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Journal of Algebra 257 (2002) 37–50

JOURNAL OF Algebra

www.academicpress.com

Relations for the Grothendieck groups of triangulated categories ☆

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Received 20 July 2001

Communicated by L. Scott

Dedicated to Professor Idun Reiten on the occasion of her 60th birthday

Abstract

A class of triangulated categories with a finiteness condition is singled out. These triangulated categories have Auslander–Reiten triangles. It is proved that the relations of the Grothendieck group of a triangulated category in this class are generated by all Auslander–Reiten triangles. Moreover, the Auslander–Reiten quivers of certain triangulated categories in this class are described in terms of Dynkin diagrams.

Keywords: Triangulated category of finite type; Grothendieck group; Auslander–Reiten triangle

0. Introduction

The notion of Auslander–Reiten triangles in a triangulated category was introduced by Happel in [5]. It has been studied in [5,6,8–10] among others. Auslander–Reiten triangles form a class of special triangles that have the same applications as Auslander–Reiten sequences in the module category of an Artin

☆ The work was supported partially by NSF 10001017 and by Scientific Foundation for Returned Overseas Chinese Scholars, Ministry of Education; was also supported in part by the bilateral project BIL99/43 "New computational geometric and algebraic methods applied to quantum groups and differential operators" of Flemish and Chinese governments.

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0021-8693/02/$ – see front matter
PII: S0021-8693(02)00122-9

algebra. Auslander–Reiten sequences were introduced in seventies in the last century by Auslander and Reiten [2].

In contrast to module categories over Artin algebras, not all triangulated categories (especially derived categories of finite dimensional algebras) have Auslander–Reiten triangles [6]. It was proved in [6] that the derived category of a finite dimensional algebra has Auslander–Reiten triangles if and only if the global dimension of the algebra is finite. Recently Reiten and Van den Bergh [10] proved that the existence of Auslander–Reiten triangles is equivalent to the existence of Serre duality in a triangulated category. In the present paper, we single out a class of triangulated categories satisfying a finite condition. These triangulated categories have Auslander–Reiten triangles. A triangulated category $\mathcal{A}$ is said to be of finite type provided $\sum_{X\in \mathrm{Obj}\,\mathcal{A}} \dim_k \mathrm{Hom}_{\mathcal{A}}(X,Y) < \infty$ and $\sum_{X\in \mathrm{Obj}\,\mathcal{A}} \dim_k \mathrm{Hom}_{\mathcal{A}}(Y,X) < \infty$ for any object Y in $\mathcal{A}$. One aim of this paper is to study the relations of the Grothendieck groups of triangulated categories of finite type by using Auslander–Reiten triangles. We prove that the relations of the Grothendieck group of a triangulated category of finite type are generated by all Auslander–Reiten triangles. Another aim is to study the Auslander–Reiten quivers of triangulated categories of finite type. For a triangulated category of finite type without loops in its Auslander–Reiten quiver, the Auslander–Reiten quiver is described in terms of Dynkin diagrams. Furthermore, for some special triangulated categories, we prove there is no loop in their Auslander–Reiten quivers. We conjecture it is also true for arbitrary triangulated categories with Auslander–Reiten triangles.

We note that for an Artin algebra of finite type, Butler [4] proved that the relations of its Grothendieck group are generated by all Auslander–Reiten sequences. Soon later Auslander added the observation that the converse is true in [3]. We also note that there is no loop in the Auslander–Reiten quiver of Artin algebras [1].

This paper is organized as follows: In Section 1, some notions which will be needed in the paper are recalled and triangulated categories of finite type are defined. Some properties of triangulated categories of finite type are given. In Section 2, the relations of the Grothendieck group of a triangulated category of finite type are proved to be generated by all Auslander–Reiten triangles. In Section 3, the conjecture that there is no loop in the Auslander–Reiten quiver of a trangulated category is posed and a partial result on it is given. The Auslander–Reiten quivers of some triangulated categories are described in terms of Dynkin diagrams.

