On cluster-tilting objects in a triangulated category with Serre duality

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Abstract

Let $\mathcal{D}$ be a Krull-Schmidt, Hom-finite triangulated category with a Serre functor and a cluster-tilting object T. We introduce the notion of an F_Λ-stable support τ-tilting module, induced by the shift functor and the Auslander-Reiten translation, in the cluster-tilted algebra $\Lambda = \mathrm{End}^{op}_{\mathcal{D}}(T)$. We show that there exists a bijection between basic cluster-tilting objects in $\mathcal{D}$ and basic F_Λ-stable support τ-tilting Λ-modules. This generalizes a result of Adachi-Iyama-Reiten [AIR]. As a consequence, we obtain that all cluster-tilting objects in $\mathcal{D}$ have the same number of non-isomorphic indecomposable direct summands.

Key words: cluster-tilting object; Serre functor; support τ-tilting module; F-stable.
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1 Introduction

The concept of cluster algebras was introduced by S. Fomin and A. Zelevinsky in [FZ]. In a series of papers they developed a theory of cluster algebras, which has influential connections with representation theory of finite dimensional algebras. In order to model some essential ingredients in the definition of cluster algebras by similar concepts in a category with additional structure, cluster categories were invented in [BMRRT] (see also [CCS] for type A_n). An important class of objects in these categories are the cluster-tilting objects, which play the role as clusters do in cluster algebras. The endomorphism algebras of cluster-tilting objects are called cluster-tilted algebras [BMR], whose module categories are closely connected with cluster categories [BMR, KR, KZ, IY].

In [Smi], the author investigated the connections between cluster-tilting objects in a cluster category $\mathcal{D}$ and tilting modules in the module category of a cluster-tilted algebra Λ. It was shown that a tilting Λ-module can be lifted to a cluster-tilting object in $\mathcal{D}$. However, cluster-tilting objects in $\mathcal{D}$ do not always correspond to tilting Λ-modules. Actually, this fact also holds for any 2-Calabi-Yau (2-CY for short) triangulated category with cluster-tilting objects [FL, HJ]. Recently, T. Adachi, O. Iyama and I. Reiten introduced the τ-tilting theory [AIR]. They established a bijection between cluster-tilting objects in $\mathcal{D}$ and so-called support τ-tilting modules in modΛ when $\mathcal{D}$ is 2-CY (see also [CZZ]). This bijection answers the question why the modules in modΛ corresponding to the cluster-tilting objects in $\mathcal{D}$ are not necessarily tilting modules. It is then natural to ask what kind of modules in modΛ correspond to the cluster-tilting objects in $\mathcal{D}$ if $\mathcal{D}$ is n-Calabi-Yau, or more generally, $\mathcal{D}$ is a triangulated category with a Serre functor.

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Let $\mathcal{D}$ be a triangulated category with a Serre functor and T be a cluster-tilting object in $\mathcal{D}$. Then $\mathcal{D}$ has an auto-equivalence $F = \tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in $\mathcal{D}$. S. Koenig and B. Zhu in [KZ] proved that there also exists an equivalence

$$\mathrm{Hom}_{\mathcal{D}}(T, -) : \mathcal{D}/\mathrm{add}T[1] \longrightarrow \mathrm{mod}\Lambda = \mathrm{mod}\ \mathrm{End}_{\mathcal{D}}^{op}(T),$$

where $\mathrm{mod}\Lambda$ is the category of finitely generated left Λ-modules. It leads us to consider the following questions:

Question 1.1. *Do cluster-tilting objects in $\mathcal{D}$ correspond to support τ-tilting modules in* $\mathrm{mod}\Lambda$ *bijectively again?*

Unfortunately, the answer is negative. We give the following counterexample. Let Q be the quiver as follows:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3\ .$$

We consider the repetitive cluster category $\mathcal{D} = D^b(kQ)/\langle \tau_Q^{-2}[2] \rangle$ introduced by B. Zhu in [Z11], whose objects are the same in $D^b(kQ)$, and whose morphisms are given by

$$\mathrm{Hom}_{D^b(kQ)/\langle \tau_Q^{-2}[2] \rangle}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{D^b(kQ)}(X, (\tau_Q^{-2}[2])^i Y).$$

It is shown in [Z11] that $\mathcal{D}$ is a triangulated category with a Serre functor $\mathbb{S}$, where τ_Q is the Auslander-Reiten translation in $D^b(kQ)$. Note that it is not 2-CY (but it is a fractional Calabi-Yau category with CY-dimension $\frac{4}{2}$). The AR-quiver of $\mathcal{D}$ is as follows:

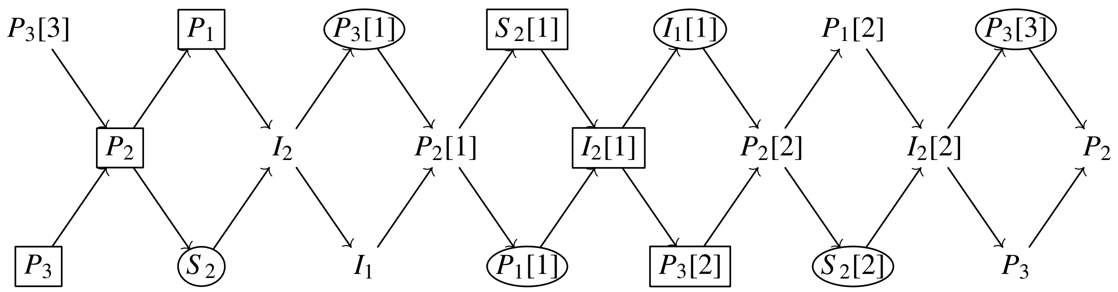
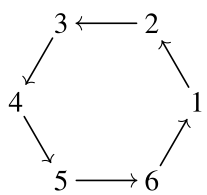


Figure 1

where P_i (respectivly, I_i, S_i) is the indecomposable projective (respectively, injective, simple) module corresponding to vertex i. Note that the direct sum $T_1 = S_2 \oplus P_3[1] \oplus P_1[1] \oplus I_1[1] \oplus S_2[2] \oplus P_3[3]$ of the encircled indecomposable objects gives a cluster-tilting object, the cluster-tilted algebra $\Lambda_1 = \mathrm{End}_{\mathcal{D}}^{op}(T_1)$ is given by the following quiver with $rad^2 = 0$.



