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Finite Repetitive Generalized Cluster Complexes and d-Cluster Categories

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Abstract. For any positive integer n, we construct an n-repetitive generalized cluster complex (a simplicial complex) associated with a given finite root system by defining a compatibility degree on the n-repetitive set of the colored root system. This simplicial complex includes Fomin-Reading's generalized cluster complex as a special case when $n=1$. We also introduce the intermediate coverings (called generalized d-cluster categories) of d-cluster categories of hereditary algebras, and study the d-cluster tilting objects and their endomorphism algebras in those categories. In particular, we show that the endomorphism algebras of d-cluster tilting objects in the generalized d-cluster categories provide the (finite) coverings of the corresponding (usual) d-cluster tilted algebras. Moreover, we prove that the generalized d-cluster categories of hereditary algebras of finite representation type provide a category model for the n-repetitive generalized cluster complexes.

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1 Introduction

Cluster algebras were introduced by Fomin-Zelevinsky [14] around 2000 in order to give an algebraic and combinatorial framework for the canonical basis and positivity. Following this, the authors in [15, 16] associated cluster complexes to the finite

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root systems which encode all combinatorics of mutations needed in the definition of cluster algebras of finite type. Later, Marsh-Reineke-Zelevinsky [22] described the quiver interpretation of cluster complex by defining the “decorated” representations of Dynkin quivers and using tilting theory. This connection leads Buan-Marsh-Reineke-Reiten-Todorov [6] to introduce cluster categories for modelling the combinatorics of cluster algebras in terms of quiver representations (see also [10] for the A_n case).

Cluster categories are triangulated categories (see [18]), which now provide a successful category model for acyclic cluster algebras and their cluster combinatorics [6, 7, 9, 11, 27, 28] (see [19] for an excellent survey of recent development in this direction). Cluster tilting theory and its combinatorics are the essential ingredients in this categorification and now become a new part of tilting theory in representation theory of algebras (see [5, 23]).

As a generalization of cluster categories, d-cluster categories were introduced by Keller [18] for $d \in N$. Since then, d-cluster categories are studied from various aspects (see [1, 3, 4, 8, 24, 25, 26, 29]). In [26] (see also [25]), d-cluster categories are proved to admit d-cluster tilting objects which take similar mutations as the cluster tilting objects do in the cluster categories shown in [6]. Generalized cluster complexes were introduced in [12] and studied in different ways in [13, 24, 29]. In [12], by generalizing the compatibility degree of roots defined by Fomin and Zelevinsky in [15] to colored roots, the authors defined the generalized cluster complexes, investigated some properties of the generalized cluster complex and made more efforts to calculate the quantitative characteristics of the simplicial complexes associated to an arbitrary finite root system and a non-negative integer m. On the other hand, it was shown in [24, 26, 29] that by using d-cluster tilting objects, one gets a category realization of the generalized cluster complexes of Fomin and Reading. In [30], the intermediate coverings for cluster categories are introduced and elements such as cluster tilting objects are considered.

The aims of this paper are as follows: First, for any pair of positive integers n, d and any finite root system Φ, we define the simplicial complex ${}^n\Delta^d(\Phi)$ which is called the n-repetitive generalized cluster complex, or simply the repetitive generalized cluster complex. When $n = 1$, this goes to the generalized cluster complex defined by Fomin and Reading [12]. Second, in order to give the category model for the repetitive generalized cluster complexes, we introduce the generalized d-cluster category $D^b(kQ)/\langle \tau^n[-nd] \rangle$. This category is triangulated by [18], and also has d-cluster tilting objects in the sense of Iyama-Yoshino [17]. We study cluster tilting theory in the generalized cluster categories. We show that there is a one-to-one correspondence among d-cluster tilting subcategories in the generalized d-cluster categories, d-cluster categories, and derived categories. The endomorphism algebra of a d-cluster tilting object in the generalized d-cluster category provides a finite covering of the corresponding d-cluster tilted algebra, generalized results in [2, 21, 30]. Third, for a Dynkin quiver Q, we show that the simplicial complex on the set of the d-rigid objects in the generalized d-cluster category $D^b(kQ)/\langle \tau^n[-nd] \rangle$ provides a category model of the corresponding repetitive generalized cluster complex, which generalizes the corresponding results of [24] and [29].

The article is organized as follows: In Section 2, some preliminaries about colored root systems (see [12]) are recalled. Following this, the n-repetitive colored root systems are introduced. Furthermore, the repetitive generalized cluster complex is constructed as a consequence of the definition of compatibility degree on it. In Section 3, basic facts about generalized d-cluster categories such as d-cluster tilting objects are investigated. The 1-1 correspondence among d-cluster tilting subcategories (objects) in $\mathcal{D}^b(H)$, in $\mathcal{C}_d^m(H)$ and in $\mathcal{C}_d(H)$ is verified. The equivalence of the three definitions: d-cluster tilting object, maximal d-rigid object and complete d-rigid object in the generalized d-cluster category is showed at the end of this section. In Section 4, the generalized d-cluster tilted algebras are considered in the generalized d-cluster categories. The generalized d-cluster tilted algebras are proved to be the Galois coverings of the corresponding d-cluster tilted algebras. Section 5 is devoted to cluster combinatorics and cluster complexes on generalized d-cluster categories. The 1-1 map from the generalized d-cluster categories to the repetitive colored almost positive roots which gives the isomorphism between the cluster complexes on generalized d-cluster categories and those on repetitive colored almost positive roots is found.

2 m-Repetitive Generalized Cluster Complex

Assume Γ is a valued graph of Dynkin type with vertices $1, \dots, n$. Φ denotes the set of roots of the corresponding Lie algebra, $\Phi_{>0}$ the set of positive roots, and $\alpha_1, \dots, \alpha_n$ the simple roots of Φ. Use $i \in [0, n]$ to denote $i \in \{0, 1, \dots, n\}$.

Definition 2.1. The set of almost positive roots of Γ is

$$\Phi_{\geq -1} = \{\alpha \mid \alpha \in \Phi_{>0}\} \cup \{-\alpha_r \mid r \in [1, n]\}.$$

Definition 2.2. [16] Let s_k be the Coxeter generator of the Weyl group of Φ corresponding to $k \in [1, n]$. Define the truncated simple reflection σ_k of $\Phi_{\geq -1}$ by

$$\sigma_k(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j, \ j \neq k, \\ s_k(\alpha) & \text{otherwise.} \end{cases}$$

This σ_k is an automorphism of $\Phi_{\geq -1}$.

Let Ω be an orientation of Γ, and $i_1, \dots, i_n$ an admissible ordering of Γ with respect to Ω (i.e., i_t is a sink with respect to $s_{i_{t-1}} \cdots s_{i_2} s_{i_1} \Omega$ for any $1 \leq t \leq n$). Denote $R_\Omega = \sigma_{i_n} \cdots \sigma_{i_1}$. It is an automorphism of $\Phi_{\geq -1}$ and does not depend on the choice of admissible ordering of Γ with respect to Ω (see [28]).

Let Ω_0 be one of the alternating orientations of Γ and Γ^+ (respectively, Γ^-), the set of sinks (respectively, sources) of (Γ, Ω_0).

Definition 2.3. Denote $\tau_\pm = \prod_{i \in \Gamma^\pm} \sigma_i$. Then $R_{\Omega_0} = \tau_- \tau_+$ (write R for short).

Denote by $n_i(\beta)$ the coefficient of the simple root α_i in the simple root expansion of β. There is a compatibility degree $(-||-)$ on $\Phi_{\geq -1}$ which is uniquely defined by the properties $(-\alpha_i||\beta) = \max\{n_i(\beta), 0\}$ and $(\alpha||\beta) = (\tau_\pm \alpha||\tau_\pm \beta)$ for any $i \in \Gamma$ and $\alpha, \beta \in \Phi_{\geq -1}$ (see [16]). We say that $\alpha, \beta \in \Phi_{\geq -1}$ are compatible if $(\alpha||\beta) = 0$.

Definition 2.4. [16] The cluster complex $\triangle(\Phi)$ is a simplicial complex on the ground set $\Phi_{\geq -1}$. Its faces are mutually compatible subsets of $\Phi_{\geq -1}$. The facets (maximal faces) of $\triangle(\Phi)$ are called the (root-)clusters associated to Φ.

• **Colored root system.** Recall some useful results from [12] first. For a positive integer d and a root $\alpha \in \Phi_{\geq -1}$, we denote by $\alpha^0, \alpha^1, \ldots, \alpha^{d-1}$ the d copies of α.

Definition 2.5. Let d be a positive integer, the set of colored almost positive roots is $\Phi^d_{\geq -1} = \{\alpha^i \mid i \in [0, d-1],\ \alpha \in \Phi_{>0}\} \cup \{(-\alpha_r)^0 \mid r \in [1, n]\}$.

The d-analogue of R in the colored root system is defined as follows.

