathrm{DExt}^1(Y,X)$ where $D = \mathrm{Hom}_k(-,k)$ and $\mathrm{Ext}^1(X,Y) = \mathcal{C}(X, Y[1])$. Examples:

1. The cluster category $\mathcal{C}_H$ for a hereditary category H (BMRRT);
2. $\mathcal{D}^b(\mathrm{mod}\Lambda)$ for a preprojective algebra of non-Dynkin type Λ (Chuang-Rouquier);

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau if there is a bifunctorial isomorphism $\mathrm{Ext}^1(X,Y) \cong \mathrm{DExt}^1(Y,X)$ where $D = \mathrm{Hom}_k(-,k)$ and $\mathrm{Ext}^1(X,Y) = \mathcal{C}(X, Y[1])$. Examples:

1. The cluster category $\mathcal{C}_H$ for a hereditary category H (BMRRT);
2. $\mathcal{D}^b(\mathrm{mod}\Lambda)$ for a preprojective algebra of non-Dynkin type Λ (Chuang-Rouquier);
3. $\mathcal{D}_{fd}(\Gamma)$ for the dg algebra $\Gamma = \mathbb{C}[t]$ with $\deg(t) = -1$ and trivial differential (Keller).

A basic algebra $\Lambda = \mathbb{C}Q/\mathcal{I}$ with Ext-symmetry, i.e., there is a bifunctorial isomorphism $\mathrm{Ext}^1_\Lambda(X,Y) \cong \mathrm{DExt}^1_\Lambda(Y,X)$, $X, Y \in \mathrm{mod}\Lambda$.

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau if there is a bifunctorial isomorphism $\mathrm{Ext}^1(X,Y) \cong \mathrm{DExt}^1(Y,X)$ where $D = \mathrm{Hom}_k(-,k)$ and $\mathrm{Ext}^1(X,Y) = \mathcal{C}(X,Y[1])$. Examples:

1. The cluster category $\mathcal{C}_H$ for a hereditary category H (BMRRT);
2. $\mathcal{D}^b(\mathrm{mod}\Lambda)$ for a preprojective algebra of non-Dynkin type Λ (Chuang-Rouquier);
3. $\mathcal{D}_{fd}(\Gamma)$ for the dg algebra $\Gamma = \mathbb{C}[t]$ with $\deg(t) = -1$ and trivial differential (Keller).

A basic algebra $\Lambda = \mathbb{C}Q/\mathcal{I}$ with Ext-symmetry, i.e., there is a bifunctorial isomorphism $\mathrm{Ext}^1_\Lambda(X,Y) \cong \mathrm{DExt}^1_\Lambda(Y,X)$, $X, Y \in \mathrm{mod}\Lambda$. For example,

1. Preprojective algebras (Geiss-Leclerc-Schroeer);

2-Calabi-Yau categories

$\mathcal{C}$ a k-linear triangulated category with the shift functor $[1]$, is 2-Calabi-Yau if there is a bifunctorial isomorphism $\mathrm{Ext}^1(X,Y) \cong \mathrm{DExt}^1(Y,X)$ where $D = \mathrm{Hom}_k(-,k)$ and $\mathrm{Ext}^1(X,Y) = \mathcal{C}(X,Y[1])$. Examples:

1. The cluster category $\mathcal{C}_H$ for a hereditary category H (BMRRT);
2. $\mathcal{D}^b(\mathrm{mod}\Lambda)$ for a preprojective algebra of non-Dynkin type Λ (Chuang-Rouquier);
3. $\mathcal{D}_{fd}(\Gamma)$ for the dg algebra $\Gamma = \mathbb{C}[t]$ with $\deg(t) = -1$ and trivial differential (Keller).

A basic algebra $\Lambda = \mathbb{C}Q/\mathcal{I}$ with Ext-symmetry, i.e., there is a bifunctorial isomorphism $\mathrm{Ext}^1_\Lambda(X,Y) \cong \mathrm{DExt}^1_\Lambda(Y,X)$, $X, Y \in \mathrm{mod}\Lambda$. For example,

1. Preprojective algebras (Geiss-Leclerc-Schroeer);
2. Deformed preprojective algebras.

The cluster character via the Caldero-Chapoton map

Q without oriented cycles, $\mathcal{C}_Q$ cluster category, $\langle -, - \rangle$ the Euler form on Rep-$\mathbb{C}Q$.