The AR-quiver of $\mathrm{mod}\Lambda_1$ is as follows:

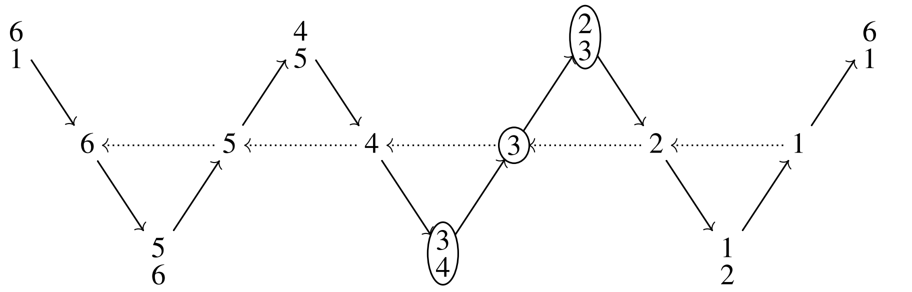


Figure 2

It is easy to see that $M_1 = \begin{smallmatrix}3\\4\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$ is a support τ-tilting Λ_1-module, but the object in $\mathcal{D}$ corresponding to M_1 is $P_1[1] \oplus I_2[1] \oplus I_1[1] \oplus I_1 \oplus S_2[1] \oplus P_1$, which is not a cluster-tilting object since it has self-extensions.

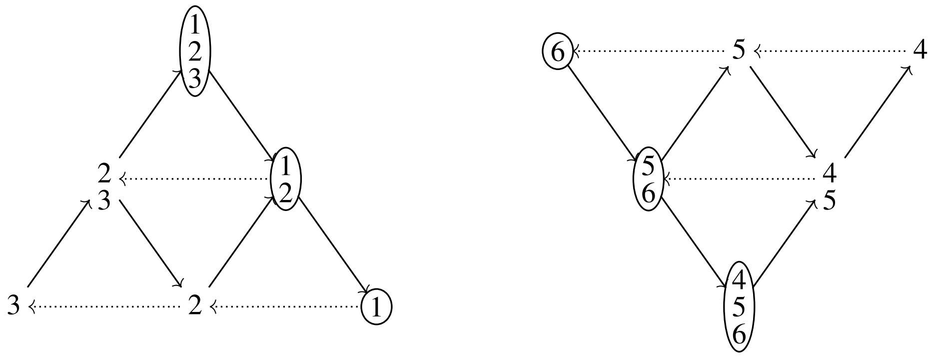
The counterexample forces us to consider the weak version of the above question.

Question 1.2. *Can tilting modules in* $\mathrm{mod}\Lambda$ *be lifted to cluster-tilting objects in* $\mathcal{D}$*?*

Let us reconsider the above example again. We take another cluster-tilting object $T_2 = P_3 \oplus P_2 \oplus P_1 \oplus S_2[1] \oplus I_2[1] \oplus P_3[2]$ (see Figure 1), the cluster-tilted algebra $\Lambda_2 = \mathrm{End}^{op}_{\mathcal{D}}(T_2)$ is not connected, it is given by the disconnected quiver:

$$1 \longrightarrow 2 \longrightarrow 3 \qquad 4 \longrightarrow 5 \longrightarrow 6$$

with no relations. The AR-quiver of $\mathrm{mod}\Lambda_2$ is given as follows



Clearly, $M_2 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 6 \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix}$ is a tilting Λ_2-module, but the object in $\mathcal{D}$ corresponding to M_2 is $P_1 \oplus I_2 \oplus I_1 \oplus S_2[1] \oplus I_2[1] \oplus P_3[2]$, which is not a cluster-tilting object, and not even a rigid object.

In order to explain why the answers to Questions 1.1 and 1.2 are negative, we need the following definition for an additive category C with an auto-equivalence G. We call an object M in C *G-stable* if $G(M) \cong M$. We introduce, in this paper, the auto-equivalence F_Λ of $\mathrm{mod}\Lambda$ induced by the auto-equivalence F of $\mathcal{D}$. The main result of this paper is a generalization of a bijection in [AIR], and our result shows that a properly modified version of Questions 1.1 and 1.2 have positive answers.

Theorem 1.3. *(see Theorem 2.14 for details). Let $\mathcal{D}$ be a triangulated category with a Serre functor and a cluster-tilting object T, and let $\Lambda = \mathrm{End}^{op}_{\mathcal{D}}(T)$ be the corresponding cluster-tilted algebra. Then there is a bijection between the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{D}$ and the set of isomorphism classes of basic F_Λ-stable support τ-tilting Λ-modules.*

We have the following direct consequence, which generalizes some results in [DK, ZZ].

Corollary 1.4. *Let $\mathcal{D}$ be a triangulated category with a Serre functor and a cluster-tilting object T, and $F = \tau^{-1}_{\mathcal{D}}[1]$. Assume that U is a basic object in $\mathcal{D}$, then the following are equivalent.*

- *U is cluster-tilting.*
- *U is F-stable and maximal rigid.*
- *U is an F-stable rigid object and $|U| = |T|$, where $|U|$ denotes the number of non-isomorphic indecomposable direct summands of U.*

In particular, all basic cluster-tilting objects in $\mathcal{D}$ have the same number of indecomposable direct summands.

We end this section with some conventions. Throughout this article, k is an algebraically closed field. By Λ, we denote a finite dimensional basic k-algebra. All modules we consider in this paper are left modules. For any triangulated category $\mathcal{D}$, we assume that it satisfies the Krull-Remak-Schmidt property. In $\mathcal{D}$, we denote the shift functor by [1] and define $\mathrm{Ext}^i_{\mathcal{D}}(X, Y) :=\mathrm{Hom}_{\mathcal{D}}(X, Y[i])$ for any objects X and Y. If $\mathcal{T}$ is a subcategory of $\mathcal{D}$, then we always assume that $\mathcal{T}$ is a full subcategory which is closed under taking isomorphisms, direct sums and direct summands. The quotient category of $\mathcal{D}$ by $\mathcal{T}$ denoted by $\mathcal{D}/\mathcal{T}$, is a category with the same objects as $\mathcal{D}$ and the space of morphisms from X to Y is the quotient of group of morphisms from X to Y in $\mathcal{D}$ by the subgroup consisting of morphisms factor through an object in $\mathcal{T}$. For an object T in $\mathcal{D}$, $\mathrm{add}T$ denotes the full subcategory consisting of direct summands of direct sum of finitely many copies of T. For two morphisms $f : M \to N$ and $g : N \to L$, the composition of f and g is denoted by $g \circ f : M \to L$.

2 Cluster-tilting objects and F_Λ-stable support τ-tilting modules

Assume that $\mathcal{D}$ is a k-linear, Hom-finite, triangulated category. Recall from [BK] a Serre functor $\mathbb{S} : \mathcal{D} \to \mathcal{D}$ is a k-linear equivalence with bifunctorial isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \cong D\mathrm{Hom}_{\mathcal{D}}(B, \mathbb{S}A)$$

for any $A, B \in \mathcal{D}$, where D is the duality over k. In [RVdB], I. Reiten and M. Van den Bergh proved that if $\mathcal{D}$ admits a Serre functor $\mathbb{S}$, then $\mathcal{D}$ has Auslander-Reiten triangles. Moreover, if $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in $\mathcal{D}$, then $\mathbb{S} \cong \tau_{\mathcal{D}}[1]$.
In the following, we always assume the category $\mathcal{D}$ has a Serre functor $\mathbb{S}$. Recall that an important class of objects in $\mathcal{D}$ are *rigid* objects: An object A in $\mathcal{D}$ is called rigid if $\mathrm{Ext}^1_{\mathcal{D}}(A, A) = 0$. With a collection of rigid objects, one can define maximal rigid objects and cluster-tilting objects as follows:

Definition 2.1. *(1) An object M in $\mathcal{D}$ is called* maximal rigid *if it is rigid and maximal with respect to the property:* $\mathrm{add}M = \{X \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(M \oplus X, M \oplus X) = 0\}$.