Definition 2.6. [29] For any $\alpha^k \in \Phi^d_{\geq -1}$, define

$$R_d(\alpha^k) = \begin{cases} \alpha^{k+1} & \alpha \in \Phi_{>0},\ k < d-1, \\ R(\alpha)^0 & \text{otherwise.} \end{cases}$$

Theorem 2.7. [12] *Let Φ be a finite root system. There is a compatibility relation on $\Phi^d_{\geq -1}$ which has the following properties:*

(1) *α^k is compatible with β^l if and only if $R_d(\alpha^k)$ is compatible with $R_d(\beta^l)$.*

(2) *$(-\alpha_i)^0$ is compatible with β^l if and only if $n_i(\beta) = 0$.*

Furthermore, these conditions uniquely determine this relation.

We have the following d-version of the truncated simple reflections on the colored almost positive roots.

Definition 2.8. [29] Let s_k be the Coxeter generator of the Weyl group of Φ corresponding to $k \in [1, n]$. Define the d-truncated simple reflection σ^d_k as

$$\sigma^d_k(\alpha^i) = \begin{cases} \alpha^d_k & i = 0,\ \alpha = -\alpha_k, \\ \alpha^{i-1}_k & \alpha = \alpha_k,\ 0 < i \leq d-1, \\ (-\alpha_j)^0 & \alpha = -\alpha_j,\ i = 0,\ j \neq k, \\ s_k(\alpha)^i & \text{otherwise.} \end{cases}$$

This is a bijection of $\Phi^d_{\geq -1}$.

• **Repetitive colored root system.** For a positive integer m, we use m copies of the colored almost positive roots $\Phi^d_{\geq -1}$ defined above to construct the m-repetitive set of colored almost positive roots as follows.

Definition 2.9. The (m-)repetitive set of colored almost positive roots is

$$\begin{aligned} {}^m\Phi^d_{\geq -1} = &\ \{{}^j\alpha^i \mid i \in [0, d-1],\ j \in [0, m-1],\ \alpha \in \Phi_{>0}\} \\ &\cup \{{}^j(-\alpha_r)^0 \mid r \in [1, n],\ j \in [0, m-1]\}. \end{aligned}$$

Remark. When $m = 1$, we get the colored almost positive roots defined in [12]. When $m = 1$ and $d = 1$, we get almost positive roots in [16].

Definition 2.10. For any ${}^i\alpha^k \in {}^m\Phi^d_{\geq -1}$, define

$$R^m_d({}^i\alpha^k) = \begin{cases} {}^i\alpha^{k+1} & \alpha \in \Phi_{>0},\ k < d-1, \\ {}^{i+1}R(\alpha)^0 & R(\alpha) \notin \{-\alpha_r\}_{r\in[1,n]},\ k = d-1,\ i < m-1, \\ {}^iR(\alpha)^0 & R(\alpha) \in \{-\alpha_r\}_{r\in[1,n]},\ k = d-1,\ i \leq m-1, \\ {}^0R(\alpha)^0 & \text{otherwise}, \end{cases}$$

where $\{-\alpha_r\}_{r\in[1,n]}$ denotes the set of negative simple roots.

Now we proceed to generalize the d-truncated simple reflection of $\Phi^d_{\geq -1}$ to the repetitive set of colored almost positive roots ${}^m\Phi^d_{\geq -1}$.

Definition 2.11. Let s_k be the Coxeter generator of the Weyl group of Φ corresponding to $k \in [1, n]$. Define the generalized d-truncated simple reflection ${}^m\sigma^d_k$ as

$${}^m\sigma^d_k({}^j\alpha^i) = \begin{cases} {}^j\alpha_k^{d-1} & i = 0,\ \alpha = -\alpha_k, \\ {}^j\alpha_k^{i-1} & \alpha = \alpha_k,\ 0 < i \leq d-1, \\ {}^j(-\alpha_r)^0 & \alpha = -\alpha_r,\ r \neq k,\ i = 0, \\ {}^{m-1}s_k(\alpha)^0 & \alpha = \alpha_k,\ i = j = 0, \\ {}^{j-1}s_k(\alpha)^0 & \alpha = \alpha_k,\ i = 0,\ 0 < j < m, \\ {}^js_k(\alpha)^i & \text{otherwise}. \end{cases}$$

It is easy to see that ${}^m\sigma^d_k$ is a bijection of ${}^m\Phi^d_{\geq -1}$.

• **Compatible degree.** For a root $\beta \in \Phi_{\geq -1}$, let $d(\beta)$ denote the smallest d such that

$$\overbrace{R(R(R(\cdots R}^{d \text{ times}}(\beta)\cdots)))$$

is a negative root. In particular, $d(\beta) = 0$ if β is a negative simple root. Define $P : {}^m\Phi^d_{\geq -1} \to \Phi^d_{\geq -1}$ such that $P({}^j\alpha^i) = {}^0\alpha^i$ for any root ${}^j\alpha^i \in {}^m\Phi^d_{\geq -1}$. Here $\Phi^d_{\geq -1}$ can be regarded as ${}^0\Phi^d_{\geq -1}$.

Definition 2.12. We say that ${}^k\alpha^i, {}^s\beta^j \in {}^m\Phi^d_{\geq -1}$ are compatible if and only if $P({}^k\alpha^i)$ and $P({}^s\beta^j)$ are compatible on $\Phi^d_{\geq -1}$. In other words, one of the following holds (see [12]):

- $i > j$, $d(\alpha) \leq d(\beta)$ and the roots $R(\alpha)$ and β are compatible on $\Phi_{\geq -1}$,
- $i < j$, $d(\alpha) \geq d(\beta)$ and the roots α and $R(\beta)$ are compatible,
- $i > j$, $d(\alpha) > d(\beta)$ and the roots α and β are compatible,
- $i < j$, $d(\alpha) < d(\beta)$ and the roots α and β are compatible,
- $i = j$, and the roots α and β are compatible,

here write ${}^0\alpha^0$, ${}^0\beta^0$ as α, β for short.

From this definition and the symmetry of the compatibility relation on $\Phi^d_{\geq -1}$, one immediately has

Lemma 2.13. *The compatibility relation on ${}^m\Phi^d_{\geq -1}$ is symmetric.*

Theorem 2.14. *The compatibility relation on ${}^m\Phi^d_{\geq -1}$ has the following properties:*

(1) ${}^{k}\alpha^{i}$ *is compatible with* ${}^{s}\beta^{j}$ *if and only if* $R_d^m({}^{k}\alpha^{i})$ *is compatible with* $R_d^m({}^{s}\beta^{j})$.
(2) ${}^{t}(-\alpha_r)^0$ *is compatible with* ${}^{s}\beta^{j}$ *if and only if* $\max\{n_r(\beta), 0\} = 0$*, here* $n_r(\beta)$ *is the coefficient of* α_r *in the simple root expansion of* β.

Furthermore, conditions (1) *and* (2) *uniquely determine this compatibility relation.*

Proof. (1) Note that the diagram below commutes

$$\begin{array}{ccc} {}^{m}\Phi^{d}_{\geq -1} & \xrightarrow{R_d^m} & {}^{m}\Phi^{d}_{\geq -1} \\ \downarrow P & & \downarrow P \\ \Phi^{d}_{\geq -1} & \xrightarrow{R_d} & \Phi^{d}_{\geq -1} \end{array}$$

Then $R_d^m({}^{k}\alpha^{i})$ is compatible with $R_d^m({}^{s}\beta^{j})$ in ${}^{m}\Phi^{d}_{\geq -1}$ if and only if $P \cdot R_d^m({}^{k}\alpha^{i}) = R_d \cdot P({}^{k}\alpha^{i})$ is compatible with $P \cdot R_d^m({}^{s}\beta^{j}) = R_d \cdot P({}^{s}\beta^{j})$ in $\Phi^{d}_{\geq -1}$, if and only if $P({}^{k}\alpha^{i})$ is compatible with $P({}^{s}\beta^{j})$ in $\Phi^{d}_{\geq -1}$, i.e., ${}^{k}\alpha^{i}$ is compatible with ${}^{s}\beta^{j}$.

(2) ${}^{t}(-\alpha_r)^0$ is compatible with ${}^{s}\beta^{j}$ if and only if $P({}^{t}(-\alpha_r)^0)$ is compatible with $P({}^{s}\beta^{j})$ in $\Phi^{d}_{\geq -1}$, if and only if $\max\{n_r(\beta), 0\} = 0$ by [12, Theorem 3.4].