The cluster character via the Caldero-Chapoton map

Q without oriented cycles, $\mathcal{C}_Q$ cluster category, $\langle -, - \rangle$ the Euler form on Rep-$\mathbb{C}Q$. The Caldero-Chapton map

$$X_?^Q : \mathrm{obj}(\mathcal{C}_Q) \to \mathbb{Q}(x_1, \cdots, x_n)$$

The cluster character via the Caldero-Chapoton map

Q without oriented cycles, $\mathcal{C}_Q$ cluster category, $\langle -,-\rangle$ the Euler form on Rep-$\mathbb{C}Q$. The Caldero-Chapton map

$$X_?^Q : \mathrm{obj}(\mathcal{C}_Q) \to \mathbb{Q}(x_1, \cdots, x_n)$$

satisfies:

1. if M is an indecomposable $\mathbb{C}Q$-module, then

$$X_M^Q = \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \mathbf{e}, s_i\rangle - \langle s_i, \mathbf{dim}M - \mathbf{e}\rangle};$$

The cluster character via the Caldero-Chapoton map

Q without oriented cycles, $\mathcal{C}_Q$ cluster category, $\langle -, - \rangle$ the Euler form on Rep-$\mathbb{C}Q$. The Caldero-Chapton map

$$X_?^Q : \mathrm{obj}(\mathcal{C}_Q) \to \mathbb{Q}(x_1, \cdots, x_n)$$

satisfies:

1. if M is an indecomposable $\mathbb{C}Q$-module, then
$$X_M^Q = \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \mathbf{e}, s_i \rangle - \langle s_i, \mathbf{dim} M - \mathbf{e} \rangle};$$
2. if $M = P_i[1]$ is the shift of the projective module associated to $i \in Q_0$, then
$$X_M^Q = x_i;$$

The cluster character via the Caldero-Chapoton map

Q without oriented cycles, $\mathcal{C}_Q$ cluster category, $\langle -, - \rangle$ the Euler form on Rep-$\mathbb{C}Q$. The Caldero-Chapton map

$$X_?^Q : \mathrm{obj}(\mathcal{C}_Q) \to \mathbb{Q}(x_1, \cdots, x_n)$$

satisfies:

1. if M is an indecomposable $\mathbb{C}Q$-module, then
$$X_M^Q = \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \mathbf{e}, s_i \rangle - \langle s_i, \mathbf{dim} M - \mathbf{e} \rangle};$$
2. if $M = P_i[1]$ is the shift of the projective module associated to $i \in Q_0$, then
$$X_M^Q = x_i;$$
3. for any two objects M, N of $\mathcal{C}_Q$, we have
$$X_{M \oplus N}^Q = X_M^Q X_N^Q.$$

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector $\mathbf{d}$.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector $\mathbf{d}$.
2. $\mathbf{c} = (c_1, \cdots, c_m) \in \{0,1\}^m$, $\mathbf{j} = (j_1, \cdots, j_m)$ is a sequence of integers.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector $\mathbf{d}$.
2. $\mathbf{c} = (c_1, \cdots, c_m) \in \{0,1\}^m$, $\mathbf{j} = (j_1, \cdots, j_m)$ is a sequence of integers.
3. $\Phi_{\mathbf{j},\mathbf{c},x}$: the variety of composition series of x of type $(\mathbf{j}, \mathbf{c})$.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector **d**.
2. $\mathbf{c} = (c_1, \cdots, c_m) \in \{0,1\}^m$, $\mathbf{j} = (j_1, \cdots, j_m)$ is a sequence of integers.
3. $\Phi_{\mathbf{j},\mathbf{c},x}$: the variety of composition series of x of type $(\mathbf{j}, \mathbf{c})$.
4. $d_{\mathbf{j},\mathbf{c}} : \mathbb{E}_{\mathbf{d}} \to \mathbb{C}$ by $d_{\mathbf{j},\mathbf{c}}(x) = \chi(\Phi_{\mathbf{j},\mathbf{c},x})$. $\mathcal{M}(\mathbf{d})$: the $\mathbb{C}$-space spanned by $d_{\mathbf{j}}$.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector $\mathbf{d}$.
2. $\mathbf{c} = (c_1, \cdots, c_m) \in \{0,1\}^m$, $\mathbf{j} = (j_1, \cdots, j_m)$ is a sequence of integers.
3. $\Phi_{\mathbf{j},\mathbf{c},x}$: the variety of composition series of x of type $(\mathbf{j}, \mathbf{c})$.
4. $d_{\mathbf{j},\mathbf{c}} : \mathbb{E}_{\mathbf{d}} \to \mathbb{C}$ by $d_{\mathbf{j},\mathbf{c}}(x) = \chi(\Phi_{\mathbf{j},\mathbf{c},x})$. $\mathcal{M}(\mathbf{d})$: the $\mathbb{C}$-space spanned by $d_{\mathbf{j}}$.