(2) We call a rigid object T in $\mathcal{D}$ cluster-tilting *if*

$$\mathrm{add}T = \{X \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(T, X) = 0\} = \{X \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(X, T) = 0\}.$$

In this situation, the algebra $\mathrm{End}^{op}_{\mathcal{D}}(T)$ *is called a* cluster-tilted algebra.

Let $\tau_{\mathcal{D}}$ be the Auslander-Reiten translation in $\mathcal{D}$, set $F = \tau^{-1}_{\mathcal{D}}[1]$, then we have the following results, which will be used frequently in this paper.

Theorem 2.2 ([KZ])**.** *Let T be a basic cluster-tilting object in $\mathcal{D}$, then $FT = T$ and the functor* $\mathrm{Hom}_{\mathcal{D}}(T, -)$ *induces an equivalence*

$$\mathcal{D}/\mathrm{add}T[1] \longrightarrow \mathrm{modEnd}^{op}_{\mathcal{D}}(T).$$

We recall in the following some definitions related to τ-tilting theory in [AIR] introduced by T. Adachi, O. Iyama and I. Reiten. Let Λ be a finite dimensional k-algebra and $\tau := \tau_{\Lambda}$ be the Auslander-Reiten translation in $\mathrm{mod}\Lambda$. We denote by $\mathrm{proj}\Lambda$ the category of finitely generated projective Λ-modules.

Definition 2.3. *Let (X, P) be a pair with $X \in \mathrm{mod}\Lambda$ and $P \in \mathrm{proj}\Lambda$.*

- *The pair (X, P) is said to be* basic *if X and P are basic.*
- *X is called* τ-rigid *if* $\mathrm{Hom}_{\Lambda}(X, \tau X) = 0$. *We say the pair (X, P) is a* τ-rigid pair *if X is τ-rigid and* $\mathrm{Hom}_{\Lambda}(P, X) = 0$.
- *We say X is* τ-tilting *if X is τ-rigid and $|X| = |\Lambda|$. A τ-rigid pair (X, P) is said to be a* support τ-tilting pair *if $|X| + |P| = |\Lambda|$. In this case, X is called a* support τ-tilting module.

We denote by τ-tiltΛ (respectively, sτ-tiltΛ) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting) modules in $\mathrm{mod}\Lambda$. The following proposition gives some characterizations of a τ-rigid pair being a support τ-tilting pair.

Proposition 2.4 ([AIR])**.** *Let (T, P) be a τ-rigid pair for Λ. Then the following are equivalent.*

- *(T, P) is a support τ-tilting pair.*
- *If $(T \oplus X, P)$ is τ-rigid for some Λ-module X, then $X \in \mathrm{add}T$.*
- *The condition* $\mathrm{Hom}_{\Lambda}(T, \tau X) = 0, \mathrm{Hom}_{\Lambda}(X, \tau T) = 0$ *and* $\mathrm{Hom}_{\Lambda}(P, X) = 0$ *implies that* $X \in \mathrm{add}T$.

Throughout this article, let $\mathcal{D}$ be a k-linear Hom-finite triangulated category with a cluster-tilting object T, and $\Lambda = \mathrm{End}^{op}_{\mathcal{D}}(T)$ the corresponding cluster-tilted algebra. By Theorem 2.2, there is an equivalence of categories:

$$\overline{(-)} := \mathrm{Hom}_{\mathcal{D}}(T, -) : \mathcal{D}/\mathrm{add}T[1] \xrightarrow{\sim} \mathrm{mod}\Lambda, \tag{1}$$

Moreover, we get a k-algebra automorphism of Λ

$$\begin{aligned} F_* : \mathrm{Hom}_{\mathcal{D}}(T, T) = \Lambda \quad &\xrightarrow{\sim} \quad \Lambda = \mathrm{Hom}_{\mathcal{D}}(FT, FT) \\ f \quad &\longmapsto \quad F(f). \end{aligned}$$

This allows us to define a functor

$$F_{\Lambda} : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda$$

as follows.

- If $M \in \mathrm{mod}\Lambda$, we set by $F_\Lambda(M)$ the vector space M endowed with the structure of left Λ-module given by
$$a \cdot m := F_*^{-1}(a)m$$
for all $m \in M$ and $a \in \Lambda$.

- If $f : M \to M'$ is a Λ-homomorphism, then we define $F_\Lambda(f)$ by
$$F_\Lambda(f)(m) := f(m),$$
for all $m \in F_\Lambda(M)$.

It is easy to show that $F_\Lambda : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda$ is a functor. Dually, we can define another functor $G_\Lambda : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda$ as follows: If $M \in \mathrm{mod}\Lambda$, we denote by $G_\Lambda(M)$ the vector space M endowed with the structure of left Λ-module given by

$$a \cdot m := F_*(a)m,$$

for all $m \in M$ and $a \in \Lambda$. If $f : M \to M'$ is a Λ-homomorphism, then we define $G_\Lambda(f)$ by

$$G_\Lambda(f)(m) := f(m),$$

for all $m \in G_\Lambda(M)$. Moreover, we have the following observation.

Proposition 2.5. *The functor* $F_\Lambda : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda$ *is an equivalence.*

Proof. It is straightforward to check that G_Λ is the quasi-inverse of F_Λ. □

From the constructions of F_Λ and G_Λ, we have the following direct observation.

Proposition 2.6. *For any object X in $\mathcal{D}$, we have* $\overline{FX} \cong F_\Lambda(\overline{X})$ *and* $\overline{F^{-1}X} \cong G_\Lambda(\overline{X})$ *in* $\mathrm{mod}\Lambda$.

Proof. Since F is an equivalence in $\mathcal{D}$, we get a k-linear isomorphism of vector spaces

$$F_\Lambda(\overline{X}) = \overline{X} = \mathrm{Hom}_{\mathcal{D}}(T, X) \cong \mathrm{Hom}_{\mathcal{D}}(FT, FX) \overset{2.2}{=\!=} \mathrm{Hom}_{\mathcal{D}}(T, FX) = \overline{FX}.$$

For any $f \in F_\Lambda(\overline{X}) = F_\Lambda(\mathrm{Hom}_{\mathcal{D}}(T, X)), a \in \Lambda = \mathrm{Hom}_{\mathcal{D}}(T, T)$,

$$F(a \cdot f) = F(f \circ F_*^{-1}(a)) = F(f) \circ F(F_*^{-1}(a)) = F(f) \circ a = aF(f),$$

where $\circ$ is the composition of morphisms in $\mathcal{D}$. Hence $F : F_\Lambda(\overline{X}) \to \overline{FX}$ is a Λ-module isomorphism. Similarly, we can show that $\overline{F^{-1}X} \cong G_\Lambda(\overline{X})$. □

The following lemma which was proved in [P] for the case $\mathcal{D}$ is a 2-CY category, can be easily generalized to our setting by the same approach.