The last assertion is immediate from the definition of compatibility relation on ${}^{m}\Phi^{d}_{\geq -1}$ and [12, Theorem 3.4]. □

Definition 2.15. Let ${}^{m}\Phi^{d}_{\geq -1}$ be the set of repetitive colored almost positive roots.
(1) A subset of ${}^{m}\Phi^{d}_{\geq -1}$ is called compatible if any two elements of it are compatible.
(2) The simplicial complex ${}^{m}\triangle^{d}(\Phi)$ associated to ${}^{m}\Phi^{d}_{\geq -1}$ is a simplicial complex on the ground set ${}^{m}\Phi^{d}_{\geq -1}$. Its simplices are compatible subsets of ${}^{m}\Phi^{d}_{\geq -1}$. We call this complex the m-repetitive generalized cluster complex associated to the root system Φ. Sometimes we simply call it the repetitive generalized cluster complex.

Note that when $m = 1$, the complex ${}^{m}\triangle^{d}(\Phi)$ is the generalized cluster complex defined by Fomin-Reading [12].

3 Generalized d-Cluster Categories

Let H be a finite dimensional hereditary algebra over a field K, and $\mathcal{H}$ the category of finite dimensional modules over H. Denote the bounded derived category of H as $\mathcal{D}^b(H)$. Then the derived category $\mathcal{D}^b(H)$ has AR triangles. The Auslander-Reiten translation τ and the shift functor [1] are automorphisms of $\mathcal{D}^b(H)$. For a positive integer d, set $F_d = \tau^{-1}[d]$, which is an automorphism of $\mathcal{D}^b(H)$. For a positive integer m, denote by $\langle F_d^m \rangle$ the (cyclic) group generated by F_d^m.

Definition 3.1. Fix two positive integers m and d. Define the generalized d-cluster category as the orbit category $\mathcal{D}^b(H)/\langle F_d^m \rangle$, the objects are the same as those in $\mathcal{D}^b(H)$ and the set of morphisms

$$\mathrm{Hom}_{\mathcal{D}^b(H)/\langle F_d^m \rangle}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i \in Z} \mathrm{Hom}_{\mathcal{D}^b(H)}(X, (F_d^m)^i Y),$$

where $\widetilde{X}$, $\widetilde{Y}$ are the corresponding objects in $\mathcal{D}^b(H)/\langle F_d^m \rangle$ for $X, Y \in \mathcal{D}^b(H)$. Denote $\mathcal{C}_d^m(H) = \mathcal{D}^b(H)/\langle F_d^m \rangle$, and write it as $\mathcal{C}_d^m$ for short.

Remark. This is a generalization of the d-cluster category $\mathcal{C}_d(H)$ which is defined as the orbit category $\mathcal{D}^b(H)/\langle F_d\rangle$ in [18], and also a generalization of generalized cluster categories studied in [30].

There is a natural projection functor $\pi_m : \mathcal{D}^b(H) \to \mathcal{D}^b(H)/\langle F_d^m\rangle = \mathcal{C}_d^m$. In particular, when $m = 1$, we get $\pi : \mathcal{D}^b(H) \to \mathcal{C}_d$, the projection to the d-cluster category. Another natural projection functor is $\rho_m : \mathcal{C}_d^m \to \mathcal{C}_d$. It is not difficult to verify $\pi = \rho_m \cdot \pi_m$.

One can identify the set $\operatorname{ind}\mathcal{C}_d(H)$ with the fundamental domain for the action of F_d on $\operatorname{ind}\mathcal{D}^b(H)$ (see [6]). Passing to the orbit category $\mathcal{C}_{F_d^m}(H)$, one can view $\operatorname{ind}\mathcal{C}_d(H)$ as a (probably not fully) subcategory of $\operatorname{ind}\mathcal{C}_{F_d^m}(H)$.

Proposition 3.2.

(1) *$\mathcal{C}_d^m$ is a triangulated category with AR triangles and Serre functor $\Sigma = \tau[1]$, where τ is an AR-translation in $\mathcal{C}_d^m$, which is induced from an AR-translation in $\mathcal{D}^b(H)$.*

(2) *$\pi_m : \mathcal{D}^b(H) \to \mathcal{C}_d^m$ and $\rho_m : \mathcal{C}_d^m \to \mathcal{C}_d$ are triangle functors and also covering functors.*

(3) *$\mathcal{C}_d^m$ has CY-dimension $\frac{m+md}{m}$.*

(4) *$\mathcal{C}_d^m$ is a Krull-Remak-Schmidt category.*

(5) $\operatorname{ind}\mathcal{C}_d^m = \bigcup_{i=0}^{m-1} \operatorname{ind} F_d^i\mathcal{C}_d$.

Proof. This proposition is just an analogy of [30, Proposition 4.3].

(1) It follows from [6, Proposition 1.3] and [18].

(2) The proof of [30, Proposition 4.3] for the situation in the cluster category $\mathcal{C}$ can be extended to the generalized d-cluster category $\mathcal{C}_d^m$.

(3) For the Serre functor $\Sigma = \tau[1]$ in $\mathcal{C}_d^m$, $\Sigma^m = \tau^m[m] = \tau^m[-md][m+md] = [m+md]$.

(4) comes from [6] and (5) is obvious. □

For any $i \in Z$, denote $\mathcal{H}[i]$ by $\mathcal{H}_i$, the copy of $\mathcal{H}$ under the i-th shift $[i]$ as a subcategory of $\mathcal{C}_d^m$.

For a triangulated category $\mathcal{D}$, define $\operatorname{Ext}^i_{\mathcal{D}}(X,Y) = \operatorname{Hom}_{\mathcal{D}}(X, Y[i])$. For $X, Y \in \mathcal{D}$, let $\operatorname{Hom}(X,Y)$ and $\operatorname{Ext}(X,Y)$ denote $\operatorname{Hom}_{\mathcal{D}}(X,Y)$ and $\operatorname{Ext}^1_{\mathcal{D}}(X,Y)$, respectively. Suppose that $\mathcal{T}$ is a functorially finite subcategory of $\mathcal{D}$, and X is an object in $\mathcal{D}$. If $\operatorname{Ext}^i(X,T) = 0$ for any $T \in \mathcal{T}$, then we write $\operatorname{Ext}^i(X,\mathcal{T}) = 0$. If $\operatorname{Ext}^i(T,T') = 0$ for any $T, T' \in \mathcal{T}$, then we write $\operatorname{Ext}^i(\mathcal{T},\mathcal{T}) = 0$. For $M \in \mathcal{D}$, let $\operatorname{add} M$ be the full subcategory of $\mathcal{D}$ consisting of direct summands of direct sums of copies of M.

Definition 3.3. (1) $\mathcal{T}$ is called d-exceptional (or equivalently, d-rigid) if $\operatorname{Ext}^i(\mathcal{T},\mathcal{T}) = 0$ for all $1 \le i \le d$. An object X is called d-exceptional if the subcategory $\operatorname{add} X$ is d-exceptional. (2) $\mathcal{T}$ is called d-cluster tilting whenever $X \in \mathcal{T}$ if and only if $\operatorname{Ext}^i(X,\mathcal{T}) = 0$ for all $1 \le i \le d$, if and only if $\operatorname{Ext}^i(\mathcal{T},X) = 0$ for all $1 \le i \le d$. An object X is called a d-cluster tilting object if $\operatorname{add} X$ is d-cluster tilting. (3) $\mathcal{T}$ is called maximal d-rigid if $\operatorname{Ext}^i(\mathcal{T},\mathcal{T}) = 0$ for all $1 \le i \le d$, and it is maximal respect to this property, i.e., if there is Y such that $\operatorname{Ext}^i(X \oplus Y, X \oplus Y) = 0$ for any $1 \leq i \leq d$ and for any $X \in \mathcal{T}$, then $Y \in \operatorname{add} X$. An object X is called maximal d-rigid if $\operatorname{add} X$ is maximal d-rigid. (4) An object X in $\mathcal{C}_d^m$ is called complete d-rigid if it is a d-exceptional object which contains exactly $n \cdot m$ non-isomorphic indecomposable direct summands in $\mathcal{C}_d^m$.

The next two lemmas hold for any triangulated category $\mathcal{D}$.

Lemma 3.4. [17] *For any d-cluster tilting subcategory $\mathcal{T}$ of a triangulated category $\mathcal{D}$, $F_d\mathcal{T} = \mathcal{T}$. In particular, F_d is an automorphism of $\mathcal{T}$.*

Lemma 3.5. [17] *$\mathcal{T} = \operatorname{add} X$ is a d-cluster tilting subcategory of $\mathcal{D}$ if and only if it satisfies the following conditions:*

(1) *$F_d\mathcal{T} = \mathcal{T}$.*

(2) *If $\operatorname{Ext}^i(Y, X) = 0$ for $1 \leq i \leq d$, then $Y \in \operatorname{add} X$.*

The d-cluster tilting subcategories (d-cluster tilting objects) have been defined in derived categories, d-cluster categories and generalized d-cluster categories. The rest of this section is devoted to the relations among them.

