Finite repetitive generalized cluster complexes and $d$–cluster categories

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Abstract

For any positive integer $n$, we construct a $n$–repetitive generalized cluster complex (a simplicial complex) associated with a given finite root system by defining a compatibility degree on the $n$–repetitive set of the colored root system. This simplicial complex includes the Fomin-Reading’s generalized cluster complex as a special case when $n = 1$. We also introduce the intermediate coverings (called generalized $d$–cluster categories) of $d$–cluster categories of hereditary algebras, and study the $d$–cluster tilting objects and their endomorphism algebras in those categories.

In particular we show that the endomorphism algebras of $d$–cluster tilting objects in the generalized $d$–cluster categories provide the (finite) coverings of the corresponding (usual) $d$–cluster tilted algebras. Moreover, we prove that the generalized $d$-cluster categories of hereditary algebras of finite representation type provide a category model for the $n$–repetitive generalized cluster complexes.

Key words: $d$–cluster categories; repetitive generalized cluster complexes; $d$–cluster tilted algebras; coverings.

Mathematics Subject Classification: 05A15, 16G20, 16G70, 17B20.

1 Introduction

Cluster algebras were introduced by Fomin-Zelevinsky [FZ1] around 2002 in order to give an algebraic and combinatorial framework for the canonical basis and positivity. Following this, the authors in [FZ2][FZ3] associated cluster complexes to the finite root systems which encode all combinatorics of mutations needed in the definition of cluster algebras of finite type. Later Marsh-Reineke-Zelevinsky [MRZ] describes the quiver interpretation of cluster complex by defining the ”decorated” representations of Dynkin quivers and using tilting theory. This connection leads Buan-Marsh-Reineke-Reiten-Todorov [BMRRT] to

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introduce cluster categories for modeling the combinatorics of cluster algebras in terms of quiver representations (see also [CCS] for \( A_n \) case).

Cluster categories are triangulated categories [K1], which now provide a successful category model for acyclic cluster algebras and their cluster combinatorics [BMRRT][BMR][CC][CK][Z1][Z2] (please see [K2] for an excellent survey of recent development in this direction). Cluster tilting theory and its combinatorics are the essential ingredients in this categorification and now become a new part of tilting theory in representation theory of algebras [BM][Rin].

d-cluster categories as a generalization of cluster categories are introduced by Keller [?] for \( d \in \mathbb{N} \). Since then, \( d \)-cluster categories are studied from various aspects [Th][Z3][BaM1][BaM2][ABST1][Wr][BT][ZZ]. In [ZZ] (see also [Wr]), \( d \)-cluster categories are proved to admit \( d \)-cluster tilting objects which take similar mutations as the cluster tilting objects do in the cluster categories shown in [BMRRT]. Generalized cluster complexes are introduced in [FR2] and studied in different ways in [FR1][Th][Z3]. In [FR2], by generalizing the compatibility degree of roots defined by Fomin and Zelevinsky in [FZ2] to colored roots, the authors defined the generalized cluster complexes, investigated some properties of the generalized cluster complex and made more efforts to calculate the quantitative characteristics of the simplicial complexes associated to an arbitrary finite root system and a nonnegative integer \( m \). On the other hand, it was shown in [Th][Z3][ZZ] that by using \( d \)-cluster tilting objects, one gets a category realization of the generalized cluster complexes of Fomin and Reading. In [Z4] the intermediate coverings for cluster categories are introduced and elements such as cluster tilting objects are considered.

The aims of this paper are the following: First, for any pair of positive integers \( n, d \) and any finite root system \( \Phi \), we define the simplicial complex \( \Delta^n_d(\Phi) \) which is called \( n \)--repetitive generalized cluster complex, or simply the repetitive generalized cluster complex. When \( n = 1 \), this repetitive generalized cluster complex goes to the generalized cluster complex defined by Fomin and Reading [FR2]. Second, in order to give the category model for the repetitive generalized cluster complexes, we introduce the generalized \( d \)--cluster category \( D^b(kQ)/<\tau^n[-nd]> \). This category is triangulated by [K1], and also has \( d \)--cluster tilting objects in the sense of Iyama-Yoshino [IY]. We study cluster tilting theory in the generalized cluster categories. We show there is a one-to-one correspondence among \( d \)--cluster tilting subcategories in the generalized \( d \)--cluster categories, \( d \)--cluster categories, and derived categories. The endomorphism algebra of \( d \)--cluster tilting object in the generalized \( d \)--cluster category provides a finite covering of the corresponding \( d \)--cluster tilted algebra, generalized results in [KZ][Z4][ABST2]. Third, for Dynkin quiver \( Q \), we show that the simplicial complex generated by the \( d \)--rigid ob-
jects in the generalized \( d \)-cluster category \( D^b(kQ)/<\tau^n[-nd]> \) provides a category model of the corresponding repetitive generalized cluster complexes, which generalized the corresponding results of Thomas [Th] and of the second named author [Z3].

The article is organized as follows: In Section 2, some preliminaries about "colored root systems"[FR2] are recalled. Following this, the \( n \)-repetitive colored root systems are introduced. Furthermore the repetitive generalized cluster complex is constructed as a consequence of the definition of compatibility degree on it. In Section 3, basic facts about generalized \( d \)-cluster categories such as \( d \)-cluster tilting objects are investigated. The 1-1 correspondence among \( d \)-cluster tilting subcategories (objects) in \( D^b(H) \), in \( C^m_d(H) \) and in \( C_d(H) \) is verified. The equivalence of the three definitions: \( d \)-cluster tilting object, maximal \( d \)-rigid object and complete \( d \)-rigid object in the generalized \( d \)-cluster category is showed at the end of this section. In Section 4, the generalized \( d \)-cluster tilted algebras are considered in the generalized \( d \)-cluster categories. The generalized \( d \)-cluster tilted algebras are proved to be the Galois coverings of the corresponding \( d \)-cluster tilted algebras. The Section 5 devotes to cluster combinatorics and cluster complexes on generalized \( d \)-cluster categories. The 1-1 map from the generalized \( d \)-cluster categories to the repetitive colored almost positive roots which gives the isomorphism between the cluster complexes on generalized \( d \)-cluster categories and those on repetitive colored almost positive roots is founded.

2 m-repetitive generalized cluster complex

We are going to make some further developments about the cluster complex following Fomin and Reading’s work in [FR2]. Assume \( \Gamma \) is a valued graph of Dynkin type with vertices \( 1, \cdots, n \). \( \Phi \) denotes the set of roots of corresponding Lie algebra and \( \Phi_{>0} \) denotes the set of positive roots. Use \( i \in [0, n] \) to denote \( i \in \{0, 1, 2 \cdots n\} \) everywhere. \( \alpha_1, \cdots, \alpha_n \) denote the simple roots of \( \Phi \).

**Definition 2.1.** The set of almost positive roots of \( \Gamma \) is

\[ \Phi_{>1} = \{\alpha | \alpha \in \Phi_{>0}\} \cup \{-\alpha_r | r \in [1, n]\} \]

**Definition 2.2.** [FZ3] Let \( s_k \) be the Coxeter generator of Weyl group of \( \Phi \) corresponding to \( k \in [1, n] \). Define the truncated simple reflection \( \sigma_k \) of \( \Phi_{>1} \) as follows

\[ \sigma_k(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j, j \neq k \\ s_k(\alpha) & \text{otherwise} \end{cases} \]

\( \sigma_k \) is an automorphism of \( \Phi_{>1} \).
Let $\Omega$ be an orientation of $\Gamma$, $i_1, i_2 \cdots i_n$ be an admissible ordering of $\Gamma$ with respect to $\Omega$ (i.e. $i_t$ is a sink with respect to $s_{i_{t-1}} \cdots s_{i_2} s_{i_1} \Omega$ for any $1 \leq t \leq n$). Denote by $R_{\Omega} = \sigma_n \cdots \sigma_{i_1}$. It is an automorphism of $\Phi_{\geq -1}$ and does not depend on the choice of admissible ordering of $\Gamma$ with respect to $\Omega$ [Z2].

Let $\Omega_0$ be one of the alternating orientations of $\Gamma$ and $\Gamma^+$ (respectively $\Gamma^-$) the set of sinks (respectively sources) of $(\Gamma, \Omega_0)$.