1. Auslander–Reiten triangles

We fix some notation and recall some definitions which will be used throughout the paper. Let k be an algebraically closed field. A category $\mathcal{A}$ is said to be a Krull–Schmidt category over k provided it is an additive category such that $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is a finite dimensional k-vector space for each pair X, Y of objects in $\mathcal{A}$, and that the endomorphism rings of indecomposable objects in $\mathcal{A}$ are local rings. It is well known that any object in a Krull–Schmidt category $\mathcal{A}$ can be written as a direct sum of indecomposable objects. The subcategory of $\mathcal{A}$ consisting of indecomposable objects is denoted by $\mathrm{ind}\,\mathcal{A}$. The composition of two maps $f : M \to N$, and $g : N \to L$ in $\mathcal{A}$ is denoted by fg. For any pair X, Y in $\mathcal{A}$, the radical of $\mathrm{Hom}_{\mathcal{A}}(X,Y)$, which is denoted by $\mathrm{rad}(X,Y)$, is the subspace consisting of morphisms f such that ufv is not a isomorphism for any section (i.e., a map with a right inverse) $u : M \to X$ and for any retraction (i.e., a map with a left inverse) $v : Y \to N$ with M, N indecomposable objects (compare [6,11]). A morphism $f \in \mathrm{rad}(X,Y)$ is called an irreducible map provided for any factorization $f = f_1 f_2$, either f_1 is a section or f_2 is a retraction [6,11]. Throughout the paper, any category is assumed a Krull–Schmidt category.

For $X \in \mathrm{ind}\,\mathcal{A}$, denote by $\mathrm{SuppHom}(X,-)$ the subcategory of $\mathcal{A}$ generated by objects Y in $\mathrm{ind}\,\mathcal{A}$ with $\mathrm{Hom}_{\mathcal{A}}(X,Y) \neq 0$. Similarly, $\mathrm{SuppHom}(-,X)$ denotes the subcategory generated by objects Y in $\mathrm{ind}\,\mathcal{A}$ with $\mathrm{Hom}_{\mathcal{A}}(Y,X) \neq 0$. If $\mathrm{SuppHom}(X,-)$ ($\mathrm{SuppHom}(-,X)$, respectively) contains only finitely many indecomposables, we say $|\mathrm{SuppHom}(X,-)| < \infty$ ($|\mathrm{SuppHom}(-,X)| < \infty$, respectively). For a triangulated category, let T denote the translation functor. We refer to [6] for the definition of triangulated categories.

Definition 1.1. Let $\mathcal{A}$ be a triangulated category. $\mathcal{A}$ is called path-connected provided for any indecomposable objects X, Y in $\mathcal{A}$, there are finitely many indecomposable objects $X_1 = X,\ X_2, \ldots, X_{n-1}, X_n = Y$ in $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(X_i, X_{i+1}) \neq 0$, or $\mathrm{Hom}_{\mathcal{A}}(X_{i+1}, X_i) \neq 0$ for each i.

Now we define the triangulated categories of finite type.

Definition 1.2. Let $\mathcal{A}$ be a triangulated category. $\mathcal{A}$ is said to be of finite type provided $|\mathrm{SuppHom}(X,-)| < \infty$ and $|\mathrm{SuppHom}(-,X)| < \infty$, for any object X in $\mathrm{ind}\,\mathcal{A}$.

From the definition, one has that the derived categories of representation-finite finite dimensional hereditary algebras and the stable module categories over finite dimensional self-injective algebras of finite type are examples of triangulated categories of finite type.

The following proposition says that the two finite conditions involved in the definition of triangulated categories of finite type can be replaced by one of them.