Lemma 2.7. *Let $\mathcal{D}$ be a triangulated category with a Serre functor $\mathbb{S}$ and a cluster-tilting object T. Then for any objects X and Y in $\mathcal{D}$, there is a bifunctorial isomorphism*

$$\mathrm{Hom}_{\mathcal{D}/\mathrm{add}T}(\tau_{\mathcal{D}}^{-1} Y, X) \cong D[T](X[-1], Y),$$

where $[T](X, Y)$ denotes the subgroup of $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ *consisting of morphisms which factor through objects in* $\mathrm{add}T$.

Proof. Since T is a cluster-tilting object, by Theorem 3.1 in [IY], we know there exists a triangle

$$T_1 \longrightarrow T_0 \longrightarrow X \xrightarrow{\eta} T_1[1]$$

in $\mathcal{D}$ with T_0 and T_1 in add$T$. Consider the morphism

$$\begin{array}{rccc} \varphi : \mathrm{Hom}_{\mathcal{D}}(T_1, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(X[-1], Y) \\ f & \longmapsto & f \circ \eta[-1]. \end{array}$$

We have

$$D[T](X[-1], Y) \cong D\mathrm{Im}\varphi \cong \mathrm{Im}D\varphi.$$

Since the category $\mathcal{D}$ has a Serre functor $\mathbb{S} \cong \tau_{\mathcal{D}}[1]$,

$$D\mathrm{Hom}_{\mathcal{D}}(T_1, Y) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}Y, T_1) \cong \mathrm{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, T_1[1]),$$

$$D\mathrm{Hom}_{\mathcal{D}}(X[-1], Y) \cong \mathrm{Hom}_{\mathcal{D}}(\mathbb{S}^{-1}Y, X[-1]) \cong \mathrm{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X).$$

Thus, $D\varphi$ is isomorphic to

$$\begin{array}{rccc} \phi : \mathrm{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, T_1[1]) \\ g & \longmapsto & \eta \circ g. \end{array}$$

Note that $\mathrm{Ker}\phi = [T](\tau_{\mathcal{D}}^{-1}Y, X)$. Hence, we have isomorphisms

$$D[T](X[-1], Y) \cong \mathrm{Im}\phi \cong \mathrm{Hom}_{\mathcal{D}}(\tau_{\mathcal{D}}^{-1}Y, X)/\mathrm{Ker}\phi \cong \mathrm{Hom}_{\mathcal{D}/\mathrm{add}T}(\tau_{\mathcal{D}}^{-1}Y, X).$$

□

By the above lemma, we can express $\mathrm{Ext}^1_{\mathcal{D}}(X, Y)$ in terms of the images $\overline{X}$ and $\overline{Y}$ in modΛ. For two Λ-modules M and N, from now on, we use $\langle X, Y\rangle$ to denote $\dim_k\mathrm{Hom}_{\Lambda}(X, Y)$ for simplicity. We first consider the following case:

Lemma 2.8. *Let X and Y be objects in $\mathcal{D}$ such that there are no non-zero indecomposable direct summands of $T[1]$ for X and Y. Then $\overline{X[1]} \cong \tau F_{\Lambda}(\overline{X})$. Moreover,*

$$\dim_k\mathrm{Ext}^1_{\mathcal{D}}(X, Y) = \langle \overline{X}, \tau F_{\Lambda}(\overline{Y})\rangle + \langle F_{\Lambda}(\overline{Y}), \tau F_{\Lambda}(\overline{X})\rangle.$$

Proof. We first prove $\overline{X[1]} \cong \tau F_{\Lambda}(\overline{X})$. By Proposition 4.7 in [KZ], the residue class of any sink (respectively, source) map in $\mathcal{D}$ is again a sink (respectively, source) map in modΛ. Combining this with Proposition 2.6, we obtain

$$\overline{X[1]} = \overline{\tau_{\mathcal{D}}FX} \cong \tau\overline{FX} \cong \tau F_{\Lambda}(\overline{X}).$$

Consider the following exact sequence,

$$0 \to [T[1]](X, Y[1]) \to \mathrm{Hom}_{\mathcal{D}}(X, Y[1]) \to \mathrm{Hom}_{\mathcal{D}/\mathrm{add}T[1]}(X, Y[1]) \to 0.$$

By the equivalence (1), we get

$$\mathrm{Hom}_{\mathcal{D}/\mathrm{add}T[1]}(X, Y[1]) \cong \mathrm{Hom}_{\Lambda}(\overline{X}, \overline{Y[1]}) \cong \mathrm{Hom}_{\Lambda}(\overline{X}, \tau F_{\Lambda}(\overline{Y})). \tag{2}$$

Using Lemma 2.7 and (2), we have

$$\begin{aligned}[T[1]](X, Y[1]) &\cong D\mathrm{Hom}_{\mathcal{D}/\mathrm{add}T[1]}(\tau_{\mathcal{D}}^{-1}Y[1], X[1]) \\ &= D\mathrm{Hom}_{\mathcal{D}/\mathrm{add}T[1]}(FY, X[1]) \\ &\overset{2.6}{\cong} D\mathrm{Hom}_{\Lambda}(F_{\Lambda}(\overline{Y}), \tau F_{\Lambda}(\overline{X})).\end{aligned}$$

Together, $0 \to D\mathrm{Hom}_{\Lambda}(F_{\Lambda}(\overline{Y}), \tau F_{\Lambda}(\overline{X})) \to \mathrm{Ext}^1_{\mathcal{D}}(X, Y) \to \mathrm{Hom}_{\Lambda}(\overline{X}, \tau F_{\Lambda}(\overline{Y})) \to 0$ is an exact sequence. Hence, $\dim_k \mathrm{Ext}^1_{\mathcal{D}}(X, Y) = \langle \overline{X}, \tau F_{\Lambda}(\overline{Y})\rangle + \langle F_{\Lambda}(\overline{Y}), \tau F_{\Lambda}(\overline{X})\rangle$. □

For the general case, we have the following proposition.