**Definition 2.3.** Denote $\tau_\pm = \prod_{i \in \Gamma^\pm} \sigma_i$. Then $R_{\Omega_0} = \tau_- \tau_+$, write as $R$ for short.

Denote by $n_i(\beta)$ the coefficient of simple root $\alpha_i$ in the simple root expansion of $\beta$. Then there is a compatibility degree $(-||-)$ on $\Phi_{\geq -1}$ which is uniquely defined by the following properties

\[
(-\alpha_i||\beta) = \max \{n_i(\beta), 0\} \quad (\star)
\]

\[
(\alpha||\beta) = (\tau_\pm \alpha||\tau_\pm \beta) \quad (\star\star)
\]

for any $\alpha, \beta \in \Phi_{\geq -1}$, $i \in \Gamma$[FZ3].

$\alpha, \beta \in \Phi_{\geq -1}$ are called compatible if $(\alpha||\beta) = 0$.

**Definition 2.4.** [FZ3] The cluster complex $\triangle(\Phi)$ is a simplicial complex on the ground set $\Phi_{\geq -1}$. Its faces are mutually compatible subsets of $\Phi_{\geq -1}$. The facets (the maximal face) of $\triangle(\Phi)$ are called the (root-)clusters associated to $\Phi$.

2.1 Colored root system

In this section let’s review some useful results from [FR2] first. For a positive integer $d$ and a root $\alpha \in \Phi_{\geq -1}$, we denote by $\alpha^0, \alpha^1, \cdots, \alpha^{d-1}$, the $d$ copies of $\alpha$.

**Definition 2.5.** Let $d$ be a positive integer, the set of colored almost positive roots is

\[\Phi_d^{\geq -1} = \{\alpha^i \mid i \in [0, d-1], \alpha \in \Phi_{>0}\} \cup \{(-\alpha_r)^0 \mid r \in [1, n]\}\]

The $d$-analogue of $R$ in the colored root system is defined as follows

**Definition 2.6.** [Z3] For any $\alpha^k \in \Phi_{\geq -1}$, define

\[R_d(\alpha^k) = \begin{cases} \alpha^{k+1} & \alpha \in \Phi_{>0} \quad k < d - 1 \\ R(\alpha)^0 & \text{otherwise} \end{cases}\]
**Theorem 2.7.** [FR2] Let $\Phi$ be a finite root system. There is a compatibility relation on $\Phi_{\geq -1}^d$ which has the following properties:

1. $\alpha^k$ is compatible with $\beta^l$ if and only if $R_d(\alpha^k)$ is compatible with $R_d(\beta^l)$.
2. $(-\alpha_i)^0$ is compatible with $\beta^l$ if and only if $n_i(\beta) = 0$.

Furthermore, these conditions uniquely determine this relation.

We have the following $d$-version of the truncated simple reflections on the colored almost positive roots.

**Definition 2.8.** [Z3] Let $s_k$ be the Coxeter generator of Weyl group of $\Phi$ corresponding to $k \in [1, n]$. Define $d$-truncated simple reflection $\sigma_k^d$ as follows

$$
\sigma_k^d(\alpha^i) = \begin{cases} 
\alpha^i_k & i = 0, \alpha = -\alpha_k \\
\alpha^i_k^{-1} & \alpha = \alpha_k, 0 < i \leq d - 1 \\
(-\alpha_j)^0 & \alpha = -\alpha_j, i = 0, j \neq k \\
s_k(\alpha)^i & \text{otherwise}
\end{cases}
$$

here $\sigma_k^d$ is a bijection of $\Phi_{\geq -1}^d$.

### 2.2 Repetitive colored root system

For a positive integer $m$, we use $m$ copies of the colored almost positive roots $\Phi_{\geq -1}^d$ defined above to construct the $m$-repetitive set of colored almost positive roots as follows.

**Definition 2.9.** The $(m-)repetitive set of colored almost positive roots is

$$m\Phi_{\geq -1}^d = \{i \alpha^i | i \in [0, d-1], j \in [0, m-1], \alpha \in \Phi_{>0} \cup \{(-\alpha_r)^0 | r \in [1, n], j \in [0, m-1]\}$$

**Remark:** When $m = 1$, we get the colored almost positive roots defined in [FR2]; When $m = 1, d = 1$, we get almost positive roots in [FZ3].

**Definition 2.10.** For any $i \alpha^k \in m\Phi_{\geq -1}^d$, define

$$R_d^m(i \alpha^k) = \begin{cases} 
i \alpha^{k+1} & \alpha \in \Phi_{>0} \quad k < d - 1 \\
^{i+1}R(\alpha)^0 & R(\alpha) \notin \{-\alpha_r\}_{r \in [1, n]} \quad k = d - 1 \quad i < m - 1 \\
i R(\alpha)^0 & R(\alpha) \in \{-\alpha_r\}_{r \in [1, n]} \quad k = d - 1 \quad i \leq m - 1 \\
0 R(\alpha)^0 & \text{otherwise}
\end{cases}$$

where $\{-\alpha_r\}_{r \in [1, n]}$ denotes the set of negative simple roots.
Now we proceed to generalize the "d-truncated simple reflection" of $\Phi_{\geq -1}$ to the repetitive set of colored almost positive roots $m\Phi_{\geq -1}$.

**Definition 2.11.** Let $s_k$ be the Coxeter generator of Weyl group of $\Phi$ corresponding to $k \in [1, n]$. Define the generalized $d$-truncated simple reflection $m\sigma^d_k$ as follows

$$m\sigma^d_k(\alpha^i) = \begin{cases} j\alpha^i_k & i = 0, \alpha = -\alpha_k \\
j\alpha^i_k & \alpha = \alpha_k, 0 < i \leq d - 1 \\
(-\alpha)^0 & \alpha = -\alpha, r \neq k, i = 0 \\
m^{-1}s_k(\alpha)^0 & \alpha = \alpha_k, i = j = 0 \\
-1s_k(\alpha)^0 & \alpha = \alpha_k, i = 0, 0 < j < m \\
j s_k(\alpha)^i & \text{otherwise} \end{cases}$$

It is easy to see that $m\sigma^d_k$ is a bijection of $m\Phi_{\geq -1}$.

**2.3 Compatible degree**

For a root $\beta \in \Phi_{\geq -1}$, let $d(\beta)$ denote the smallest $d$ such that $d$ times

$$\underbrace{R(R(R(\cdots R(\beta) \cdots)))}_{\text{d times}}$$

is a negative root. In particular, $d(\beta) = 0$ if $\beta$ is negative simple root. Define $P : m\Phi_{\geq -1} \rightarrow \Phi_{\geq -1}$ such that $P(\alpha^i) = 0^0\alpha^i$, for any root $\alpha^i \in m\Phi_{\geq -1}$. Here $\Phi_{\geq -1}$ can be regarded as $0\Phi_{\geq -1}$.

**Definition 2.12.** $k\alpha^i, s\beta^j \in m\Phi_{\geq -1}$ are called compatible if and only if $P(k\alpha^i), P(s\beta^j)$ are compatible on $\Phi_{\geq -1}$. In other words one of the followings holds [FR2]:

- $i > j$, $d(\alpha) \leq d(\beta)$ and the roots $R(\alpha)$ and $\beta$ are compatible on $\Phi_{\geq -1}$
- $i < j$, $d(\alpha) \geq d(\beta)$ and the roots $\alpha$ and $R(\beta)$ are compatible
- $i > j$, $d(\alpha) > d(\beta)$ and the roots $\alpha$ and $\beta$ are compatible
- $i < j$, $d(\alpha) < d(\beta)$ and the roots $\alpha$ and $\beta$ are compatible
- $i = j$, and the roots $\alpha$ and $\beta$ are compatible

Here write $0^0\alpha^0, 0^0\beta^0$ as $\alpha, \beta$ for short.

From this definition and the symmetry of compatibility relation on $\Phi_{\geq -1}$, one immediately have
Lemma 2.13. The compatibility relation on \( m \Phi \geq -1 \) is symmetric.

Theorem 2.14. The compatibility relation on \( m \Phi \geq -1 \) has the following properties:
1) \( k \alpha \) is compatible with \( s \beta \) if and only if \( R_d^m(k \alpha) \) is compatible with \( R_d^m(s \beta) \).
2) \( t(-\alpha)^0 \) is compatible with \( s \beta \) if and only if \( \max\{n_r(\beta), 0\} = 0 \), here \( n_r(\beta) \) is the coefficient of \( \alpha_r \) in the simple root expansion of \( \beta \).