Define the evaluation form $\delta_x : \mathcal{M}(\mathbf{d}) \to \mathbb{C}$ by $\delta_x(f) = f(x)$.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector **d**.
2. $\mathbf{c} = (c_1, \cdots, c_m) \in \{0, 1\}^m$, $\mathbf{j} = (j_1, \cdots, j_m)$ is a sequence of integers.
3. $\Phi_{\mathbf{j},\mathbf{c},x}$: the variety of composition series of x of type $(\mathbf{j}, \mathbf{c})$.
4. $d_{\mathbf{j},\mathbf{c}} : \mathbb{E}_{\mathbf{d}} \to \mathbb{C}$ by $d_{\mathbf{j},\mathbf{c}}(x) = \chi(\Phi_{\mathbf{j},\mathbf{c},x})$. $\mathcal{M}(\mathbf{d})$: the $\mathbb{C}$-space spanned by $d_{\mathbf{j}}$.

Define the evaluation form $\delta_x : \mathcal{M}(\mathbf{d}) \to \mathbb{C}$ by $\delta_x(f) = f(x)$.
Set $\langle x \rangle = \{y \in \mathbb{E}_{\mathbf{d}} \mid \delta_x = \delta_y\}$.

The cluster character via evaluation forms

$\Lambda = \mathbb{C}Q/\mathcal{I}$ for a finite quiver Q with Ext-symmetry.

1. $\mathbb{E}_{\mathbf{d}}$ the variety of dimension vector **d**.
2. $\mathbf{c} = (c_1, \cdots, c_m) \in \{0, 1\}^m$, $\mathbf{j} = (j_1, \cdots, j_m)$ is a sequence of integers.
3. $\Phi_{\mathbf{j},\mathbf{c},x}$: the variety of composition series of x of type $(\mathbf{j}, \mathbf{c})$.
4. $d_{\mathbf{j},\mathbf{c}} : \mathbb{E}_{\mathbf{d}} \to \mathbb{C}$ by $d_{\mathbf{j},\mathbf{c}}(x) = \chi(\Phi_{\mathbf{j},\mathbf{c},x})$. $\mathcal{M}(\mathbf{d})$: the $\mathbb{C}$-space spanned by $d_{\mathbf{j}}$.

Define the evaluation form $\delta_x : \mathcal{M}(\mathbf{d}) \to \mathbb{C}$ by $\delta_x(f) = f(x)$. Set $\langle x \rangle = \{y \in \mathbb{E}_{\mathbf{d}} \mid \delta_x = \delta_y\}$. Then there exists a finite subset $S(\mathbf{d})$ of $\mathbb{E}_{\mathbf{d}}$ such that

$$\mathbb{E}_{\mathbf{d}} = \bigsqcup_{x \in S(\mathbf{d})} \langle x \rangle.$$

The cluster multiplication theorems for cluster categories

The cluster multiplication theorems for cluster categories

(1) (Caldero-Keller, 2005) Q Dynkin quiver, $M, N \in \mathcal{C}_Q$, $\mathrm{Ext}^1_{\mathcal{C}_Q}(M, N) \neq 0$,

$$\chi_c(\mathbb{P}\mathrm{Ext}^1(M,N))X_M X_N$$

$$= \sum_Y (\chi_c(\mathbb{P}\mathrm{Ext}^1(M,N)_Y) + \chi_c(\mathbb{P}\mathrm{Ext}^1(N,M)_Y))X_Y$$