Proposition 2.9. *Let $X = X' \oplus X''$ and $Y = Y' \oplus Y''$ be objects in $\mathcal{D}$ such that X'' and Y'' are the maximal direct summands of X and Y respectively, which belong to* add$T[1]$. *Then*

$$\dim_k \mathrm{Ext}^1_{\mathcal{D}}(X, Y) = \langle \overline{X'}, \tau F_{\Lambda}(\overline{Y'})\rangle + \langle F_{\Lambda}(\overline{Y'}), \tau F_{\Lambda}(\overline{X'})\rangle + \langle \overline{X''[-1]}, \overline{Y'}\rangle + \langle \overline{Y''[-1]}, G_{\Lambda}(\overline{X'})\rangle.$$

Proof. Since X'' and Y'' belong to add$T[1]$, we have

$$\mathrm{Ext}^1_{\mathcal{D}}(X, Y) \cong \mathrm{Ext}^1_{\mathcal{D}}(X', Y') \oplus \mathrm{Ext}^1_{\mathcal{D}}(X'', Y') \oplus \mathrm{Ext}^1_{\mathcal{D}}(X', Y''). \tag{3}$$

By Lemma 2.8, we get

$$\dim_k \mathrm{Ext}^1_{\mathcal{D}}(X', Y') = \langle \overline{X'}, \tau F_{\Lambda}(\overline{Y'})\rangle + \langle F_{\Lambda}(\overline{Y'}), \tau F_{\Lambda}(\overline{X'})\rangle. \tag{4}$$

Moreover we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{D}}(X'', Y') &\cong \mathrm{Hom}_{\mathcal{D}}(X''[-1], Y') \\ &\cong \mathrm{Hom}_{\mathcal{D}/\mathrm{add}T[1]}(X''[-1], Y') \\ &\cong \mathrm{Hom}_{\Lambda}(\overline{X''[-1]}, \overline{Y'}).\end{aligned} \tag{5}$$

Since $\mathcal{D}$ has a Serre functor $\mathbb{S}$ and similarly as for equation (5) we obtain

$$\mathrm{Ext}^1_{\mathcal{D}}(X', Y'') \cong D\mathrm{Hom}_{\mathcal{D}}(Y''[1], \mathbb{S}X') \cong D\mathrm{Hom}_{\Lambda}(\overline{Y''[-1]}, \overline{\mathbb{S}X'[-2]}). \tag{6}$$

By Proposition 2.6, we have

$$\overline{\mathbb{S}X'[-2]} \cong \overline{F^{-1}X'} = G_{\Lambda}(\overline{X'}).$$

Thus, (6) can be written as

$$\mathrm{Ext}^1_{\mathcal{D}}(X', Y'') \cong D\mathrm{Hom}_{\Lambda}(\overline{Y''[-1]}, \overline{\mathbb{S}X'[-2]}) \cong D\mathrm{Hom}_{\Lambda}(\overline{Y''[-1]}, G_{\Lambda}(\overline{X'})). \tag{7}$$

Thus the assertion follows from (3), (4), (5) and (7) immediately. □

In order to state our main result in this paper, we need the following definition.

Definition 2.10. *Let C be an additive category with an auto-equivalence G. Assume that M and P are two objects in C.*

*(1) An object M is said to be G-*stable *if $G(M) \cong M$.*

*(2) We call a pair (M, P) is G-*stable *if $G(M) \cong M$ and $G(P) \cong P$.*

We denote by $\mathrm{iso}\mathcal{D}$ the set of isomorphism classes of objects in $\mathcal{D}$. By the equivalence (1), we get a bijection

$$\begin{aligned} \widetilde{(-)} : \mathrm{iso}\mathcal{D} &\longleftrightarrow \mathrm{iso}(\mathrm{mod}\Lambda) \times \mathrm{iso}(\mathrm{proj}\Lambda) \\ X = X' \oplus X'' &\longmapsto \widetilde{X} := (\overline{X'}, \overline{X''[-1]}), \end{aligned} \tag{8}$$

where X'' is a maximal direct summand of X which belongs to $\mathrm{add}T[1]$.

From now on, we denote by F-$\mathrm{rigid}\mathcal{D}$ the set of isomorphism classes of basic F-stable rigid objects in $\mathcal{D}$ and by F_Λ-τ-$\mathrm{rigid}\Lambda$ the set of isomorphism classes of basic F_Λ-stable τ-rigid pairs for Λ.
For the proof of our main result, the following lemma is needed.

Lemma 2.11. *Let X be an object in $\mathcal{D}$ and $\widetilde{(-)}$ be the bijection in (8). Then $X = X' \oplus X''$ is F-stable in $\mathcal{D}$ if and only if $\widetilde{X} = (\overline{X'}, \overline{X''[-1]})$ is F_Λ-stable in* $\mathrm{mod}\Lambda$.

Proof. If X is F-stable in $\mathcal{D}$, then $FX' \oplus FX'' \cong X' \oplus X''$. By Theorem 2.2, we have

$$FX'' \in \mathrm{add}T[1] \text{ and } \mathrm{add}FX' \cap \mathrm{add}T[1] = 0.$$

Thus, $FX' \cong X'$ and $FX'' \cong X''$. Therefore,

$$F_\Lambda(\overline{X'}) \cong \overline{FX'} \cong \overline{X'} \text{ and } F_\Lambda(\overline{X''[-1]}) \cong \overline{FX''[-1]} \cong \overline{X''[-1]}.$$

Conversely, if $\widetilde{X} = (\overline{X'}, \overline{X''[-1]})$ is F_Λ-stable in $\mathrm{mod}\Lambda$, then $\overline{FX'} \cong F_\Lambda(\overline{X'}) \cong \overline{X'}$ and $\overline{FX''[-1]} \cong F_\Lambda(\overline{X''[-1]}) \cong \overline{X''[-1]}$. Thus $FX' \cong X'$ and $FX'' \cong X''$ in $\mathcal{D}$, which implies that $FX = FX' \oplus FX'' \cong X' \oplus X'' = X$. □

Theorem 2.12. *Let $\mathcal{D}$ be a k-linear Hom-finite triangulated category with a Serre functor $\mathbb{S}$ and a cluster-tilting object T, and $\Lambda = \mathrm{End}^{op}_{\mathcal{D}}(T)$. Then the bijection $\widetilde{(-)}$ in (8) induces a bijection*

$$F\text{-}\mathrm{rigid}\mathcal{D} \longleftrightarrow F_\Lambda\text{-}\tau\text{-}\mathrm{rigid}\Lambda$$

Proof. Let X be a basic F-stable rigid object in $\mathcal{D}$, then by Proposition 2.9, we have

$$\mathrm{Hom}_\Lambda(\overline{X'}, \tau F_\Lambda(\overline{X'})) = \mathrm{Hom}_\Lambda(\overline{X''[-1]}, \overline{X'}) = 0. \tag{9}$$

Using Lemma 2.11, we know $\widetilde{X}$ is F_Λ-stable. Thus, (9) implies that $\widetilde{X}$ is a τ-rigid pair for Λ. Conversely, let $\widetilde{X}$ be a basic F_Λ-stable τ-rigid pair for Λ, then

$$G_\Lambda(\overline{X'}) \cong G_\Lambda \circ F_\Lambda(\overline{X'}) \cong \overline{X'}.$$

Similarly, by Proposition 2.9 and Lemma 2.11, we can easily check that X is a basic F-stable rigid object in $\mathcal{D}$. □

We denote by F_Λ-$s\tau$-$\mathrm{tpair}\Lambda$ the set of isomorphism classes of basic F_Λ-stable support τ-tilting pairs for Λ and by F_Λ-$s\tau$-$\mathrm{tilt}\Lambda$ the set of isomorphism classes of basic F_Λ-stable support τ-tilting Λ-modules. For a basic support τ-tilting pair (X, P), the following observation shows that if X is F_Λ-stable, then so is P.