Furthermore, conditions 1), 2) uniquely determine this compatibility relation.

Proof. 1) Note that the diagram below commutes

\[
\begin{array}{ccc}
\Phi \geq -1 & \xrightarrow{R_d^m} & \Phi \geq -1 \\
\downarrow P & & \downarrow P \\
\Phi \geq -1 & \xrightarrow{R_d} & \Phi \geq -1 \\
\end{array}
\]

\( P \cdot R_d^m = R_d \cdot P \), then \( R_d^m(k \alpha) \) is compatible with \( R_d^m(s \beta) \) in \( m \Phi \geq -1 \) if and only if \( P \cdot R_d^m(k \alpha) = R_d \cdot P(k \alpha) \) is compatible with \( P \cdot R_d^m(s \beta) = R_d \cdot P(s \beta) \) in \( m \Phi \geq -1 \) if and only if \( P(k \alpha) \) is compatible with \( P(s \beta) \) in \( m \Phi \geq -1 \) and this is exactly \( k \alpha \) is compatible with \( s \beta \).

2) \( t(-\alpha)^0 \) is compatible with \( s \beta \) if and only if \( P^i(-\alpha)^0 \) is compatible with \( P(s \beta) \) in \( m \Phi \geq -1 \) if and only if \( \max\{n_r(\beta), 0\} = 0 \) by [FR2] Theorem 3.4.

The last assertion is immediate from the definition of compatibility relation on \( m \Phi \geq -1 \) and Theorem 3.4. in [FR2]. \( \square \)

Definition 2.15. Let \( m \Phi \geq -1 \) be the set of repetitive colored almost positive roots.
1) A subset of \( m \Phi \geq -1 \) is called compatible if any two elements of it are compatible.
2) The simplicial complex \( m \Delta_d(\Phi) \) associated to \( m \Phi \geq -1 \) is a simplicial complex on the ground set \( m \Phi \geq -1 \). Its simplices are compatible subsets of \( m \Phi \geq -1 \). We call this complex the \( m \)-repetitive generalized cluster complex associated to the root system \( \Phi \). Sometimes we simply call it repetitive generalized cluster complex.

Note that when \( m = 1 \), the complex \( m \Delta_d(\Phi) \) is the generalized cluster complex defined by Fomin-Reading [FR2].

3 Generalized \( d \)-cluster categories

Let \( H \) be a finite dimensional hereditary algebra over a field \( K \), \( \mathcal{H} \) the category of finite dimensional modules over \( H \). Denote the bounded derived category of \( H \) as \( D^b(H) \).
Then the derived category $\mathcal{D}^b(H)$ has AR triangles. The Auslander-Reiten translation $\tau$ and the shift functor $[1]$ are the automorphisms of $\mathcal{D}^b(H)$. For a positive integer $d$, set $F_d = \tau^{-1}[d]$, $F_d$ is an automorphism of $\mathcal{D}^b(H)$. For a positive integer $m$, denote by $F_d^m = \tau - 1 < d >$, $F_d^m$ is an automorphism of $\mathcal{D}^b(H)$. For a positive integer $m$, denote by $\langle F_d^m \rangle$ the (cyclic) group generated by $F_d^m$.

**Definition 3.1.** Fix two positive integers $m, d$. Define the generalized $d$-cluster category as orbit category $\mathcal{D}^b(H)/ < F_d^m >$, the objects are the same as those in $\mathcal{D}^b(H)$, the set of morphisms $\text{Hom}_{\mathcal{D}^b(H)/ < F_d^m >} (\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(H)}(X, (F_d^m)^i Y)$ where $\tilde{X}, \tilde{Y}$ are the corresponding objects in $\mathcal{D}^b(H)/ < F_d^m >$ for $X, Y \in \mathcal{D}^b(H)$. Denote $\mathcal{C}_d^m(H) = \mathcal{D}^b(H)/ < F_d^m >$, and write it as $\mathcal{C}_d^m$ for short.

**Remark:** The definition is a generalization of $d$-cluster category $\mathcal{C}_d(H)$ which is defined as orbit category $\mathcal{D}^b(H)/ < F_d^m >$ in [K1], and also a generalization of generalized cluster categories studied in [Z4].

There is a natural projection functor $\pi_m : \mathcal{D}^b(H) \to \mathcal{D}^b(H)/ < F_d^m >$, especially when $m=1$, we get $\pi : \mathcal{D}^b(H) \to \mathcal{C}_d$ the projection to $d$-cluster category. Another natural projection functor is $\rho_m : \mathcal{C}_d^m \to \mathcal{C}_d$. It’s not difficult to verify $\pi = \rho_m \cdot \pi_m$.

One can identify the set $\text{ind}\mathcal{C}_d^m(H)$ with the fundamental domain for the action of $F_d$ on $\text{ind}\mathcal{D}^b(H)$ [BMRRT]. Passing to the orbit category $\mathcal{C}_F^m(H)$, one can view $\text{ind}\mathcal{C}_d^m(H)$ as a (probably not fully) subcategory of $\text{ind}\mathcal{C}_F^m(H)$.

Some basic properties of the generalized $d$-cluster categories are collected below.

**Proposition 3.2.** 1) $\mathcal{C}_d^m$ is a triangulated category with AR triangles and serre functor $\Sigma = \tau[1]$, where $\tau$ is AR-translate in $\mathcal{C}_d^m$, which is induced from AR-translate in $\mathcal{D}^b(H)$.

2) $\pi_m : \mathcal{D}^b(H) \to \mathcal{C}_d^m$ and $\rho_m : \mathcal{C}_d^m \to \mathcal{C}_d$ are triangle functors and also covering functors.

3) $\mathcal{C}_d^m$ has CY-dimension $\frac{m+md}{m}$.

4) $\mathcal{C}_d^m$ is a Krull-Remak-Schmidt category.

5) $\text{ind}\mathcal{C}_d^m = \bigcup_{i=0}^{m-1} \text{ind}F_d^i \mathcal{C}_d$.

**Proof.** This proposition is just an analogy of proposition 4.3 in [Z4] 1) is from prop. 1.3 [BMRRT] and [K1].

2) The proof for situation in cluster category $\mathcal{C}$ in proposition 4.3 [Z4] can be extended in the generalized $d$-cluster category $\mathcal{C}_d^m$.

3) For the serre functor $\Sigma = \tau[1]$ in $\mathcal{C}_d^m, \Sigma^m = \tau^m[m] = \tau^m[-md][m+md] = [m+md]$.

4) It’s from [BMRRT] and 5) is obvious. ☐
For any $i \in \mathbb{Z}$, denote $\mathcal{H}_i$ by $\mathcal{H}[i]$, the copy of $\mathcal{H}$ under the $i$-th shift $[i]$ as a subcategory of $\mathcal{C}_m^d$.

For a triangulated category $\mathcal{D}$, define $\text{Ext}^i_{\mathcal{D}}(X,Y) = \text{Hom}_{\mathcal{D}}(X,Y[i])$. For $X,Y \in \mathcal{D}$, write $\text{Hom}(X,Y)$, $\text{Ext}(X,Y)$ to denote $\text{Hom}_{\mathcal{D}}(X,Y)$ and $\text{Ext}^i_{\mathcal{D}}(X,Y)$ respectively. Suppose that $\mathcal{T}$ is a functorially finite subcategory of $\mathcal{D}$, and $X$ is an object $X \in \mathcal{D}$. If $\text{Ext}^i(X,T) = 0$, for any $T \in \mathcal{T}$, then we say that $\text{Ext}^i(X,T) = 0$. If $\text{Ext}^i(T,T') = 0$, for any $T, T' \in \mathcal{T}$, then we say that $\text{Ext}^i(T,T) = 0$. For any $M \in \mathcal{D}$, $\text{add}M$ denotes the full subcategory of $\mathcal{D}$ consisting of direct summands of direct sums of copies of $M$.

**Definition 3.3.**
1) $\mathcal{T}$ is called $d$-exceptional (or equivalently, $d$-rigid) if $\text{Ext}^i(T,T) = 0$ for all $1 \leq i \leq d$. An object $X$ is called $d$-exceptional if the subcategory $\text{add}X$ is $d$-exceptional.