The cluster multiplication theorems for cluster categories

(1) (Caldero-Keller, 2005) Q Dynkin quiver, $M, N \in \mathcal{C}_Q$, $\mathrm{Ext}^1_{\mathcal{C}_Q}(M, N) \neq 0$,

$$\chi_c(\mathbb{P}\mathrm{Ext}^1(M, N))X_M X_N$$
$$= \sum_Y (\chi_c(\mathbb{P}\mathrm{Ext}^1(M, N)_Y) + \chi_c(\mathbb{P}\mathrm{Ext}^1(N, M)_Y))X_Y$$

(2) (Xiao-Xu, 2007) Q without oriented cycles, $M, N \in \mathcal{C}_Q$, $\mathrm{Ext}^1_{\mathcal{C}_Q}(M, N) \neq 0$,

$$\chi_c(\mathbb{P}\mathrm{Ext}^1(M, N))X_M X_N$$
$$= \sum_{Y \in \mathcal{Y}} (\chi_c(\mathbb{P}\mathrm{Ext}^1(M, N)_{\langle Y \rangle}) + \chi_c(\mathbb{P}\mathrm{Ext}^1(N, M)_{\langle Y \rangle}))X_Y$$

where $\mathcal{Y}$ is a finite set.

(3) (Palu, 2009) $\mathcal{C}$ 2-Calabi-Yau triangulated category with tilting objects and some constructible conditions,

$$\chi(\mathbb{P}\mathcal{C}(M,N[1]))X_L^T X_M^T$$

$$=\sum_{Y\in\mathcal{Y}}(\chi(\mathbb{P}\mathcal{C}(M,N[1])_{\langle Y\rangle})+\chi(\mathbb{P}\mathcal{C}(N,M[1])_{\langle Y\rangle}))X_Y^T.$$

where $\mathcal{Y}$ is a finite set.

The multiplication theorems for 2-Calabi-Yau algebras

The multiplication theorems for 2-Calabi-Yau algebras

1. (Geiss-Leclerc-Schroeer, 2007) Λ preprojective algebra of a connected quiver without loops, x', x'' finite dim. nilpotent Λ-modules,

$$\chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x', x''))\, \delta_{x' \oplus x''}$$
$$= \sum_{x \in \mathcal{X}} \left(\chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x', x'')_{\langle x \rangle}) + \chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x'', x')_{\langle x \rangle})\right) \delta_x$$

where $\mathcal{X}$ is a finite set.

The multiplication theorems for 2-Calabi-Yau algebras

1. (Geiss-Leclerc-Schroeer, 2007) Λ preprojective algebra of a connected quiver without loops, x', x'' finite dim. nilpotent Λ-modules,
$$\chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x', x''))\,\delta_{x'\oplus x''}$$
$$= \sum_{x\in\mathcal{X}} \left(\chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x', x'')_{\langle x\rangle}) + \chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x'', x')_{\langle x\rangle})\right)\delta_x$$
where $\mathcal{X}$ is a finite set.
2. (Xiao-Xu, 2009) $\Lambda = \mathbb{C}Q/\mathcal{I}$ with Ext-symmetry for a finite connected quiver Q, x', x'' finite dim. nilpotent Λ-modules,
$$\chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x', x''))\,\delta_{x'\oplus x''}$$
$$= \sum_{x\in\mathcal{X}} \left(\chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x', x'')_{\langle x\rangle}) + \chi(\mathbb{P}\mathrm{Ext}^1_\Lambda(x'', x')_{\langle x\rangle})\right)\delta_x$$
where $\mathcal{X}$ is a finite set.