Lemma 2.13. *There exists a bijection*

$$\begin{array}{rcl}\xi : F_\Lambda\text{-s}\tau\text{-tpair}\Lambda & \longleftrightarrow & F_\Lambda\text{-s}\tau\text{-tilt}\Lambda \\ (X,P) & \longmapsto & X.\end{array}$$

Proof. Clearly, ξ is well-defined. For any basic F_Λ-stable support τ-tilting Λ-module X, by Proposition 2.3 in [AIR] there exists a unique basic Λ-module $P \in \text{proj}\Lambda$ such that (X,P) is a basic support τ-tilting pair for Λ. We may assume that $P \cong \overline{T_1}$, where $T_1 \in \text{add}T$. Thus,

$$F_\Lambda(P) = F_\Lambda(\overline{T_1}) = \overline{FT_1} \in \text{proj}\Lambda.$$

Moreover we have

$$\text{Hom}_\Lambda(F_\Lambda(P),X) \cong \text{Hom}_\Lambda(F_\Lambda(P),F_\Lambda(X)) \cong \text{Hom}_\Lambda(P,X) = 0,$$

and

$$|F_\Lambda(P)| + |X| = |P| + |X| = |\Lambda|.$$

Therefore, $(X,F_\Lambda(P))$ is also a basic support τ-tilting pair. By the uniqueness of P, we have $F_\Lambda(P) \cong P$ and (X,P) is a basic F_Λ-stable support τ-tilting pair for Λ. □

We denote by F-m-rigid$\mathcal{D}$ (respectively, c-tilt$\mathcal{D}$) the set of isomorphism classes of basic F-stable maximal rigid (respectively, cluster-tilting) objects in $\mathcal{D}$ and by F_Λ-sτ-tiltΛ the set of isomorphism classes of basic F_Λ-stable support τ-tilting Λ-modules. The following result gives a close relationship between the cluster-tilting objects in $\mathcal{D}$ and support τ-tilting Λ-modules.

Theorem 2.14. *Let $\mathcal{D}$ be a k-linear Hom-finite triangulated category with a Serre functor $\mathbb{S}$ and a cluster-tilting object T, and $\Lambda = \text{End}_{\mathcal{D}}^{op}(T)$. Then the bijection $\widetilde{(-)}$ in (8) induces the following bijections*

$$F\text{-m-rigid}\mathcal{D} \overset{(a)}{\longleftrightarrow} F_\Lambda\text{-s}\tau\text{-tilt}\Lambda$$

$$\text{c-tilt}\mathcal{D} \overset{(b)}{\longleftrightarrow} F_\Lambda\text{-s}\tau\text{-tilt}\Lambda$$

Proof. (a) By Lemma 2.13, it suffices to show that $\widetilde{(-)}$ induces a bijection

$$F\text{-m-rigid}\mathcal{D} \longleftrightarrow F_\Lambda\text{-s}\tau\text{-tpair}\Lambda.$$

Let $X = X' \oplus X''$ be a basic F-stable maximal rigid object in $\mathcal{D}$, where X'' is the maximal direct summands of X which belong to add$T[1]$. By Theorem 2.12, we know that $\widetilde{X}$ is a basic F_Λ-stable τ-rigid pair.
If $(\overline{X'} \oplus \overline{M}, \overline{X''[-1]})$ is τ-rigid for some Λ-module $\overline{M}$, then $X \oplus M$ is rigid by Theorem 2.12. Since X is maximal rigid, we have $\overline{M} \in \text{add}\overline{X} = \text{add}\overline{X'}$. Thus $\widetilde{X}$ is a support τ-tilting pair by Proposition 2.4.
Conversely, we assume that $\widetilde{X}$ is a basic F_Λ-stable support τ-tilting pair. By Theorem 2.12 we need to show that X is maximal rigid. If $X \oplus M = (X' \oplus M') \oplus (X'' \oplus M'')$ is a rigid object, then $(\overline{X'} \oplus \overline{M'}, \overline{X''[-1]} \oplus \overline{M''[-1]})$ is a τ-rigid pair. Thus,

$$M' \in \text{add}X' \ \text{ and } \ M''[-1] \in \text{add}X''[-1],$$

which imply that $M \in \text{add}X$.

(b) Similarly, we only need to show that $\widetilde{(-)}$ induces a bijection c-tilt$\mathcal{D} \leftrightarrow F_\Lambda$-s$\tau$-tpair$\Lambda$. If X is a basic cluster-tilting object in $\mathcal{D}$, then $\widetilde{X}$ is a basic F_Λ-stable support τ-tilting pair by Theorem 2.2 and (a).
Conversely, if $\widetilde{X}$ is a basic F_Λ-stable support τ-tilting pair, then X is F-stable and maximal rigid by (a). Thus we only need to show that

$$\{Y \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(X,Y)=0\} \subseteq \mathrm{add}X \supseteq \{Z \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(Z,X)=0\}.$$

(1) Assume that $\mathrm{Ext}^1_{\mathcal{D}}(X,Y)=0$ for some object $Y \in \mathcal{D}$. Then by Proposition 2.9, we have

$$\langle \overline{X'}, \tau F_\Lambda(\overline{Y'})\rangle = \langle F_\Lambda(\overline{Y'}), \tau F_\Lambda(\overline{X'})\rangle = \langle \overline{X''[-1]}, \overline{Y'}\rangle = \langle \overline{Y''[-1]}, G_\Lambda(\overline{X'})\rangle = 0.$$

Since $\widetilde{X}$ is F_Λ-stable and F_Λ is an auto-equivalence of modΛ, we get

$$\mathrm{Hom}_\Lambda(\overline{X'}, \tau F_\Lambda(\overline{Y'})) = \mathrm{Hom}_\Lambda(F_\Lambda(\overline{Y'}), \tau\overline{X'}) = \mathrm{Hom}_\Lambda(\overline{X''[-1]}, F_\Lambda(\overline{Y'})) = 0, \tag{10}$$

$$\mathrm{Hom}_\Lambda(\overline{Y''[-1]}, \overline{X'}) \cong \mathrm{Hom}_\Lambda(\overline{Y''[-1]}, G_\Lambda \circ F_\Lambda(\overline{X'})) \cong \mathrm{Hom}_\Lambda(\overline{Y''[-1]}, G_\Lambda(\overline{X'})) = 0. \tag{11}$$

By Proposition 2.4 and (10), we obtain $F_\Lambda(\overline{Y'}) \in \mathrm{add}\overline{X'} = \mathrm{add}F_\Lambda(\overline{X'})$, which implies that $Y' \in \mathrm{add}X'$. From (11), we have $\overline{Y''[-1]} \in \mathrm{add}\overline{X''[-1]}$, then $Y'' \in \mathrm{add}X''$. Therefore, $Y \in \mathrm{add}X$.