2) $\mathcal{T}$ is called $d$-cluster tilting if $X \in \mathcal{T}$ if and only if $\text{Ext}^i(X,T) = 0$ for all $1 \leq i \leq d$ if and only if $\text{Ext}^i(T,X) = 0$ for all $1 \leq i \leq d$. An object $X$ is called a $d$-cluster tilting object if $\text{add}X$ is $d$-cluster tilting.

3) $\mathcal{T}$ is called maximal $d$-rigid if $\text{Ext}^i(T,T) = 0$ for all $1 \leq i \leq d$, and it is maximal respect to this property, i.e. if there is $Y$ such that $\text{Ext}^i(X \oplus Y, X \oplus Y) = 0$ for any $1 \leq i \leq d$ and for any $X \in \mathcal{T}$, then $Y \in \text{add}(X)$. An object $X$ is called maximal $d$-rigid if $\text{add}X$ is maximal $d$-rigid.

4) An object $X$ in $\mathcal{C}_m^d$ is called complete $d$-rigid if it is a $d$-exceptional object which contains exactly $n \cdot m$ non-isomorphic indecomposable direct summands in $\mathcal{C}_m^d$.

The next lemmas are from [IY 3.4, 3.5] and hold for any triangulated category $\mathcal{D}$.

**Lemma 3.4.** [IY]For any $d$-cluster tilting subcategory $\mathcal{T}$ of triangulated category $\mathcal{D}$, $F_d\mathcal{T} = \mathcal{T}$. In particular $F_d$ is an automorphism of $\mathcal{T}$.

The following lemma can reduce the condition to verify a $d$-cluster tilting subcategory.

**Lemma 3.5.** [IY]If $\mathcal{T} = \text{add}X$ is a $d$-cluster tilting subcategory of $\mathcal{D}$ if and only if it satisfies

1. $F_d\mathcal{T} = \mathcal{T}$,

2. if $\text{Ext}^i(Y,X) = 0$, $1 \leq i \leq d$ then $Y \in \text{add}X$.

The $d$-cluster tilting subcategories ($d$-cluster tilting objects) have been defined in derived category, in $d$-cluster category and in generalized $d$-cluster category. The rest of this section devotes to the relations among them.
Theorem 3.6. An object $T$ in $\mathcal{C}^m_d$ is a $d$-cluster tilting object if and only if $\pi_m^{-1}(\text{add}T)$ is a $d$-cluster tilting subcategory of $D^b(H)$.

Proof. Let $\mathcal{T}$ denote the $d$-cluster tilting subcategory in $D^b(H)$. Use lemma 3.4 and without loss of generality, set $T' = \{ \oplus_{i=0}^{d-1} \mathcal{H}_i \oplus H[d] \} \cap \mathcal{T}$. Then $\mathcal{T} = \oplus_{n \in \mathbb{Z}} F^m_dT'$, $\pi_m(\mathcal{T}) = \oplus_{i=0}^{m-1} F^m_dT' = \text{add}T$ denoted by $\mathcal{T}_1$. For any $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{T}$, there are $T_1, T_2 \in \mathcal{T}$ such that $\mathcal{T}_1 = F^m_d(\pi_m(T_1)), \mathcal{T}_2 = F^m_d(\pi_m(T_2))$, $0 \leq t, s \leq m - 1$. For $1 \leq i \leq d$

$$\text{Ext}^i(\mathcal{T}_1, \mathcal{T}_2) = \text{Hom}(\mathcal{T}_1, \mathcal{T}_2)[i]$$

$$= \oplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(H)}(F^m_dT_1, (F^m_d)^n F^m_dT_2[i])$$

$$= \oplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(H)}(T_1, F^m_d, F^m_d, s, t)F^m_dT_2[i])$$

Obviously if $mn + s - t \notin \{0, -1\}$, $\text{Hom}_{D^b(H)}(T_1, F^m_d, s, t)F^m_dT_2[i]) = 0$.

When $mn + s - t = -1$, $\text{Hom}_{D^b(H)}(T_1, F^m_d, s, t)F^m_dT_2[i]) = 0$.

$$\text{Hom}_{D^b(H)}(T_1, F^m_d, s, t)F^m_dT_2[i]) = \text{Hom}_{D^b(H)}(T_1, \tau T_2[i - d])$$

$$\cong D\text{Ext}_{D^b(H)}(\tau T_2[i - d], T_1)$$

$$\cong D\text{Ext}_{D^b(H)}(T_2[i - d], T_1)$$

$$= 0$$

the last equality holds because $1 \leq i \leq d$ and $\mathcal{T}$ is $d$-cluster tilting subcategory. When $mn + s - t = 0, \text{Hom}_{D^b(H)}(T_1, T_2[i]) = \text{Ext}_{D^b(H)}(T_1, T_2) = 0$ for $1 \leq i \leq d$ by the same reason, so $\mathcal{T}_1$ is exceptional in $\mathcal{C}^m_d$.

If there is an indecomposable object $\tilde{X} = \pi_m(X) \in \mathcal{C}^m_d$ with $X \in D^b(H)$ such that $\text{Ext}^i(\mathcal{T}_1, \tilde{X}) = 0$, $1 \leq i \leq d$ then $\text{Ext}^i_{D^b(H)}(\oplus_{n \in \mathbb{Z}} F^m_dT', X) = \text{Ext}^i_{D^b(H)}(\mathcal{T}, X) = 0$, $1 \leq i \leq d$. Thus $X \in \mathcal{T}$ by $\mathcal{T}$ is $d$-cluster tilting. Then $\tilde{X} \in \mathcal{T}_1$. Finally $\mathcal{T}_1 = \pi_m(\mathcal{T})$ is a $d$-cluster tilting subcategory in $\mathcal{C}^m_d$.

On the other hand, assume $\mathcal{T}_1$ is a $d$-cluster tilting subcategory in $\mathcal{C}^m_d$. As $\text{Ext}^k_{D^b(H)}(\mathcal{T}_1, T_1) = \oplus_{n \in \mathbb{Z}} \text{Ext}^k_{D^b(H)}(\oplus_{i=0}^{m-1} F^m_dT', (F^m_d)^n((\oplus_{i=0}^{m-1} F^m_dT'))) = 0$, $0 < k \leq d$, $\text{Ext}^k_{D^b(H)}(\mathcal{T}', F^m_dT') = 0$ for $1 \leq k \leq d$, $n \in \mathbb{Z}$. Since $\mathcal{T} = \oplus_{n \in \mathbb{Z}} F^m_dT'$, $\mathcal{T}$ is orthogonal. If $X \in D^b(H)$ satisfies $\text{Ext}^k_{D^b(H)}(\mathcal{T}, X) = 0$ for $1 \leq k \leq d$, $\text{Ext}^k(F^m_dT', \tilde{X}) = 0$ for $0 \leq i < m$. Therefore $\tilde{X} \in \mathcal{T}_1$, hence $X \in \mathcal{T}$. □

As a special case when $m = 1$, we get the following result, which generalized Lemma 4.14 in [KZ].

Corollary 3.7. An object $T$ in $\mathcal{C}_d(H)$ is a $d$-cluster tilting object if and only if $\pi^{-1}(\text{add}T)$ is a $d$-cluster tilting subcategory of $D^b(H)$.
With the above propositions, an 1-1 correspondence between the $d$-cluster tilting subcategories in $\mathcal{D}^b(H)$ and the $d$-cluster tilting subcategories in $\mathcal{C}^m_d$ is established via the triangle functor $\pi_m$, so is an 1-1 correspondence between the $d$-cluster tilting subcategories in $\mathcal{D}^b(H)$ and the $d$-cluster tilting subcategories in $\mathcal{C}_d$ via triangle functor $\pi$.

**Theorem 3.8.** Let $H$ be a hereditary algebra, $\mathcal{C}_d(H)$ be the $d$-cluster category and $\mathcal{C}^m_d(H)$ be the generalized $d$-cluster category. Then

1) $T'$ is a $d$-cluster tilting subcategory in $\mathcal{C}_d(H)$ if and only if $\rho_m^{-1}(T')$ is a $d$-cluster tilting subcategory in $\mathcal{C}^m_d(H)$ if and only if $\pi^{-1}(T')$ is a $d$-cluster tilting subcategory in $\mathcal{D}^b(H)$.