Theorem 3.6. *An object T in $\mathcal{C}_d^m$ is a d-cluster tilting object if and only if $\pi_m^{-1}(\operatorname{add} T)$ is a d-cluster tilting subcategory of $\mathcal{D}^b(H)$.*

Proof. Let $\mathcal{T}$ denote the d-cluster tilting subcategory in $\mathcal{D}^b(H)$. By Lemma 3.4, without loss of generality, set $\mathcal{T}' = \{\bigoplus_{i=0}^{d-1} \mathcal{H}_i \oplus H[d]\} \cap \mathcal{T}$. Then $\mathcal{T} = \bigoplus_{n\in Z} F_d^n \mathcal{T}'$, and $\pi_m(\mathcal{T}) = \bigoplus_{i=0}^{m-1} F_d^i \mathcal{T}' = \operatorname{add} T$, denoted by $\mathcal{T}_1$. For any $\widetilde{T_1}, \widetilde{T_2} \in \mathcal{T}_1$, there are $T_1, T_2 \in \mathcal{T}$ such that $\widetilde{T_1} = F_d^t(\pi_m(T_1))$ and $\widetilde{T_2} = F_d^s(\pi_m(T_2))$, where $0 \leq t, s \leq m-1$. For $1 \leq i \leq d$, we have

$$\begin{aligned} &\operatorname{Ext}^i(\widetilde{T_1}, \widetilde{T_2}) = \operatorname{Hom}(\widetilde{T_1}, \widetilde{T_2}[i]) \\ = &\bigoplus_{n\in Z} \operatorname{Hom}_{\mathcal{D}^b(H)}(F_d^t T_1, (F_d^m)^n F_d^s T_2[i]) \\ = &\bigoplus_{n\in Z} \operatorname{Hom}_{\mathcal{D}^b(H)}(T_1, F_d^{mn+s-t} T_2[i]). \end{aligned}$$

Obviously, if $mn + s - t \notin \{0, -1\}$, then $\operatorname{Hom}_{\mathcal{D}^b(H)}(T_1, F_d^{mn+s-t} T_2[i]) = 0$.

When $mn + s - t = -1$,

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{D}^b(H)}(T_1, F_d^{-1} T_2[i]) = \operatorname{Hom}_{\mathcal{D}^b(H)}(T_1, \tau T_2[i-d]) \\ \cong\ &D\operatorname{Ext}_{\mathcal{D}^b(H)}(T_2[i-d], T_1) \cong D\operatorname{Ext}_{\mathcal{D}^b(H)}^{d-i+1}(T_2, T_1) = 0. \end{aligned}$$

The last equality holds because $1 \leq i \leq d$ and $\mathcal{T}$ is a d-cluster tilting subcategory. When $mn + s - t = 0$, $\operatorname{Hom}_{\mathcal{D}^b(H)}(T_1, T_2[i]) = \operatorname{Ext}^i_{\mathcal{D}^b(H)}(T_1, T_2) = 0$ for $1 \leq i \leq d$ by the same reason, so $\mathcal{T}_1$ is exceptional in $\mathcal{C}_d^m$.

If there is an indecomposable $\widetilde{X} = \pi_m(X) \in \mathcal{C}_d^m$ with $X \in \mathcal{D}^b(H)$ such that $\operatorname{Ext}^i(\mathcal{T}_1, \widetilde{X}) = 0$ for $1 \leq i \leq d$, then $\operatorname{Ext}^i_{\mathcal{D}^b(H)}\big(\bigoplus_{n\in Z} F_d^n \mathcal{T}', X\big) = \operatorname{Ext}^i_{\mathcal{D}^b(H)}(\mathcal{T}, X)$ $= 0$ for $1 \leq i \leq d$. Thus, $X \in \mathcal{T}$ since $\mathcal{T}$ is d-cluster tilting. Then $\widetilde{X} \in \mathcal{T}_1$. Finally, $\mathcal{T}_1 = \pi_m(\mathcal{T})$ is a d-cluster tilting subcategory in $\mathcal{C}_d^m$.

On the other hand, assume $\mathcal{T}_1$ is a d-cluster tilting subcategory in $\mathcal{C}_d^m$. Since $\mathrm{Ext}^k(\mathcal{T}_1, \mathcal{T}_1) = \bigoplus_{n\in Z} \mathrm{Ext}^k_{\mathcal{D}^b(H)}\big(\bigoplus_{i=0}^{m-1} F_d^i \mathcal{T}', (F_d^m)^n\big(\bigoplus_{i=0}^{m-1} F_d^i\mathcal{T}'\big)\big) = 0$ for $0 < k \leq d$, $\mathrm{Ext}^k_{\mathcal{D}^b(H)}(\mathcal{T}', F_d^n\mathcal{T}') = 0$ for $1 \leq k \leq d$ and $n \in Z$. Since $\mathcal{T} = \bigoplus_{n\in Z} F_d^n \mathcal{T}'$, $\mathcal{T}$ is orthogonal. If $X \in \mathcal{D}^b(H)$ satisfies $\mathrm{Ext}^k_{\mathcal{D}^b(H)}(\mathcal{T}, X) = 0$ for $1 \leq k \leq d$, then $\mathrm{Ext}^k(F_d^i\mathcal{T}', \widetilde{X}) = 0$ for $0 \leq i < m$. Therefore, $\widetilde{X} \in \mathcal{T}_1$, hence $X \in \mathcal{T}$. □

As a special case when $m = 1$, we get the following result, which generalizes [21, Lemma 4.14].

Corollary 3.7. *An object T in $\mathcal{C}_d(H)$ is a d-cluster tilting object if and only if $\pi^{-1}(\mathrm{add}\, T)$ is a d-cluster tilting subcategory of $\mathcal{D}^b(H)$.*

With the above results, a 1-1 correspondence between the d-cluster tilting subcategories in $\mathcal{D}^b(H)$ and the d-cluster tilting subcategories in $\mathcal{C}_d^m$ is established via the triangle functor π_m, so is a 1-1 correspondence between the d-cluster tilting subcategories in $\mathcal{D}^b(H)$ and the d-cluster tilting subcategories in $\mathcal{C}_d$ via the triangle functor π.

Theorem 3.8. *Let H be a hereditary algebra, $\mathcal{C}_d(H)$ be the d-cluster category and $\mathcal{C}_d^m(H)$ be the generalized d-cluster category. Then*

(1) *$\mathcal{T}'$ is a d-cluster tilting subcategory in $\mathcal{C}_d(H)$ if and only if $\rho_m^{-1}(\mathcal{T}')$ is a d-cluster tilting subcategory in $\mathcal{C}_d^m(H)$, if and only if $\pi^{-1}(\mathcal{T}')$ is a d-cluster tilting subcategory in $\mathcal{D}^b(H)$.*

(2) *For any d-cluster tilting subcategory $\mathcal{T}'$ in $\mathcal{C}_d(H)$, $\bigoplus_{i=0}^{m-1} F_d^i\mathcal{T}'$ is a d-cluster tilting subcategory in $\mathcal{C}_d^m(H)$. And any d-cluster tilting subcategory in $\mathcal{C}_d^m(H)$ arises in this way.*

Proof. (1) Since $\pi = \rho_m \cdot \pi_m$, $\rho_m^{-1}(\mathcal{T}')$ is a d-cluster tilting subcategory in $\mathcal{C}_d^m(H)$ if and only if $\pi_m^{-1}\rho_m^{-1}(\mathcal{T}')$ is a d-cluster tilting subcategory in $\mathcal{D}^b(H)$. Since $\mathcal{T}' = \pi(\pi_m^{-1}\rho_m^{-1}(\mathcal{T}'))$, $\rho_m^{-1}(\mathcal{T}')$ is d-cluster tilting if and only if $\mathcal{T}'$ is d-cluster tilting.

(2) It is a direct consequence of (1) and Lemma 3.4. □

Theorem 3.9. *Let X be a d-rigid object in $\mathcal{C}_d^m(H)$. Then the following statements are equivalent:*

(1) *X is d-cluster tilting.*

(2) *X is maximal d-rigid.*

(3) *X is complete d-rigid.*

Proof. (2)⇒(1) Since X is maximal d-rigid in $\mathcal{C}_d^m$, $\rho_m(X)$ is also maximal d-rigid in $\mathcal{C}_d$. Then $\rho_m(X)$ is a d-cluster tilting object in $\mathcal{C}_d$ by [26]. Using the theorems above, X is a d-cluster tilting object in $\mathcal{C}_d^m$.

(1)⇒(2) It is straightforward by definition.

(2)⇒(3) By Lemma 3.4, since a d-cluster tilting object in $\mathcal{C}_d$ has n non-isomorphic indecomposable direct summands, the maximal d-rigid object in $\mathcal{C}_d^m$ has nm non-isomorphic indecomposable direct summands.

(3)⇒(2) If the complete d-rigid object with nm non-isomorphic indecomposable direct summands is not maximal, it can be extended to a maximal d-rigid object with at least $nm+1$ non-isomorphic indecomposable direct summands. It is a contradiction. $\square$

4 Generalized d-Cluster Tilted Algebra

In this section, we study the endomorphism algebras of d-cluster tilting objects in the generalized d-cluster categories. These algebras generalize cluster tilted algebras (see [7, 30]) and d-cluster tilted algebras (see [20]).