(2) If $\mathrm{Ext}^1_{\mathcal{D}}(Z,X)=0$ for some object $Z \in \mathcal{D}$, then

$$\mathrm{Ext}^1_{\mathcal{D}}(X, \tau_{\mathcal{D}}Z[-1]) = \mathrm{Hom}_{\mathcal{D}}(X[1], \mathbb{S}Z) \cong D\mathrm{Ext}^1_{\mathcal{D}}(Z,X) = 0.$$

By (1), we know that $\tau_{\mathcal{D}}Z[-1] = F^{-1}Z \in \mathrm{add}X$, then $Z \in \mathrm{add}(FX) = \mathrm{add}X$.

Together, X is a cluster-tilting object in $\mathcal{D}$.

□

As an application of Theorem 2.14, we give the following corollary to characterize cluster-tilting objects in $\mathcal{D}$.

Corollary 2.15. *In a triangulated category $\mathcal{D}$ with a Serre functor and a cluster-tilting object T, a basic object U is cluster-tilting if and only if it is F-stable and maximal rigid if and only if it is an F-stable rigid object and $|U| = |T|$.*

Proof. The first "if and only if" follows from bijections in Theorem 2.14 immediately. Let X be an object in $\mathcal{D}$, we define

$$|\widetilde{X}| = |(\overline{X'}, \overline{X''[-1]})| := |\overline{X'}| + |\overline{X''[-1]}|.$$

Clearly, $|\widetilde{X}| = |X|$. If X is a basic cluster-tilting object in $\mathcal{D}$, then X is an F-stable rigid object and $\widetilde{X}$ is a basic F_Λ-stable support τ-tilting pair by Theorem 2.14. Hence

$$|X| = |\widetilde{X}| = |\Lambda| = |T|.$$

Conversely, if $U \in F$-rigid$\mathcal{D}$ and $|U| = |T|$, then $\widetilde{U} \in F_\Lambda$-$\tau$-rigid$\Lambda$. Since $|\widetilde{U}| = |U| = |T| = |\Lambda|$, we know that $\widetilde{U}$ is a basic F_Λ-stable support τ-tilting pair. Hence U is a cluster-tilting object in $\mathcal{D}$ by Theorem 2.14. □

Remark 2.16. *(a) When $\mathcal{D}$ is 2-CY, then $F = \tau_{\mathcal{D}}^{-1}[1] = \mathrm{Id}$. Thus we have*

$$\text{c-tilt}\mathcal{D} = \text{m-rigid}\mathcal{D} = \{U \in \text{rigid}\mathcal{D} \mid |U| = |T|\},$$

this was proved in [AIR, DK, ZZ].

(b) This corollary implies that all basic cluster-tilting objects have the same number of indecomposable direct summands for a triangulated category $\mathcal{D}$ with a Serre functor.

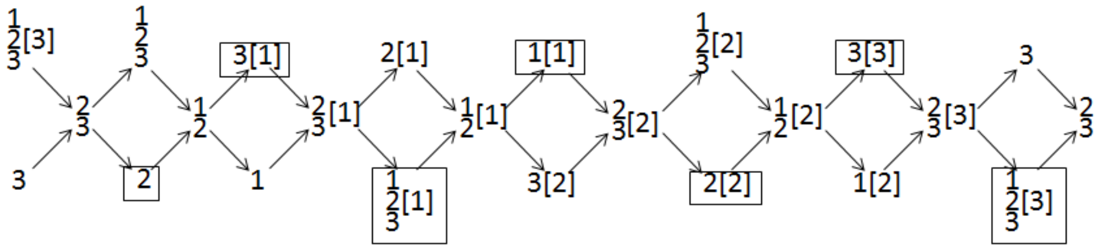
We denote by $\text{c-tilt}_T\mathcal{D}$ the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{D}$ which do not have non-zero direct summands in $\text{add}T[1]$ and by F_Λ-τ-tiltΛ the set of isomorphism classes of basic F_Λ-stable τ-tilting Λ-modules. Immediately, we have the following result.

Corollary 2.17. *Let $\mathcal{D}$ be a k-linear Hom-finite triangulated category with a Serre functor $\mathbb{S}$ and a cluster-tilting object T, and $\Lambda = \mathrm{End}_{\mathcal{D}}^{op}(T)$. Then the bijection $\widetilde{(-)}$ in (8) induces a bijection*

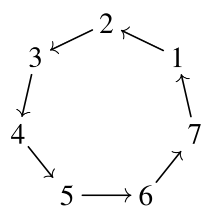
$$\text{c-tilt}_T\mathcal{D} \longleftrightarrow F_\Lambda\text{-}\tau\text{-tilt}\Lambda$$

Remark 2.18. *Theorem 2.12, Theorem 2.14 and Corollary 2.17 generalize a result of Adachi-Iyama-Reiten [AIR], They proved this theorem in case $\mathcal{D}$ is a 2-CY category.*

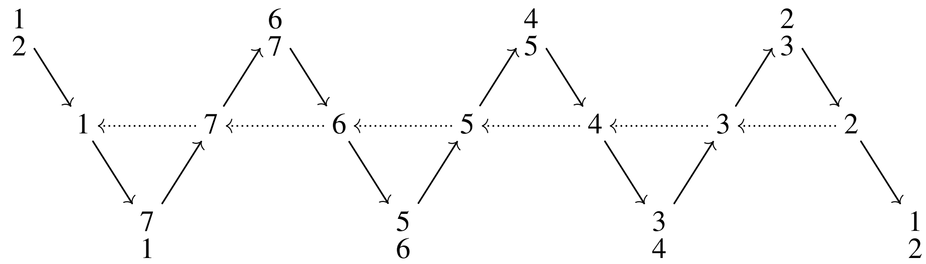
Example 2.19. Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $\mathcal{D} = D^b(kQ)/\tau_Q^{-1}[3]$ be the 3-cluster category of type A_3, where τ_Q is the Auslander-Reiten translation in $D^b(kQ)$. Then $\mathcal{D}$ is a 4-Calabi-Yau triangulated category (see [K1, K2, T, Z08] for details) and its AR-quiver is as follows:



The direct sum $T = 2 \oplus 3[1] \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1] \oplus 1[1] \oplus 2[2] \oplus 3[3] \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}[3]$ gives a cluster-tilting object, the cluster-tilted algebra $\Lambda = \mathrm{End}_{\mathcal{D}}^{op}(T)$ is given by the following quiver with $rad^2 = 0$.



The AR-quiver of modΛ is as follows:



For an indecomposable module M in modΛ, the smallest positive integer n such that $F^n_\Lambda(M) \cong M$ is called the F_Λ-*period* of M, denoted by $o(M)$. In this example, it is easy to see that any indecomposable Λ-module M is τ-rigid and satisfies $o(M) = 7 = |\Lambda|$. Thus, F_Λ-sτ-tilt$\Lambda = \{0, \Lambda\}$. Using our results, we obtain c-tilt$\mathcal{D} = \{T[1], T\}$.