2) For any $d$-cluster tilting subcategory $T'$ in $\mathcal{C}_d(H)$, $\oplus_{i=0}^{m-1} F^i T'$ is a $d$-cluster tilting subcategory in $\mathcal{C}^m_d(H)$. And any $d$-cluster tilting subcategory in $\mathcal{C}^m_d(H)$ arises in this way.

**Proof.** 1) Since $\pi = \rho_m \cdot \pi_m$, $\rho_m^{-1}(T')$ is a $d$-cluster tilting subcategory in $\mathcal{C}^m_d(H)$ if and only if $\pi^{-1}(T')$ is a $d$-cluster tilting subcategory in $\mathcal{D}^b(H)$. By $\pi(\pi^{-1}(\rho_m^{-1}(T'))) = T'$, $\rho_m^{-1}(T')$ is $d$-cluster tilting if and only if $T'$ is $d$-cluster tilting.

2) It’s direct sequences from the first part and Lemma 3.4.

Now let’s conclude this section by considering the relation among several definitions mentioned above.

**Theorem 3.9.** Let $X$ be a $d$-rigid object in $\mathcal{C}^m_d(H)$. Then the following statements are equivalent.

1) $X$ is $d$-cluster tilting.

2) $X$ is maximal $d$-rigid.

3) $X$ is complete $d$-rigid.

**Proof.** 2) $\Rightarrow$ 1) Since $X$ is maximal $d$-rigid in $\mathcal{C}^m_d$, $\rho_m(X)$ is also maximal $d$-rigid in $\mathcal{C}_d$, then $\rho_m(X)$ is a $d$-cluster tilting object in $\mathcal{C}_d$ by [ZZ]. Use the theorems above, $X$ is $d$-cluster tilting object in $\mathcal{C}^m_d$.

1) $\Rightarrow$ 2) is straight by definition.

2) $\Rightarrow$ 3) By Lemma 3.4 and the fact $d$-cluster tilting object in $\mathcal{C}_d$ has $n$ non-isomorphic indecomposable direct summands, the maximal $d$-rigid object in $\mathcal{C}^m_d$ has $nm$ non-isomorphic indecomposable direct summands.

3) $\Rightarrow$ 2) If the complete $d$-rigid object with $nm$ non-isomorphic indecomposable direct summands is not maximal, it can be extended to a maximal $d$-rigid object with at least $nm + 1$ non-isomorphic indecomposable direct summands. It is contradiction.
4 Generalized $d$-cluster tilted algebra

In this section we study the endomorphism algebras of $d$–cluster tilting objects in the generalized $d$–cluster categories. These algebras generalize cluster tilted algebras [BMR][Z4] and $d$–cluster tilted algebras [KR]. More useful notations and results are listed below.

Let $T$ be a full subcategory of a triangulated category $D$ which is closed under taking direct sums and direct summands, in other words for any two objects $X, Y \in D$, $X \oplus Y \in T$ if and only if $X, Y \in T$. Then the quotient category $A := D/T$ has the same objects as $D$ and morphisms from $X$ to $Y$ in $A$ are the residue classes of $D$-morphisms modulo the subgroup of morphisms factoring through some object in $T$.

By a $T$-module, we mean a contravariant $K$-linear functor from $T$ to the category of vector spaces. Denote $\text{mod}T$ the category of finitely presented $T$-module, by $F : D \to \text{mod}T$ the functor which sends $X \in D$ to the module $T \to \text{Hom}(T, X)$. Since idempotents split in $T$, this functor induces an equivalence from $T$ to the category of projectives in $\text{mod}T$.

**Definition 4.1.** Let $T$ be a $d$-cluster tilting object in the generalized $d$-cluster category $C^m_d$. The endomorphism algebra $\text{End}C^m_d T$ of $T$ is called the generalized $d$-cluster tilted algebra.

Now let’s recall some results from [IY].

**Lemma 4.2.** [IY] Let $T$ be a $(d-1)$-cluster tilting subcategory of a triangulated category $D$. For any $0 < l < d$ denote $Z_l$ the following set

$$\bigcap_{0 < i < d \atop i \neq l} T[-i] \subseteq \bigcap_{0 < i < d \atop i \neq n-l} ^\perp T[i]$$

Then there exists equivalence $G_l : Z_l/T \to \text{mod}T$ defined by

$$G_l X := \text{Hom}_T(-, X[l])$$

which preserves 2-rigidity. In particular $Z_1/T \cong Z_2/T \ldots \cong Z_{d-1}/T$ is an abelian category.

Let $T', T_1$ and $T$ be the $d$-cluster tilting subcategories in $C_d, C^m_d$ and $D^b(H)$ respectively as stated in Theorem 3.8. By the discussion in the last section one can assume $T_1 = \bigoplus_{i=0}^{m-1} F^i T'$ and $T = \bigoplus_{i \in \mathbb{Z}} F^i T'$. Construct $Z'_l = \bigcap_{0 < i \leq d \atop i \neq l} T'[-i] \perp$ in $C_d$, similarly $Z'_l$ in $C^m_d$ and $Z_l$ in $D^b(H)$. 

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**Theorem 4.3.** Let $T$ be a $d$–cluster tilting object in $C_d^m$ with the generalized $d$–cluster tilted algebra $\tilde{A} = \text{End}_{C_d^m} T$. Denote by $T_1 = \text{add} T$, the $d$-cluster tilting subcategory in $C_d^m$, and by $T'$, $T$ the corresponding $d$–cluster tilting subcategories in $C_d$ and $D^b(H)$ respectively. Then

1) $\tilde{A}$ has a Galois covering induced by the restriction of the projection $\pi_m : D^b(H) \longrightarrow C_d^m$ to $T$ 

2) The projection $\pi_m$ induces a push-down functor $\tilde{\pi}_m : Z_1/T[1] \longrightarrow \tilde{A} - \text{mod}$.

**Proof.** 1) As $\pi_m(T) = T_1$ and $Z_1 = \bigcap_{i=2}^{m} \perp T[i]$ by Theorem 3.6, Theorem 3.8 and the definition of projection $\pi_m$

$$Z_{1*} = \bigcap_{i=2}^{m} \perp T_1[i] = \pi_m(Z_1)$$

Use lemma 4.2, there is an equivalence

$$Z_{1*}/T_1[1] \longrightarrow \text{mod} T_1$$

$$X \longrightarrow (\cdot, X)$$

and $\text{mod} T_1$ is the $\tilde{A} - \text{mod}$. The projection $\pi_m$ sends $D^b(H)$ to $C_d^m$, $\pi_m|_T : T \longrightarrow T_1$ is a Galois covering with Galois groups generated by $F_m$.

2) There are equivalences $Z_1/T[1] \longrightarrow \text{mod} T$ and $Z_{1*}/T_1[1] \longrightarrow \text{mod} T_1$ by lemma 4.2. Define $\tilde{\pi}_m(X) := \pi_m|_Z_1(X)$ for any $X \in Z_1/T[1]$ and $\tilde{\pi}_m(f) := \pi_m|_Z_1 f$ for any morphism $f : X \longrightarrow Y$ in $Z_1/T[1]$. It’s easy to see $\tilde{\pi}_m$ is well defined and the following diagram commutes

$$\begin{array}{ccc}
Z_1 & \xrightarrow{\pi_m|Z_1} & Z_{1*} \\
\downarrow P_1 & & \downarrow P_2 \\
Z_1/T[1] & \xrightarrow{\tilde{\pi}_m|Z_1} & Z_{1*}/T_1[1]
\end{array}$$

here $P_1, P_2$ are natural quotient functors. Then $\tilde{\pi}_m|Z_1$ is a covering functor from $Z_1/T[1]$ to $Z_{1*}/T_1[1]$. By lemma 4.2 $Z_{1*}/T_1[1]$ is equivalent to $\text{mod} T_1$ which is the $\tilde{A} - \text{mod}$, the assertion is true.

**Theorem 4.4.** Let $A = \text{End}_{C_d} T'$ and $\tilde{A} = \text{End}_{C_d^m} T_1$, where $T_1 = \rho_m^{-1}(T')$. Let $T' = \text{add} T'$, $T_1 = \text{add} T_1$. Then

1) The restriction of $\rho_m : C_d^m \longrightarrow C_d$ to $T_1$ induces a Galois covering of $A$.