Outline

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

1. (Sherman-Zelevinsky, 2003) Canonical bases in rank 2 cluster algebras;

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

1. (Sherman-Zelevinsky, 2003) Canonical bases in rank 2 cluster algebras;
2. (Caldero-Keller, 2005) Integral bases in cluster algebras of Dynkin type;

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

1. (Sherman-Zelevinsky, 2003) Canonical bases in rank 2 cluster algebras;
2. (Caldero-Keller, 2005) Integral bases in cluster algebras of Dynkin type;
3. (Dupont, 2008) Integral bases in cluster algebras of affine type $\widetilde{A}$;

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

1. (Sherman-Zelevinsky, 2003) Canonical bases in rank 2 cluster algebras;
2. (Caldero-Keller, 2005) Integral bases in cluster algebras of Dynkin type;
3. (Dupont, 2008) Integral bases in cluster algebras of affine type $\widetilde{A}$;
4. (Ding-Xiao-Xu, 2008) Integral bases in cluster algebras of affine types $\widetilde{A}, \widetilde{D}$ and $\widetilde{E}$;

Bases of acyclic cluster algebras

Bases of acyclic cluster algebras

1. (Sherman-Zelevinsky, 2003) Canonical bases in rank 2 cluster algebras;
2. (Caldero-Keller, 2005) Integral bases in cluster algebras of Dynkin type;
3. (Dupont, 2008) Integral bases in cluster algebras of affine type $\widetilde{A}$;
4. (Ding-Xiao-Xu, 2008) Integral bases in cluster algebras of affine types $\widetilde{A}, \widetilde{D}$ and $\widetilde{E}$;
5. (Geiss-Leclerc-Schroeer, 2009) Integral bases in acyclic cluster algebras.

Undergoing: Cluster algebra for $Coh\mathbb{P}^1$?

$\mathcal{C} = \mathcal{D}^b(Coh\mathbb{P}^1)/S \circ [-2]$: the cluster category of $Coh\mathbb{P}^1$ is a triangulated category (Keller). $\langle -, - \rangle$ the Euler form. The analog of the Caldero-Chapton map

$$X_? : \mathrm{obj}(\mathcal{C}) \to \hat{\mathbb{Q}}[(x_1, \cdots, x_n)$$

Undergoing: Cluster algebra for $Coh\mathbb{P}^1$?

$\mathcal{C} = \mathcal{D}^b(Coh\mathbb{P}^1)/S \circ [-2]$: the cluster category of $Coh\mathbb{P}^1$ is a triangulated category (Keller). $\langle -, - \rangle$ the Euler form. The analog of the Caldero-Chapton map

$$X_? : \mathrm{obj}(\mathcal{C}) \to \hat{\mathbb{Q}}[(x_1, \cdots, x_n)$$

satisfies:

$$X_M = \sum_{\mathbf{e}} \chi^{na}(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \mathbf{e}, s_i \rangle - \langle s_i, \mathbf{dim} M - \mathbf{e} \rangle};$$

where $\hat{\mathbb{Q}}[(x_1, \cdots, x_n)$ is the completion of $\mathbb{Q}[(x_1, \cdots, x_n)$. χ^{na} is the naïve Euler characteristic defined by D. Joyce.

The cluster algebra $\mathcal{A}(Coh\mathbb{P}^1)$ is generated by X_M for rigid $M \in Coh\mathbb{P}^1$, i.e., $\mathrm{Ext}^1(M, M) = 0$

The cluster algebra $\mathcal{A}(Coh\mathbb{P}^1)$ is generated by X_M for rigid $M \in Coh\mathbb{P}^1$, i.e., $\mathrm{Ext}^1(M, M) = 0$

K: the Kronecker algebra of the quiver $1 \rightrightarrows 2$.

The cluster algebra $\mathcal{A}(Coh\mathbb{P}^1)$ is generated by X_M for rigid $M \in Coh\mathbb{P}^1$, i.e., $\mathrm{Ext}^1(M,M)=0$

K: the Kronecker algebra of the quiver $1 \rightrightarrows 2$.

Question: The derived equivalence $\mathcal{D}^b(K) \cong \mathcal{D}^b(Coh\mathbb{P}^1)$ induces the isomorphism between cluster algebras of K and $Coh\mathbb{P}^1$?