Let $\mathcal{T}$ be a full subcategory of a triangulated category $\mathcal{D}$ which is closed under taking direct sums and direct summands, in other words, for any objects $X, Y \in \mathcal{D}$, $X \oplus Y \in \mathcal{T}$ if and only if $X, Y \in \mathcal{T}$. The quotient category $\mathcal{A} := \mathcal{D}/\mathcal{T}$ has the same objects as $\mathcal{D}$ and morphisms from X to Y in $\mathcal{A}$ are the residue classes of $\mathcal{D}$-morphisms modulo the subgroup of morphisms factoring through some object in $\mathcal{T}$.

By a $\mathcal{T}$-module, we mean a contravariant K-linear functor from $\mathcal{T}$ to the category of vector spaces. Denote by $\operatorname{mod}\mathcal{T}$ the category of finitely presented $\mathcal{T}$-modules, by $F : \mathcal{D} \to \operatorname{mod}\mathcal{T}$ the functor which sends $X \in \mathcal{D}$ to the module $T \to \operatorname{Hom}(T, X)$. Since idempotents split in $\mathcal{T}$, this functor induces an equivalence from $\mathcal{T}$ to the category of projectives in $\operatorname{mod}\mathcal{T}$.

Definition 4.1. Let T be a d-cluster tilting object in the generalized d-cluster category $\mathcal{C}_d^m$. The endomorphism algebra $\operatorname{End}_{\mathcal{C}_d^m}T$ of T is called the generalized d-cluster tilted algebra.

Lemma 4.2. [17] *Let $\mathcal{T}$ be a $(d-1)$-cluster tilting subcategory of a triangulated category $\mathcal{D}$. For any $0 < l < d$, denote*

$$Z_l = \bigcap_{0<i<d,\ i\neq l} \mathcal{T}[-i]^{\perp} = \bigcap_{0<i<d,\ i\neq n-l} {}^{\perp}\mathcal{T}[i].$$

Then there exists an equivalence $G_l : Z_l/\mathcal{T} \to \operatorname{mod}\mathcal{T}$ by $G_l X := \operatorname{Hom}_{\mathcal{T}}(-, X[l])$ which preserves 2-rigidity. In particular, $Z_1/\mathcal{T} \cong Z_2/\mathcal{T} \cong \cdots \cong Z_{d-1}/\mathcal{T}$ is an abelian category.

Let $\mathcal{T}', \mathcal{T}_1$ and $\mathcal{T}$ be the d-cluster tilting subcategories in $\mathcal{C}_d, \mathcal{C}_d^m$ and $D^b(H)$, respectively, as stated in Theorem 3.8. By the discussion in the last section, one can assume $\mathcal{T}_1 = \bigoplus_{i=0}^{m-1} F_d^i \mathcal{T}'$ and $\mathcal{T} = \bigoplus_{i\in Z} F_d^i \mathcal{T}'$. Construct $Z_l' = \bigcap\limits_{0<i\leq d,\ i\neq l} \mathcal{T}'[-i]^{\perp}$ in $\mathcal{C}_d$, similarly Z_{l*} in $\mathcal{C}_d^m$ and Z_l in $D^b(H)$.

Theorem 4.3. *Let T be a d-cluster tilting object in $\mathcal{C}_d^m$ with the generalized d-cluster tilted algebra $\widetilde{A} = \operatorname{End}_{\mathcal{C}_d^m}T$. Denote by $\mathcal{T}_1 = \operatorname{add}T$ the d-cluster tilting subcategory in $\mathcal{C}_d^m$, and by $\mathcal{T}'$ and $\mathcal{T}$ the corresponding d-cluster tilting subcategories in $\mathcal{C}_d$ and $D^b(H)$, respectively. Then*

(1) *$\widetilde{A}$ has a Galois covering induced by the restriction of the projection $\pi_m : D^b(H) \to \mathcal{C}_d^m$ to $\mathcal{T}$.*

(2) *The projection π_m induces a push-down functor $\tilde{\pi}_m : Z_1/\mathcal{T}[1] \to \widetilde{A}$-mod.*

Proof. (1) As $\pi_m(\mathcal{T}) = \mathcal{T}_1$ and $Z_1 = \bigcap\limits_{i=2}^{m} {}^{\perp}\mathcal{T}[i]$ by Theorems 3.6, 3.8 and the definition of projection π_m, we have $Z_{1*} = \bigcap_{i=2}^{m} {}^{\perp}\mathcal{T}_1[i] = \pi_m(Z_1)$. Using Lemma 4.2, there is an equivalence $Z_{1*}/\mathcal{T}_1[1] \to \operatorname{mod}\mathcal{T}_1$ by $X \to (-,X)$, and $\operatorname{mod}\mathcal{T}_1$ is $\widetilde{A}$-mod. The projection π_m sends $D^b(H)$ to $\mathcal{C}_d^m$, and $\pi_m|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_1$ is a Galois covering with Galois group generated by F_d^m.

(2) There are equivalences $Z_1/\mathcal{T}[1] \to \operatorname{mod}\mathcal{T}$ and $Z_{1*}/\mathcal{T}_1[1] \to \operatorname{mod}\mathcal{T}_1$ by Lemma 4.2. Define $\bar{\pi}_m(X) := \pi_m|_{Z_1}(X)$ for any $X \in Z_1/\mathcal{T}[1]$ and $\bar{\pi}_m(f) := \pi_m|_{Z_1} f$ for any morphism $f : X \to Y$ in $Z_1/\mathcal{T}[1]$. It is easy to see that $\bar{\pi}_m$ is well defined and the following diagram commutes

$$\begin{array}{ccc} Z_1 & \xrightarrow{\pi_m|_{Z_1}} & Z_{1*} \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} \\ Z_1/\mathcal{T}[1] & \xrightarrow{\bar{\pi}_m|_{Z_1}} & Z_{1*}/\mathcal{T}_1[1] \end{array}$$

Here P_1, P_2 are natural quotient functors. Then $\bar{\pi}_m|_{Z_1}$ is a covering functor from $Z_1/\mathcal{T}[1]$ to $Z_{1*}/\mathcal{T}_1[1]$. By Lemma 4.2, $Z_{1*}/\mathcal{T}_1[1]$ is equivalent to $\operatorname{mod}\mathcal{T}_1$ which is $\widetilde{A}$-mod, the assertion is true. □

Theorem 4.4. *Let* $A = \operatorname{End}_{\mathcal{C}_d} T'$ *and* $\widetilde{A} = \operatorname{End}_{\mathcal{C}_d^m} T_1$, *where* $T_1 = \rho_m^{-1}(T')$. *Let* $\mathcal{T}' = \operatorname{add} T'$ *and* $\mathcal{T}_1 = \operatorname{add} T_1$. *Then*

(1) *The restriction of* $\rho_m : \mathcal{C}_d^m \to \mathcal{C}_d$ *to* $\mathcal{T}_1$ *induces a Galois covering of* A.

(2) *The functor* ρ_m *also induces a push-down functor* $\widetilde{\rho}_m : \widetilde{A}$-mod $\to A$-mod.

Proof. (1) Following the proof of Theorem 4.3, by Lemma 4.2, there is an equivalence between $Z_{1*}/\mathcal{T}_1[1]$ and $\operatorname{mod}\mathcal{T}_1$. For any $X \in Z_{1*}/\mathcal{T}_1[1]$, the image under the equivalence functor is $(-,X) \in \operatorname{mod}\mathcal{T}_1$. The triangle functor $\rho_m|_{\mathcal{T}_1}(\mathcal{T}_1) = \mathcal{T}'$, thus $\rho_m|_{\mathcal{T}_1}$ is a Galois covering.