2) The functor $\rho_m$ also induces a push-down functor $\tilde{\rho}_m : \tilde{A} - \text{mod} \longrightarrow A - \text{mod}$.

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Proof. 1) Follow the proof of the Theorem 4.3, by lemma 4.2, there is an equivalence between $Z_{1s}/T_1[1]$ and $\text{mod} T_1$, for any $X \in Z_{1s}/T_1[1]$ the image under the equivalence functor is $(-, X) \in \text{mod} T_1$. The triangle functor $\rho_m|_{T_1}(T_1) = T'$, thus $\rho_m|_{T_1}$ is a Galois covering.

2) By lemma 4.2 $Z'_1/T'[1] \cong A - \text{mod}$ and $Z_{1s}/T_1[1] \cong \tilde{A} - \text{mod}$. Define induced functor $\tilde{\rho}_m$ such that $\tilde{\rho}_m(X) := \rho_m X$ for $X \in Z_{1s}/T_1[1]$, $\tilde{\rho}_m(f) := \rho_m f$ for $f : X \rightarrow Y$ in $Z_{1s}/T_1[1]$. By direct check $\tilde{\rho}_m$ is well defined and the following diagram commutes

![Diagram](image)

here $P_1, P_2$ are the natural quotient functors. Then $\tilde{\rho}_m$ is a covering functor from $Z_{1s}/T_1[1]$ to $Z'_1/T'[1]$ which is $A - \text{mod}$.

Example 4.5. Let us consider the derived category of path algebra $A_5$

Use $P_i$, $E_j$ to denote the projective and simple modules respectively. Consider $F_d$ for $d = 2$. If we choose the 2-cluster tilting subcategory $T \subset \mathcal{D}^b(A_5)$ generated by $\{F_d^a P_1, F_d^a P_2, F_d^a P_5, F_d^a E_5, F_d^a E_4[1]\}_{n \in \mathbb{Z}}$, $\mathcal{D}^b(A_5)/T[1] \cong A_\infty - \text{mod}$, where the Gabriel quiver of $A_\infty$ is

![Diagram](image)

Take $m = 2$. In the generalized 2-cluster category $\mathcal{C}_2^2$, we choose the corresponding 2-cluster tilting object $T$ generated by $\{F_d^a P_1, F_d^a P_2, F_d^a P_5, F_d^a E_5, F_d^a E_4[1]\}_{n=0,1}$. Then $\mathcal{C}_2^2(A_5)/\text{(add}T)[1]$ is isomorphic to the module category of algebra of quiver

![Diagram](image)

Take $m = 1$. In the 2-cluster category $\mathcal{C}_2$, we choose the corresponding 2-cluster tilting
object $T$ generated by $\{F^\alpha_d P_1, F^\alpha_d P_2, F^\alpha_d P_3, F^\alpha_d E_5, F^\alpha_d E_4[1]\}_{n=0}$. Then $C_2(A_5)/(\text{add}\,T)[1]$ is isomorphic to the module category of algebra of quiver

\[
\begin{array}{c}
\xymatrix{P_1 \ar[r] & P_2 \ar[d] & P_3 \ar[l] \\
& E_4[1] & E_5 \ar[u] \ar[l] }
\end{array}
\]

As $F_d : C_d^m \to C_d^m$ is automorphism with $F_d^m = \text{id}$, $F_d|_T$ is an automorphism of $T$. Then the restriction of $F_d$ on $Z_l$ is still an automorphism. So there is the induced automorphism $\tilde{F}_d : \text{mod}\,T \to \text{mod}\,T$, since $Z_l/T[l] \cong \text{mod}\,T$ by Lemma 4.2.

**Proposition 4.6.** $\tilde{F}_d : \text{mod}\,T \to \text{mod}\,T$ is naturally induced from $F_d|_T : T \to T$.

**Proof.** By $F_d|_T$ is an autoisomorphism of $T$ and $F_d|_{Z_l}$ is an autoisomorphism of $Z_l$, $F_d$ induces $\tilde{F}_d : Z_l/T[l] \to Z_l/T[l]$. The diagram below can illustrate this.

\[
\begin{array}{c}
\xymatrix{T \ar[r] & Z_l \ar[d] & Z_l/T[l] \ar[l] \\
& F_d|_T & \tilde{F}_d \ar[l] \\
\tilde{T} \ar[r] & \tilde{Z}_l \ar[d] & \tilde{Z}_l/T[l] \ar[l] \\
& F_d|_{Z_l} & \tilde{F}_d \ar[l] \\
\tilde{T} \ar[r] & \tilde{Z}_l \ar[d] & \tilde{Z}_l/T[l] \ar[l] \\
& \tilde{F}_d & \\
& \text{mod}\,T \ar[r] & \text{mod}\,T \\
& F^*_d & \\
& \text{mod}\,T \ar[r] & \text{mod}\,T \\
& g & g \\
mod\,T \ar[r] & \text{mod}\,T \\
& F^*_d \ar[r] & \text{mod}\,T \\
\end{array}
\]

If $\tilde{X} = \tilde{Y} \in \tilde{Z}_l/T[l]$, in $\tilde{Z}_l X = Y + T[l]$. Then $F_d(X) = F_d(Y) + F_dT[l] = F_d(Y) + T[l]$, so in $\tilde{Z}_l/T[l] \tilde{F}_d \tilde{X} = \tilde{F}_d \tilde{Y}$, thus $\tilde{F}_d$ is well defined. By Lemma 4.2 for any object in $T$ it corresponds one element in $\tilde{Z}_l/T[l] \cong \text{mod}\,T$. For $(-, X) \in \text{mod}\,T$, $\tilde{F}_d(-, X) = (-, F_dX)$ and for $(-, f) : (-, X) \to (-, Y)$, $\tilde{F}_d(-, f) = (-, F_df)$.

On the other hand, $F_d|_T$ sends $X \in T$ to $F_dX$. By natural functor $g : T \to \text{mod}\,T$ which sends $X \in T$ to $(-, X)$, $F_d|_T$ induces the functor $F^*_d : \text{mod}\,T \to \text{mod}\,T$ with the diagram below commutes

\[
\begin{array}{c}
\xymatrix{T \ar[r]^{F_d} & \text{mod}\,T \ar[d]^{g} \\
\text{mod}\,T \ar[r]^{F^*_d} & \text{mod}\,T \ar[d]^{g} \\
\end{array}
\]

then $g \cdot F_d = F^*_d \cdot g$. In other words $g \cdot F_d(X) = (-, F_dX) = F^*_d(-, X) = F^*_d \cdot g(X)$ for $X \in T$, and $g \cdot F_d(f) = (-, F_df) = F^*_d(-, f) = F^*_d \cdot g(f)$ for $f : X \to Y$. Therefore $F^*_d = \tilde{F}_d$, the statement is true. \qed
5 Relation between generalized $d$-cluster category and repetitive colored root system

In this section cluster complexes of generalized $d$-cluster category will be defined, then an 1-1 correspondence between generalized $d$-cluster category and repetitive colored almost positive roots is constructed. With this correspondence, the cluster complex of generalized $d$-cluster category is isomorphic to the $m$-repetitive generalized cluster complex defined above.

Note that any indecomposable object $X$ in $C^m_d(H)$ is of the form $F^i_dM[i]$, here $M \in \text{ind} \, H$ and if $i = d$, $M = P$, the indecomposable projective module. From now on we denote $^iM^k := F^i_dM[k]$ for any module.