Example 2.20. Let $A = kQ/I$ be a self-injective algebra given by the quiver Q: $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $I =< \alpha\beta\alpha\beta, \beta\alpha\beta\alpha >$. Let $\mathcal{D}$ be the stable module category $\underline{\text{mod}}A$ of A, then it is a triangulated category with a Serre functor and it is not 2-CY. We describe the AR-quiver of mod$A$ in the following picture:

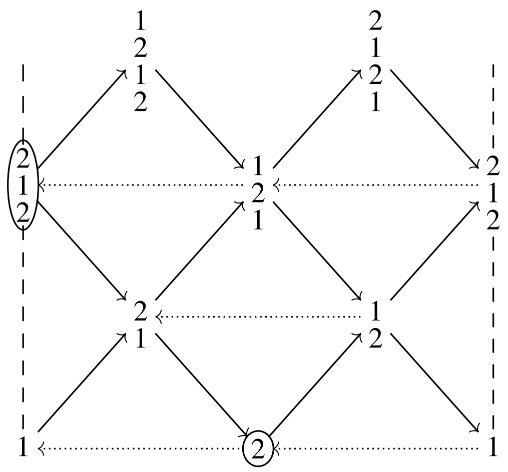
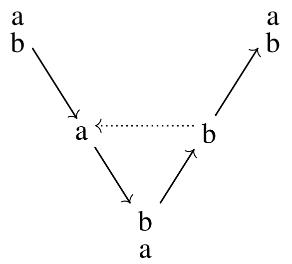


Figure 3

where the leftmost and rightmost columns are identified. Thus, we also get the AR-quiver of $\underline{\text{mod}}A$ by deleting the first row in Figure 3. The direct sum $T = 2 \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix}$ is a cluster-tilting object in $\mathcal{D}$. The cluster-tilted algebra $\Lambda = \text{End}^{op}_{\mathcal{D}}(T) = kQ'/I'$ is given by the quiver Q': $\text{a} \underset{\delta}{\overset{\gamma}{\rightleftarrows}} \text{b}$ and $I' =< \gamma\delta, \delta\gamma >$. The AR-quiver of mod$\Lambda$ is



Hence, F_Λ-sτ-tilt$\Lambda = \{0, \Lambda\}$ and c-tilt$\mathcal{D} = \{T[1], T\}$.

Finally, We want to mention that one can also investigate the connections between cluster-tilting objects in $\mathcal{D}$, functorially finite torsion classes in modΛ and two-term silting complexes for Λ (see [AIR, M]). Note that the auto-equivalence F_Λ of modΛ induces an auto-equivalence of $\mathrm{K^b}(\mathrm{proj}\Lambda)$, and we denote it by F_Λ again. By using a similar approach in [M], we get the following

Remark 2.21. *Let $\mathcal{D}$ be a triangulated category with a Serre functor and a cluster-tilting object T, and let $\Lambda = \mathrm{End}^{op}_{\mathcal{D}}(T)$ be the corresponding cluster-tilted algebra. Then we have bijective correspondences between the set of isomorphism classes of basic cluster-tilting objects in $\mathcal{D}$, the set of isomorphism classes of functorially finite torsion classes in* modΛ *which are stable under F_Λ, and the set of isomorphism classes of two-term silting complexes in* $\mathrm{K^b}(\mathrm{proj}\Lambda)$ *which are stable under F_Λ.*

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References

[AIR] T. Adachi, O. Iyama and I. Reiten. τ-tilting theory. Compos. Math. **150**(3), 415-452, 2014.

[BK] A. I. Bondal and M. M. Kapranov. Representable functors, Serre functors, and mutations. Math. USSR-Izv. **35**(3), 519-541, 1990.

[BMRRT] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov. Tilting theory and cluster combinatorics. Advances in Math. **204**, 572-618, 2006.

[BMR] A. B. Buan, R. J. Marsh and I. Reiten. Cluster-tilted algebras. Trans. Amer. Math. Soc. **359**(1), 323-332, 2007.

[CCS] P. Caldero, F. Chapoton and R. Schiffler. Quivers with relations arising from clusters (A_n case). Trans. Am. Math. Soc. **358**(3), 1347-1364, 2006.

[CZZ] W. Chang, J. Zhang and B. Zhu. On support τ-tilting modules over endomorphism algebras of rigid objects. To appear in Acta mathematica Sinca, English series.

[DK] R. Dehy and B. Keller. On the combinatorics of rigid objects in 2-Calabi-Yau categories. Int. Math. Res. Not. IMRN (11), Art. ID rnn029, 2008.

[FL] C. Fu and P. Liu. Lifting to cluster-tilting objects in 2-Calabi-Yau triangulated categories. Comm. Algebra **37**(7), 2410-2418, 2009.

[FZ] S. Fomin and A. Zelevinsky. Cluster algebras I: Foundations. J. Amer. Math. Soc. **15**(2), 497-529, 2002.

[HJ] T. Holm and P. Jørgensen. On the relation between cluster and classical tilting. J. Pure. Appl. Algebra. **214**, 1523-1533, 2010.

[IY] O. Iyama and Y. Yoshino. Mutation in triangulated categories and rigid Cohen-Macaulay modules. Invent. Math., **172**(1), 117-168, 2008.

[KR] B. Keller and I. Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math., **211**, 123-151, 2007.

[KZ] S. Koenig and B. Zhu. From triangulated categories to abelian categories: cluster tilting in a general framework. Math. Zeit. **258**, 143-160, 2008.

[K1] B. Keller. On triangulated orbit categories. Doc. Math. **10**, 551-581, 2005.

[K2] B. Keller. Calabi-Yau triangulated categories. Trends in Representation Theory of Algebras and Related Topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zrich, 467-489, 2008.

[M] Y. Mizuno. ν-stable τ-tilting modules. Comm. in algebra (to appear), arXiv:1210.8322.

[P] Y. Palu. Cluster characters for 2-Calabi-Yau triangulated categories. Ann. Inst. Fourier, Grenoble **58**(6), 2221-2248, 2008.

[RVdB] I. Reiten and M. Van den Bergh. Noetherian hereditary abelian categories satisfying Serre duality. J. Am. Math. Soc. **15**(2), 295-366, 2002.

[Smi] D. Smith. On tilting modules over cluster-tilted algebras. Illinois J. Math. **52**(4), 1223-1247, 2008.

[T] H. Thomas. Defining an m-cluster category. J. Algebra **318**, 37-46, 2007.

[Z08] B. Zhu. Generalized cluster complexes via quiver representations. J. Algebr. Comb. **27**, 35-54, 2008.

[Z11] B. Zhu. Cluster-tilted algebras and their intermediate coverings. Comm. in algebra **39**, 2437-2448, 2011.

[ZZ] Y. Zhou and B. Zhu. Maximal rigid subcategories in 2-Calabi-Yau triangulated categories. J. Algebra **348**, 49-60, 2011.

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