(2) By Lemma 4.2, $Z_1'/\mathcal{T}'[1] \cong A$-mod and $Z_{1*}/\mathcal{T}_1[1] \cong \widetilde{A}$-mod. Define the induced functor $\bar{\rho}_m$ such that $\bar{\rho}_m(X) := \rho_m X$ for $X \in Z_{1*}/\mathcal{T}_1[1]$ and $\bar{\rho}_m(f) := \rho_m f$ for $f : X \to Y$ in $Z_{1*}/\mathcal{T}_1[1]$. Clearly, $\bar{\rho}_m$ is well defined and the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}_d^m & \xrightarrow{\rho_m} & \mathcal{C}_d \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} \\ Z_{1*}/\mathcal{T}_1[1] & \xrightarrow{\bar{\rho}_m} & Z_1'/\mathcal{T}'[1] \end{array}$$

Here P_1, P_2 are the natural quotient functors. Then $\bar{\rho}_m$ is a covering functor from $Z_{1*}/\mathcal{T}_1[1]$ to $Z_1'/\mathcal{T}'[1]$ which is A-mod. □

Example 4.5. Let us consider the derived category of the path algebra A_5

$$5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1.$$

Use P_i and E_j to denote the projective and simple modules, respectively. Consider F_d for $d = 2$. If we choose the 2-cluster tilting subcategory $\mathcal{T} \subset \mathcal{D}^b(A_5)$ generated by $\{F_d^nP_1, F_d^nP_2, F_d^nP_5, F_d^nE_5, F_d^nE_4[1]\}_{n\in Z}$, then $\mathcal{D}^b(A_5)/\mathcal{T}[1] \cong A_\infty$-mod, where the Gabriel quiver of A_∞ is

$$\begin{array}{ccccccccccccccc} & & P_1 & & & & & & & & F_dP_1 & & & & \\ & & \downarrow & & & & & & & & \downarrow & & & & \\ \cdots F_d^{-1}E_4[1] & \longrightarrow & P_2 & \longrightarrow & P_5 & \longrightarrow & E_5 & \longrightarrow & E_4[1] & \longrightarrow & F_dP_2 & \longrightarrow & F_dP_5 & \longrightarrow & \cdots \end{array}$$

Take $m = 2$. In the generalized 2-cluster category $\mathcal{C}_2^2$, choose the corresponding 2-cluster tilting object T generated by $\{F_d^nP_1, F_d^nP_2, F_d^nP_5, F_d^nE_5, F_d^nE_4[1]\}_{n=0,1}$. Then $\mathcal{C}_2^2(A_5)/(\operatorname{add} T)[1]$ is isomorphic to the module category of the algebra of quiver

$$\begin{array}{ccccccccccc} P_1 & \longrightarrow & P_2 & \longrightarrow & P_5 & \longrightarrow & E_5 & \longrightarrow & E_4[1] & & \\ & & \uparrow & & & & & & \downarrow & & \\ & & F_dE_4[1] & \longleftarrow & F_dE_5 & \longleftarrow & F_dP_5 & \longleftarrow & F_dP_2 & \longleftarrow & F_dP_1 \end{array}$$

Take $m = 1$. In the 2-cluster category $\mathcal{C}_2$, we choose the corresponding 2-cluster tilting object T generated by $\{F_d^nP_1, F_d^nP_2, F_d^nP_5, F_d^nE_5, F_d^nE_4[1]\}_{n=0}$. Therefore, $\mathcal{C}_2(A_5)/(\operatorname{add} T)[1]$ is isomorphic to the module category of the algebra of quiver

$$\begin{array}{ccccc} P_1 & \longrightarrow & P_2 & \longrightarrow & P_5 \\ & & \uparrow & & \downarrow \\ & & E_4[1] & \longleftarrow & E_5 \end{array}$$

As $F_d : \mathcal{C}_d^m \to \mathcal{C}_d^m$ is an automorphism with $F_d^m = \mathrm{id}$, $F_d|_\mathcal{T}$ is an automorphism of $\mathcal{T}$. The restriction of F_d on Z_l is still an automorphism. So there is an induced automorphism $\widetilde{F_d} : \operatorname{mod}\mathcal{T} \to \operatorname{mod}\mathcal{T}$ since $Z_l/\mathcal{T}[l] \cong \operatorname{mod}\mathcal{T}$ by Lemma 4.2.

Proposition 4.6. $\widetilde{F_d} : \operatorname{mod}\mathcal{T} \to \operatorname{mod}\mathcal{T}$ *is naturally induced from* $F_d|_\mathcal{T} : \mathcal{T} \to \mathcal{T}$.

Proof. Since $F_d|_\mathcal{T}$ is an automorphism of $\mathcal{T}$ and $F_d|_{Z_l}$ is an automorphism of Z_l, F_d induces $\widetilde{F_d} : Z_l/\mathcal{T}[l] \to Z_l/\mathcal{T}[l]$, which can be illustrated by the diagram below:

$$\begin{array}{ccccc} \mathcal{T} & \longrightarrow & Z_l & \longrightarrow & Z_l/\mathcal{T}[l] \\ \downarrow{\scriptstyle F_d|_\mathcal{T}} & & \downarrow{\scriptstyle F_d|_{Z_l}} & & \downarrow{\scriptstyle \widetilde{F_d}} \\ \mathcal{T} & \longrightarrow & Z_l & \longrightarrow & Z_l/\mathcal{T}[l] \end{array}$$

If $\widetilde{X} = \widetilde{Y} \in Z_l/\mathcal{T}[l]$, then in Z_l, $X = Y + \mathcal{T}[l]$. Hence, $F_d(X) = F_d(Y) + F_d\mathcal{T}[l]$ $= F_d(Y) + \mathcal{T}[l]$, so in $Z_l/\mathcal{T}[l]$, $\widetilde{F_d}\widetilde{X} = \widetilde{F_d}\widetilde{Y}$, thus $\widetilde{F_d}$ is well defined. By Lemma 4.2, any object in $\mathcal{T}$ corresponds one element in $Z_l/\mathcal{T}[l] \cong \operatorname{mod}\mathcal{T}$. For $(-, X) \in \operatorname{mod}\mathcal{T}$, $\widetilde{F_d}(-, X) = (-, F_dX)$, and for $(-, f) : (-, X) \to (-, Y)$, $\widetilde{F_d}(-, f) = (-, F_df)$.

On the other hand, $F_d|_\mathcal{T}$ sends $X \in \mathcal{T}$ to F_dX. By the natural functor $g : \mathcal{T} \to \operatorname{mod}\mathcal{T}$ which sends $X \in \mathcal{T}$ to $(-, X)$, $F_d|_\mathcal{T}$ induces a functor $F_d^* : \operatorname{mod}\mathcal{T} \to \operatorname{mod}\mathcal{T}$

such that the diagram below commutes

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F_d} & \mathcal{T} \\ \downarrow g & & \downarrow g \\ \operatorname{mod}\mathcal{T} & \xrightarrow{F_d^*} & \operatorname{mod}\mathcal{T} \end{array}$$

In other words, $g \cdot F_d(X) = (-, F_d X) = F_d^*(-, X) = F_d^* \cdot g(X)$ for $X \in \mathcal{T}$, and $g \cdot F_d(f) = (-, F_d f) = F_d^*(-, f) = F_d^* \cdot g(f)$ for $f : X \to Y$. Therefore, $F_d^* = \widetilde{F_d}$, the statement is true. □

5 Relation Between the Generalized d-Cluster Category and the Repetitive Colored Root System

In this section, we define cluster complexes of generalized d-cluster categories, and then construct a 1-1 correspondence between the generalized d-cluster category and repetitive colored almost positive roots. With this correspondence, the cluster complex of the generalized d-cluster category is isomorphic to the m-repetitive generalized cluster complex defined above.

Note that any indecomposable object X in $\mathcal{C}_d^m(H)$ is of the form $F_d^j M[i]$, here $M \in \operatorname{ind}\mathcal{H}$ and if $i = d$, then $M = P_r$ is the indecomposable projective module. From now on, we denote ${}^i M^k := F_d^i M[k]$ for any module M.

• **BGP-reflection functors.** A triangle functor $G : \mathcal{D}^b(H) \to \mathcal{D}^b(H')$ with some special properties induces a well defined triangle functor $\widetilde{G} : \mathcal{C}_d^m(H) \to \mathcal{C}_d^m(H')$ by [18, 28], especially the following diagram commutes with two rows triangle equivalent

$$\begin{array}{ccc} \mathcal{D}^b(H) & \xrightarrow{G} & \mathcal{D}^b(H') \\ \downarrow & & \downarrow \\ \mathcal{C}_d^m(H) & \xrightarrow{\widetilde{G}} & \mathcal{C}_d^m(H') \end{array}$$

Let k be a vertex in the valued quiver (Γ, Ω). The reflection of (Γ, Ω) at k is the valued quiver $(\Gamma, s_k\Omega)$, where $s_k\Omega$ is the orientation of Γ obtained from Ω by reversing all arrows starting or ending at k. The corresponding category of representations of $(\Gamma, s_k\Omega, \mathcal{M})$ is denoted simply by $s_k\mathcal{H}$ and the corresponding tensor algebra is denoted by s_kH. If k is a sink in the valued quiver (Γ, Ω), then k is a source of $(\Gamma, s_k\Omega)$, and the reflection of $(\Gamma, s_k\Omega)$ at k is (Γ, Ω).