5.1 BGP-reflection functors

A triangle functor $G : D^b(H) \to D^b(H')$ with some special properties induces a well defined triangle functor $\tilde{G} : C^m_d(H) \to C^m_d(H')$ by [K1] [Z2], especially the following diagram commutes with two rows triangle equivalent.

\[
\begin{array}{ccc}
D^b(H) & \xrightarrow{G} & D^b(H') \\
\downarrow & & \downarrow \\
C^m_d(H) & \xrightarrow{\tilde{G}} & C^m_d(H')
\end{array}
\]

Let $k$ be a vertex in the valued quiver $(\Gamma, \Omega)$, the reflection of $(\Gamma, \Omega)$ at $k$ is the valued quiver $(\Gamma, s_k\Omega)$, where $s_k\Omega$ is the orientation of $\Gamma$ obtained from $\Omega$ by reversing all arrows starting or ending at $k$. The corresponding category of representations of $(\Gamma, s_k\Omega, M)$ is denoted simply by $s_kH$ and the corresponding tensor algebra is denoted by $s_kH$. If $k$ is a sink in the valued quiver $(\Gamma, \Omega)$, then $k$ is a source of $(\Gamma, s_k\Omega)$, and the reflection of $(\Gamma, s_k\Omega)$ at $k$ is $(\Gamma, \Omega)$.

For a sink $k$ of the valued quiver $(\Gamma, \Omega)$, consider the derived BGP-reflection functor $S^+_k : D^b(H) \to D^b(s_kH)$ induced from the BGP-reflection functor $H \to s_kH$, for more details refer to [Z2]. For any $X[i] \in \text{ind}D^b(H)$

\[
S^+_k(X[i]) = \begin{cases} 
S^+_k(X)[i] & X \not\cong P_k, i \in \mathbb{Z} \\
E^+_k[i-1] & \text{otherwise}
\end{cases}
\]

where $P_k$ is the indecomposable projective representation corresponding to $k$, $E^+_k$ is the simple representation of quiver $(\Gamma, s_k\Omega)$ of vertex $k$. In $D^b(s_kH) E^+_k[i - 1] = \tau P'_k[i]$.
As proved in [K1, Z1, Z2], the derived BGP-reflection functor $S_k^+ : \mathcal{D}^b(H) \to \mathcal{D}^b(s_kH)$ induces a triangle equivalence from $\mathcal{C}^m_{\delta}(H)$ to $\mathcal{C}^m_{\delta}(s_kH)$, which is also denoted by $S_k^+$.

The following result can be verified in a similar way as the proof of Proposition 2.7 in [Z2].

**Proposition 5.1.** Let $k$ be a sink of valued quiver $(\Gamma, \Omega)$, $Y$ be an indecomposable object in $\mathcal{C}^m_{\delta}(H)$. Then $Y = j X^i$ for an indecomposable object $X$ in $\mathcal{H}$, and

$$
S_k^+(j X^i) = \begin{cases}
  m^{-1} P_k^i[d] & X \cong P_k \cong E_k, i = j = 0 \\
  j^{-1} P_k^i[d] & X \cong P_k \cong E_k, i = 0, j \neq 0 \\
  j E_k^i[i - 1] & X \cong P_k \cong E_k, 0 < i \leq d \\
  j P_k^i[d] & X \cong P_r \ncong P_k, i = d \\
  j S_k^+(X)[i] & \text{otherwise}
\end{cases}
$$

**Remark:** Set $m = 1$, this result degenerates to the case considered in [Z2].

### 5.2 Construction of the relation

Let $\mathcal{C}^m_{\delta}(H)$ be the generalized $d$-cluster category of $H$, where $H$ is a finite dimensional hereditary algebra of Dynkin type. Let $\text{ind}H$ be the category of the isomorphism classes of indecomposable $H$–modules.

**Definition 5.2.** The cluster complex $m \triangle^d(H)$ of $\mathcal{C}^m_{\delta}(H)$ is a simplicial complex which has $\text{ind}\mathcal{C}^m_{\delta}(H)$ as the set of vertices and $d$-exceptional subsets in $\mathcal{C}^m_{\delta}(H)$ as its simplices.

**Definition 5.3.** Now define a map $m \gamma^d$ from $\text{ind}\mathcal{C}^m_{\delta}(H)$ to $m \Phi^d_{\geq -1}$ for $k M^i \in \mathcal{C}^m_{\delta}(H)$

$$
m \gamma^d(k M^i) = \begin{cases}
  k \dim M^i & k M^i \in F^k_\delta(\text{ind}H[i]), i \in [0, d - 1], k \in [0, m - 1] \\
  k(-\alpha_j) \supseteq & k M^i \in F^k_\delta(\text{ind}H[i]), i = d
\end{cases}
$$

**Remark:** This is a bijection between $\text{ind}\mathcal{C}^m_{\delta}(H)$ and $m \Phi^d_{\geq -1}$. So denote $M_\beta$ the unique indecomposable exceptional module in $\mathcal{H}$ whose dimension vector is $\beta$.

One of the applications for this map is to describe the translation $R^m_{\beta}$ defined on the generalized colored root system $m \Phi^d_{\geq -1}$ with corresponding objects in $\mathcal{C}^m_{\delta}$.

**Example 5.4.** For Dynkin diagram of type $A_3$, $d = 2$, $m = 2$, the $\mathcal{C}^m_{\delta}$ is as follows:

```
    0 P_3^0  0 P_1^1  1 P_3^0
   / \         / \         / \       
0 P_1^0 0 P_1^2 1 P_1^2
```
By the definition of $R^m_d$, it is easy to get
\[
R^m_d(0 \dim P^1_1) = 0 \dim P^1_1 = m \gamma^d(0 P^1_1)  \\
R^m_d(0 \dim P^1_1) = 0 R(\dim P^1_1) = 0 - \alpha_0^1 = m \gamma^d(0 P^1_2)  \\
R^m_d(m \gamma^d(0 P^1_2)) = R^m_d(0 - \alpha_0^1) = 1 R(\alpha_0^1) = 1 \dim P^3_3 = m \gamma^d(1 P^3_3)  \\
R^m_d(m \gamma^d(1 P^1_2)) = R^m_d(1 - \alpha_0^1) = 0 R(\alpha_0^1) = 0 \dim P^3_0 = m \gamma^d(0 P^3_0)
\]

We are going to define "the generalized d-compatible degree" in generalized d-cluster category $\mathcal{C}^m_d$.

**Definition 5.5.** For any pair of $M_\alpha, M_\beta \in \mathcal{C}^m_d$, the generalized d-compatible degree of $M_\alpha, M_\beta$ is defined as follows

\[
(M_\alpha \parallel M_\beta)^m_d = \dim_{\text{End} M_\alpha}(\oplus_{i \in [1, d]} \text{Ext}^i(M_\alpha, M_\beta))
\]

where $\dim_{\text{End} M_\alpha}(\oplus_{i \in [1, d]} \text{Ext}^i(M_\alpha, M_\beta))$ denotes the length of $\oplus_{i \in [1, d]} \text{Ext}^i(M_\alpha, M_\beta)$ as a right $\text{End} M_\alpha - \text{mod}$.

Of course it is natural to induce a generalized d-compatibility on m-repetitive colored almost positive roots through the bijection $m \gamma^d$. In fact many properties could be translated from one side to another.

**Lemma 5.6.** The generalized d-compatible degree on $\mathcal{C}^m_d$ naturally induces a compatibility $(-\parallel-)_{d, H}$ on $m \Phi^d_{\geq -1}$, furthermore $(\alpha \parallel \beta)^m_d = (R^m_d \alpha \parallel R^m_d \beta)^m_{d, H}$

**Proof.** Let’s start to check the following diagram commutes

\[
\begin{array}{ccc}
\text{ind} \mathcal{C}^m_d & \overset{[1]}{\longrightarrow} & \text{ind} \mathcal{C}^m_d \\
\downarrow m \gamma^d & & \downarrow m \gamma^d \\
m \Phi^d_{\geq -1} & \overset{R^m_d}{\longrightarrow} & m \Phi^d_{\geq -1}
\end{array}
\]

Any indecomposable object in $\mathcal{C}^m_d$ is $j^i X^j$, $X \in \text{ind} \mathcal{H}$, $0 \leq i \leq d - 1$, $0 \leq j \leq m - 1$ or $j^i P^d_k$, $0 \leq j \leq m - 1$. Denote $\dim X = \alpha$. If $i \leq d - 2$

\[
R^m_d(m \gamma^d(j X^i)) = R^m_d(j \alpha^i) = j \alpha^{i+1} = m \gamma^d[1](j X^i)
\]

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When \( i = d - 1 \), \( j < m - 1 \)

Subcase a) If \( X \) is a projective module \( P_k \), then

\[
R^m_d (m\gamma^d (j X^i)) = R^m_d (j \dim P_k^{d-1}) = j R(\dim P_k)^0 = j (-\alpha_k)^0
\]

and \( m\gamma^d[1](j P_k^{d-1}) = m\gamma^d(j P_k^d) = j (-\alpha_k)^0 = R^m_d (m\gamma^d (j X^i)) \)

Subcase b) \( X \) is not projective. Then

\[
R^m_d (m\gamma^d (j X^{d-1})) = R^m_d (j \alpha^{d-1}) = j+1 R(\alpha)^0
\]

and \( m\gamma^d[1](j X^i) = m\gamma^d(j X^d) = m\gamma^d(j+1 \tau X^0) = j+1 (\dim X)^0 = j+1 R(\alpha)^0 \).