For a sink k of the valued quiver (Γ, Ω), consider the derived BGP-reflection functor $S_k^+ : \mathcal{D}^b(H) \to \mathcal{D}^b(s_kH)$ induced from the BGP-reflection functor $\mathcal{H} \to s_k\mathcal{H}$ (refer to [28] for more details). For any $X[i] \in \operatorname{ind}\mathcal{D}^b(H)$,

$$S_k^+(X[i]) = \begin{cases} S_k^+(X)[i] & X \ncong P_k,\ i \in Z, \\ E_k'[i-1] & \text{otherwise}, \end{cases}$$

where P_k is the indecomposable projective representation corresponding to k, E_k' is the simple representation of the quiver $(\Gamma, s_k\Omega)$ of vertex k. In $\mathcal{D}^b(s_kH)$, $E_k'[i-1] =$

$\tau P'_k[i]$. As proved in [18, 27, 28], the derived BGP-reflection functor $S_k^+ : \mathcal{D}^b(H) \to \mathcal{D}^b(s_k H)$ induces a triangle equivalence from $\mathcal{C}_d^m(H)$ to $\mathcal{C}_d^m(s_k H)$, which is also denoted by S_k^+.

The following result can be verified in a similar way as the proof of [28, Proposition 2.7].

Proposition 5.1. *Let k be a sink of the valued quiver (Γ, Ω), and Y an indecomposable object in $\mathcal{C}_d^m(H)$. Then $Y = {}^jX^i$ for an indecomposable object X in $\mathcal{H}$, and*

$$S_k^+({}^jX^i) = \begin{cases} {}^{m-1}P'_k[d] & X \cong P_k \cong E_k,\ i = j = 0, \\ {}^{j-1}P'_k[d] & X \cong P_k \cong E_k,\ i = 0,\ j \neq 0, \\ {}^jE'_k[i-1] & X \cong P_k \cong E_k,\ 0 < i \leq d, \\ {}^jP'_r[d] & X \cong P_r \not\cong P_k,\ i = d, \\ {}^jS_k^+(X)[i] & \text{otherwise.} \end{cases}$$

Remark. When $m = 1$, this result degenerates to the case considered in [28].

• **Construction of the relation.** Let $\mathcal{C}_d^m(H)$ be the generalized d-cluster category of H, where H is a finite dimensional hereditary algebra of Dynkin type. Let $\operatorname{ind} H$ be the category of the isomorphism classes of indecomposable H-modules.

Definition 5.2. The cluster complex ${}^m\triangle^d(H)$ of $\mathcal{C}_d^m(H)$ is a simplicial complex which has $\operatorname{ind}\mathcal{C}_d^m(H)$ as the set of vertices and d-exceptional subsets in $\mathcal{C}_d^m(H)$ as its simplices.

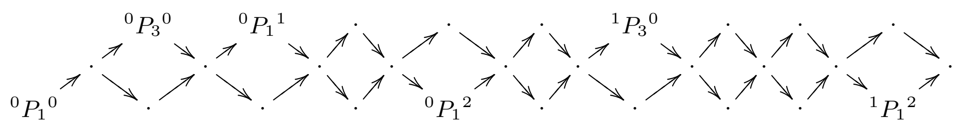
Definition 5.3. Define a map ${}^m\gamma^d$ from $\operatorname{ind}\mathcal{C}_d^m(H)$ to ${}^m\Phi^d_{\geq -1}$:

$${}^m\gamma^d({}^kM^i) = \begin{cases} {}^k\underline{\dim}\, M^i & {}^kM^i \in F_d^k(\operatorname{ind} H[i]),\ i \in [0, d-1],\ k \in [0, m-1], \\ {}^k(-\alpha_j)^0 & {}^kM^i \in F_d^k(\operatorname{ind} H[i]),\ i = d. \end{cases}$$

Remark. This is a bijection between $\operatorname{ind}\mathcal{C}_d^m(H)$ and ${}^m\Phi^d_{\geq -1}$. So denote by M_β the unique indecomposable exceptional module in $\mathcal{H}$ whose dimension vector is β.

One of the applications for this map is to describe the translation R_d^m defined on the generalized colored root system ${}^m\Phi^d_{\geq -1}$ with corresponding objects in $\mathcal{C}_d^m$.

Example 5.4. For the Dynkin diagram of type A_3, $d = 2$ and $m = 2$, $\mathcal{C}_d^m$ is as follows:



By the definition of R_d^m, it is easy to get

$$\begin{aligned} R_d^m({}^0\underline{\dim}\, P_1{}^0) &= {}^0\underline{\dim}\, P_1{}^1 = {}^m\gamma^d({}^0P_1{}^1), \\ R_d^m({}^0\underline{\dim}\, P_1{}^1) &= {}^0R(\underline{\dim}\, P_1)^0 = {}^0{-\alpha_1}{}^0 = {}^m\gamma^d({}^0P_1{}^2), \\ R_d^m({}^m\gamma^d({}^0P_1{}^2)) &= R_d^m({}^0{-\alpha_1}{}^0) = {}^1R(-\alpha_1)^0 = {}^1\underline{\dim}\, P_3{}^0 = {}^m\gamma^d({}^1P_3{}^0), \\ R_d^m({}^m\gamma^d({}^1P_1{}^2)) &= R_d^m({}^1{-\alpha_1}{}^0) = {}^0R(-\alpha_1)^0 = {}^0\underline{\dim}\, P_3{}^0 = {}^m\gamma^d({}^0P_3{}^0). \end{aligned}$$

Definition 5.5. For any pair $M_\alpha, M_\beta \in \mathcal{C}_d^m$, the generalized d-compatible degree of M_α, M_β is defined as $(M_\alpha||M_\beta)_d^m = \dim_{\mathrm{End}M_\alpha}\big(\bigoplus_{i\in[1,d]}\mathrm{Ext}^i(M_\alpha, M_\beta)\big)$, the length of $\bigoplus_{i\in[1,d]}\mathrm{Ext}^i(M_\alpha, M_\beta)$ as a right $\mathrm{End}M_\alpha$-mod.

Of course, it is natural to induce a generalized d-compatibility on m-repetitive colored almost positive roots through the bijection ${}^m\gamma^d$. In fact, many properties could be translated from one side to another.

Lemma 5.6. *The generalized d-compatible degree on $\mathcal{C}_d^m$ naturally induces a compatibility $(-||-)_{d,H}^m$ on ${}^m\Phi_{\geq -1}^d$, furthermore, $(\alpha||\beta)_{d,H}^m = (R_d^m\alpha||R_d^m\beta)_{d,H}^m$.*

Proof. Let us start to check that the following diagram commutes

$$\begin{array}{ccc} \mathrm{ind}\,\mathcal{C}_d^m & \xrightarrow{[1]} & \mathrm{ind}\,\mathcal{C}_d^m \\ \downarrow {\scriptstyle {}^m\gamma^d} & & \downarrow {\scriptstyle {}^m\gamma^d} \\ {}^m\Phi_{\geq -1}^d & \xrightarrow{R_d^m} & {}^m\Phi_{\geq -1}^d \end{array}$$

Any indecomposable object in $\mathcal{C}_d^m$ is of the form ${}^jX^i$, where $X \in \mathrm{ind}\,\mathcal{H}$, $0 \leq i \leq d-1$ and $0 \leq j \leq m-1$, or ${}^jP_k^d$, where $0 \leq j \leq m-1$. Denote $\underline{\dim}\,X = \alpha$. If $i \leq d-2$, then $R_d^m({}^m\gamma^d({}^jX^i)) = R_d^m({}^j\alpha^i) = {}^j\alpha^{i+1} = {}^m\gamma^d[1]({}^jX^i)$.

Assume $i = d-1$ and $j < m-1$. If X is a projective module P_k, then

$$R_d^m({}^m\gamma^d({}^jX^i)) = R_d^m({}^j\underline{\dim}\,P_k^{d-1}) = {}^jR(\underline{\dim}\,P_k)^0 = {}^j(-\alpha_k)^0$$

and ${}^m\gamma^d[1]({}^jP_k^{d-1}) = {}^m\gamma^d({}^jP_k^d) = {}^j(-\alpha_k)^0 = R_d^m({}^m\gamma^d({}^jX^i))$. If X is not projective, then $R_d^m({}^m\gamma^d({}^jX^{d-1})) = R_d^m({}^j\alpha^{d-1}) = {}^{j+1}R(\alpha)^0$ and ${}^m\gamma^d[1]({}^jX^i) = {}^m\gamma^d({}^jX^d) = {}^m\gamma^d({}^{j+1}\tau X^0) = {}^{j+1}(\underline{\dim}\,\tau X)^0 = {}^{j+1}R(\alpha)^0$.

Assume $i = d-1$ and $j = m-1$. If X is projective, then

$$\begin{aligned} &R_d^m({}^m\gamma^d({}^{m-1}P_k^{d-1})) = R_d^m({}^{m-1}\underline{\dim}\,P_k^{d-1}) = {}^{m-1}R(\underline{\dim}\,P_k)^0 \\ =\ &{}^{m-1}(-\alpha_k)^0 = {}^m\gamma^d({}^{m-1}P_k^d) = {}^m\gamma^d[1]({}^{m-1}P_k^{d-1}). \end{aligned}$$

If X is not projective, then $R_d^m({}^m\gamma^d({}^{m-1}X^{d-1})) = R_d^m({}^{m-1}\alpha^{d-1}) = {}^0R(\alpha)^0$ and

$$\begin{aligned} &{}^m\gamma^d[1]({}^{m-1}X^{d-1}) = {}^m\gamma^d({}^{m-1}X^d) = {}^m\gamma^d({}^0(F_d^m\tau X)^0) \\ =\ &{}^m\gamma^d({}^0(\tau X)^0) = {}^0(\underline{\dim}\,\tau X)^0 = {}^0R(\alpha)^0. \end{aligned}$$

When $i = d$, a similar discussion is enough, and the diagram is commutative. Since the shift functor [1] is an auto-equivalence,

$$\begin{aligned} &(R_d^m(\alpha)||R_d^m(\beta))_{d,H}^m = \dim_{\mathrm{End}M_\alpha[1]}\bigoplus_{i\in[1,d]}\mathrm{Ext}^i(M_\alpha[1], M_\beta[1]) \\ =\ &\dim_{\mathrm{End}M_\alpha}\bigoplus_{i\in[1,d]}\mathrm{Ext}^i(M_\alpha, M_\beta) = (\alpha||\beta)_{d,H}^m. \end{aligned}$$ □

Now it is very convenient to show the correspondence between the generalized d-compatibility relation on $\mathcal{C}_d^m$ and the compatibility relation on repetitive colored almost positive roots introduced above.

Theorem 5.7. *The compatibility degree $(-||-)^m_{d,H}$ on ${}^m\Phi^d_{\geq -1}$ induced by the generalized d-compatible degree on $\mathcal{C}^m_d$ is identical to the compatibility degree defined in Definition 2.12.*

Proof. Using Lemma 5.6, the theorem holds if the following assertion is verified: For any almost positive real Schur root ${}^j\beta^i \in {}^m\Phi^d_{\geq -1}$, $({}^k(-\alpha_r)^0||{}^j\beta^i)^m_d = 0$ if and only if $\max\{n_r(\beta), 0\} = 0$.

For the sufficiency, if $\beta \neq -\alpha_r$, then $n_r(\beta) = 0$ by $\max\{n_r(\beta), 0\} = 0$. In other words, $\mathrm{Hom}_{\mathcal{D}}(P_r, M_\beta) = 0$. Now

$$\begin{aligned} &\mathrm{Ext}^l(F^k_d P_r[d], F^j_d M_\beta[i]) = \mathrm{Ext}^l_{\mathcal{D}}(P_r[d], F^{j-k+t}_d M_\beta[i]) \\ =\ &\mathrm{Hom}_{\mathcal{D}}(P_r, F^{j-k+t}_d M_\beta[i-d+l]), \end{aligned}$$

here $1 \leq l \leq d$ and $0 \leq i \leq d$, thus $1 \leq l+i \leq 2d$ and $1-d \leq l+i-d \leq d$. Then $\mathrm{Hom}_{\mathcal{D}}(P_r, F^{j-k+t}_d M_\beta[i-d+l]) = 0$ when $j+t-k \notin \{-1, 0, 1\}$.

If $j+t-k = 0$, then $\mathrm{Ext}^l(F^k_d P_r[d], F^j_d M_\beta[i]) = \mathrm{Hom}_{\mathcal{D}}(P_r, M_\beta[i-d+l])$. Since $1-d \leq i+l-d \leq d$ and $\mathrm{Hom}_{\mathcal{D}}(P_r, M_\beta) = 0$, $\mathrm{Hom}_{\mathcal{D}}(P_r, M_\beta[i+l-d]) = 0$.

If $j+t-k = -1$, then $\mathrm{Ext}^l(F^k_d P_r[d], F^j_d M_\beta[i]) = \mathrm{Hom}_{\mathcal{D}}(P_r, \tau M_\beta[i-2d+l])$. Since $1-2d \leq i+l-2d \leq 0$, one can check it case by case. When $i+l-2d < 0$, $\mathrm{Hom}_{\mathcal{D}}(P_r, \tau M_\beta[i-2d+l]) = 0$. When $i+l-2d = 0$, it means $i = l = d$, then $M_\beta = P_\beta \ncong P_r$, therefore $\mathrm{Hom}_{\mathcal{D}}(P_r, \tau P_\beta) = 0$.

If $j+t-k = 1$, then $\mathrm{Ext}^l(F^k_d P_r[d], F^j_d M_\beta[i]) = \mathrm{Hom}_{\mathcal{D}}(P_r, \tau^{-1} M_\beta[i+l])$. Since $1 \leq i+l \leq 2d$, $\mathrm{Hom}_{\mathcal{D}}(P_r, \tau^{-1} M_\beta[i+l]) = 0$.

For $\beta = -\alpha_r$, all the cases discussed above hold because now $i = d$ except $j+t-k = -1$ and $l-d = 0$, fortunately at this time, $\mathrm{Ext}^l(F^k_d P_r[d], F^j_d M_\beta[i]) = \mathrm{Hom}_{\mathcal{D}}(P_r, \tau P_r) = 0$.

On the other hand, let β be an almost real Schur root with $({}^k(-\alpha_r)^0||{}^j\beta^i)^m_d = 0$, we want to prove $\max\{n_r(\beta), 0\} = 0$. If $\beta = -\alpha_r$, the statement is clear. When $\beta \neq -\alpha_r$, $\mathrm{Ext}^l({}^kP^d_r, {}^jM^i_\beta) = 0$ for $1 \leq l \leq d$, $0 \leq k, j \leq m-1$, $0 \leq i \leq d-1$ and $1 \leq r \leq n$. Especially, $\mathrm{Ext}^l_{\mathcal{D}}(P^d_r, M^i_\beta) = 0$ for $0 \leq i \leq d$ and $1 \leq l \leq d$. Then $\mathrm{Ext}^l_{\mathcal{D}}(P_r[d], M_\beta[i]) \cong \mathrm{Hom}_{\mathcal{D}}(P_r, M_\beta[i+l-d]) = 0$. Choose l such that $i+l-d = 0$, then $\mathrm{Hom}_{\mathcal{D}}(P_r, M_\beta) = 0$. Therefore, $n_r(\beta) = \mathrm{Hom}(P_r, M_\beta) = 0$ □

Proposition 5.8. *For any pair $\alpha, \beta \in {}^m\Phi^d_{\geq -1}$ and a sink (or source) v, we have $(\alpha||\beta)^m_{d,H} = ({}^m\sigma^d_v\alpha||{}^m\sigma^d_v\beta)^m_{d,s_vH}$.*

Proof. If v is a sink, then the following diagram commutes

$$\begin{array}{ccc} \mathrm{ind}\,\mathcal{C}^m_d(H) & \xrightarrow{S^+_v} & \mathrm{ind}\,\mathcal{C}^m_d(S^+_v H) \\ \downarrow {\scriptstyle {}^m\gamma^d_H} & & \downarrow {\scriptstyle {}^m\gamma^d_{s_vH}} \\ {}^m\Phi^d_{\geq -1} & \xrightarrow{{}^m\sigma^d_v} & {}^m\Phi^d_{\geq -1} \end{array}$$

Hence,

$$({}^m\sigma_v^d\alpha||{}^m\sigma_v^d\beta)^m_{d,s_vH} = \dim_{\mathrm{End}S_v^+(M_\alpha)} \mathrm{Ext}(S_v^+M_\alpha, \bigoplus_{i=0}^{d-1} S_v^+M_\beta[i])$$
$$= \dim_{\mathrm{End}M_\alpha} \mathrm{Ext}\big(M_\alpha, \bigoplus_{i=0}^{d-1} M_\beta[i]\big) = (\alpha||\beta)^m_{d,H}. \qquad \square$$

Theorem 5.9. *Let Γ be a valued Dynkin graph and Φ the corresponding root system. Let Ω be an orientation of Γ. Then ${}^m\gamma^d$ provides an isomorphism from the generalized d-cluster complex ${}^m\triangle^d(H)$ to the repetitive generalized cluster complex ${}^m\triangle^d(\Phi)$, which sends vertices to vertices and k-faces to k-faces.*

Proof. Just by generalizing the proof of [29, Theorem 5.7(1)]. $\square$

From the correspondence ${}^m\gamma^d$ and the fact that $(F_d)^m = (\tau^{-1}[d])^m$ is identity on the generalized d-cluster category, the following proposition is straightforward:

Proposition 5.10. *The functor $\digamma := ({}^m\sigma_n^d \cdots {}^m\sigma_2^{dm}\sigma_1^d)^{-1}(R_d^m)^d$ is an automorphism of the repetitive generalized cluster complex ${}^m\triangle^d(\Phi)$ and $\digamma^m = \mathrm{id}$.*

Proof. One observation is enough. Under the correspondence ${}^m\gamma^d$ from the generalized d-cluster category to the repetitive colored almost positive roots, the AR-translation τ corresponds to ${}^m\sigma_n^d \cdots {}^m\sigma_2^{dm}\sigma_1^d$. $\square$

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