When \( i = d - 1 \), \( j = m - 1 \)

Subcase a) \( X \) is projective, then

\[
R^m_d (m\gamma^d (m-1 P_k^{d-1})) = R^m_d (m-1 \dim P_k^{d-1}) = R^m_d (m-1 \dim P_k)^0 = m\gamma^d(m-1 P_k)
\]

Subcase b) \( X \) is not projective, then \( R^m_d (m\gamma^d (m-1 X^{d-1})) = R^m_d (m-1 \alpha^{d-1}) = 0 R(\alpha)^0 \)

and

\[
m\gamma^d[1](m-1 X^i) = m\gamma^d(m-1 X^d) = m\gamma^d(0(F^m_d \tau X)^0) = m\gamma^d(0(\tau X)^0) = 0 (\dim \tau X)^0 = 0 R(\alpha)^0
\]

When \( i = d \), the similar discussion is enough, the diagram is commutative. Since the shift functor \([1]\) is an auto-equivalence

\[
(R^m_d(\alpha)||R^m_d(\beta))_{d,H}^m = \dim_{\End M_{\alpha}[1] \oplus_{i \in [1,d]} \Ext^i (M_{\alpha}[1], M_{\beta}[1])}
\]

Now it is very convenient to show the correspondence between generalized \( d \)-compatibility relation on \( C^m_d \) and the compatibility relation on repetitive colored almost positive roots introduced above.

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Theorem 5.7. The compatibility degree \((-\|\|)_{d,H}^m\) on \(\Phi_{-1}^d\) induced by generalized \(d\)-compatible degree on \(C_d^m\) is identical to the compatibility degree defined in Definition 2.12 in Section 2.

Proof. Use lemma 5.6, the theorem holds if the following assertion is verified.

For any almost positive real Schur root \(i\beta^i \in \Phi_{-1}^d\), \((k(-\alpha_r)^0\|i\beta^i)^m_d = 0\) if and only if \(\max\{n_r(\beta), 0\} = 0\).

For the sufficiency, if \(\beta \neq -\alpha_r\), \(n_r(\beta) = 0\) by \(\max\{n_r(\beta), 0\} = 0\). In other words, \(\Hom_D(P_r, M_\beta) = 0\).

\[
\Ext^1(F^k_dP_r[d], F^i_dM_\beta[i]) = \Ext^1_D(P_r[d], F^{k-k+1}_dM_\beta[i]) = \Hom_D(P_r, F^{k-k+1}_dM_\beta[i-d+l])
\]

here \(1 \leq l \leq d, 0 \leq i \leq d\), thus \(1 \leq l+i \leq 2d, 1 \leq l \leq l+i-d \leq d\). Then \(\Hom_D(P_r, F^{k-k+1}_dM_\beta[i-d+l] = 0\) when \(j+t-k \notin \{-1, 0, 1\}\).

If \(j+t-k = 0\), \(\Ext^1(F^k_dP_r[d], F^i_dM_\beta[i]) = \Hom_D(P_r, M_\beta[i-d+l])\). Since \(1-d \leq i+l-d \leq d\) and \(\Hom_D(P_r, M_\beta) = 0\), \(\Hom_D(P_r, M_\beta[i+l-d]) = 0\).

If \(j+t-k = -1\), \(\Ext^1(F^k_dP_r[d], F^i_dM_\beta[i]) = \Hom_D(P_r, \tau M_\beta[i-2d+l])\). Since \(1 \leq i+l \leq 2d\), \(\Hom_D(P_r, \tau M_\beta[i+l]) = 0\).

For \(\beta = -\alpha_r\), all the cases discussed above hold because now \(i = d\) except \(j+t-k = -1\) and \(l-d = 0\), fortunately \(\Ext^1(F^k_dP_r[d], F^i_dM_\beta[i]) = \Hom_D(P_r, \tau P_r) = 0\) at this time.

On the other hand, let \(\beta\) is an almost real Schur root with \((k(-\alpha_r)^0\|i\beta^i)^m_d = 0\), we want to prove \(\max\{n_r(\beta), 0\} = 0\). If \(\beta = -\alpha_r\), the statement is straight. When \(\beta \neq -\alpha_r\), \(\Ext^1(F^k_dP_r, M_\beta) = 0\) for \(1 \leq l \leq d, 0 \leq k, j \leq m-1, 0 \leq i \leq d-1\) and \(1 \leq r \leq n\).

Especially \(\Ext^1(D^r_dP_r, M_\beta) = 0\) \(0 \leq i \leq d, 1 \leq l \leq d\). Then \(\Ext^1(D^r_dP_r, M_\beta) \cong \Hom_D(P_r, M_\beta[i+l-d]) = 0\), choose \(l\) such that \(i+l-d = 0\), \(\Hom_D(P_r, M_\beta) = 0\).

Therefore \(n_r(\beta) = \Hom(P_r, M_\beta) = 0\).

Meanwhile under this compatibility degree the generalized \(d\)-truncated simple reflection \(\sigma^d_k\) has the following property.

Proposition 5.8. For any pair \(\alpha, \beta \in \Phi_{-1}^d\), \(v\) is a sink(or source), then \(\sigma^d_\alpha \sigma^d_\beta^m = \sigma^d_\alpha^m \sigma^d_\beta^m\) in Section 2.
Proof. If \( v \) is a sink, the following diagram commutes

\[
\begin{array}{ccc}
\text{ind}C^m_d(H) & \xrightarrow{S'_v} & \text{ind}C^m_d(S'_v^+H) \\
\downarrow^{m\gamma_H} & & \downarrow^{m\sigma'_{v^+H}} \\
m\Phi^d \geq -1 & \xrightarrow{m\sigma^d} & m\Phi^d \geq -1
\end{array}
\]

Hence

\[
(m\sigma^d \alpha || m\sigma^d \beta)^m_{d,v,H} = \dim_{\text{End}S'_v^+(M_\alpha)} \text{Ext}(S'_v^+M_\alpha, \oplus_{i=0}^{d-1} S'_v^+ M_\beta[i])
\]

\[
= \dim_{\text{End}M_\alpha} \text{Ext}(M_\alpha, \oplus_{i=0}^{d-1} M_\beta[i])
\]

\[
= (\alpha || \beta)^m_{d,H}
\]

\[\square\]

**Theorem 5.9.** Let \( \Gamma \) be a valued Dynkin graph and \( \Phi \) the corresponding root system. Let \( \Omega \) be an orientation on \( \Gamma \). Then \( m\gamma^d \) provides an isomorphism from generalized \( d \)-cluster complex \( m\Delta^d(H) \) to repetitive generalized cluster complex \( m\Delta^d(\Phi) \), which sends vertices to vertices, \( k \)-faces to \( k \)-faces.

**Proof.** Just generalize the proof of Theorem 5.7 (1). in [Z3]. \[\square\]

From the correspondence \( m\gamma^d \) and the fact that \( (F_d)^m = (\tau^{-1}[d])^m \) is identity on the generalized \( d \)-cluster category, the following proposition is straight.

**Proposition 5.10.** The functor

\[
F := (m\sigma^d_1 \cdots m\sigma^d_2 m\sigma^d_1)^{-(d-1)} (R_d^m)^d
\]

is an automorphism of repetitive generalized cluster complex \( m\Delta^d(\Phi) \) and \( F^m = \text{id} \).

**Proof.** One observation is enough. Under the correspondence \( m\gamma^d \) from generalized \( d \)-cluster category to repetitive colored almost positive roots, the correspondent of AR-translate \( \tau \) is \( m\sigma^d_1 \cdots m\sigma^d_2 m\sigma^d_1 \). \[\square\]
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