Proposition 1.1. *A triangulated category $\mathcal{A}$ is of finite type if and only if $|\mathrm{SuppHom}(X,-)| < \infty$ for any object X in $\mathrm{ind}\,\mathcal{A}$.*

Proof. The necessity follows from the definition. We prove the sufficiency. Let Y be an indecomposable object in $\mathrm{SuppHom}(-, X)$ with $Y \ncong X$. There is a non-zero non-invertible map $f \in \mathrm{Hom}_{\mathcal{A}}(Y, X)$. Assume that f is embedded in the following triangle:

$$Y \xrightarrow{f} X \xrightarrow{g} Z \xrightarrow{h} TY.$$

We decompose $Z = \bigoplus_1^n Z_i$ as a direct sum of indecomposable objects. Then the maps g and h can be written as $g = (g_1, \ldots, g_n)$, $h = (h_1, \ldots, h_n)^{\mathrm{t}}$, where $g_i : X \to Z_i$ and $h_i : Z_i \to TY$. It follows from Lemma 1.3 in [12] that for any i, g_i and h_i are non-zero non-invertible. Therefore $TY \in \mathrm{SuppHom}(Z_i, -) \subseteq \bigcup_{N \in \mathrm{SuppHom}(X,-)} \mathrm{SuppHom}(N, -)$. The right-hand side, by definition, contains only finitely many indecomposable objects. Then $|\mathrm{SuppHom}(-, X)| < \infty$ for any object X in $\mathrm{ind}\,\mathcal{A}$. The proof is finished. □

Let $\mathcal{A}$ be a triangulated category and F the free abelian group generated by representatives of the isomorphism classes of objects in $\mathcal{A}$. We denote by $[X]$ such a representative. Let F_0 be the subgroup of F generated by elements of the forms: $[X] - [Y] + [Z]$ for all triangles $X \to Y \to Z \to TX$ in $\mathcal{A}$.

Definition 1.3. The Grothendieck group, which is denoted by $K_0(\mathcal{A})$, of a triangulated category is the factor group F/F_0 of F by F_0 (compare [6]).

For any $X \in \mathcal{A}$, we still denote by $[X]$ the element in $K_0(\mathcal{A})$ corresponding to X.

Remark 1.1. There are examples in [13] which show that the Grothendieck group $K_0(\mathcal{A})$ of a non-zero triangulated category may be zero.

The analogue of an Auslander–Reiten sequence for a triangulated category was introduced by Happel (cf. [6]).

Definition 1.4. Let $\mathcal{A}$ be a triangulated category. A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\mathcal{A}$ is called an Auslander–Reiten triangle if the following conditions are satisfied:

(AR1) X and Z are indecomposable.
(AR2) $w \neq 0$.
(AR3) If $f : W \to Z$ is not a retraction (i.e., there is not any $g : Z \to W$ with $gf = 1_Z$), there exists $f' : W \to Y$ such that $f'v = f$.

In this case, the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\mathcal{A}$ is called, for simplicity, an AR-triangle.

Remark 1.2. The maps u and v in an AR-triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ are irreducible maps [6]. Sometimes, X is denoted by τZ, and Z is denoted by $\tau^{-1}X$, where τ is the Auslander–Reiten translation.

We say that a triangulated category $\mathcal{A}$ has Auslander–Reiten triangles if for any indecomposable object $Z \in \operatorname{Obj}\mathcal{A}$, there exist an AR-triangle ending at $Z: X \to Y \to Z \to TX$, and an AR-triangle starting at $Z: Z \to M \to N \to TZ$. In this case, $\mathcal{A}$ is also called a triangulated category with Auslander–Reiten triangles.

The next lemma and proposition concern the elementary properties of triangulated categories of finite type. They are easy to prove, but are needed in Sections 2 and 3. We present them here for the completeness.

Lemma 1.2. *Let $\mathcal{A}$ be a triangulated category of finite type. Then for any object X in $\mathcal{A}$, there exists a natural number n (m, respectively) such that* $\operatorname{rad}^n(-, X) = 0$ ($\operatorname{rad}^m(X, -) = 0$*, respectively*).

Proof. Otherwise, we have an infinite chain, denoted by $(*)$, of non-zero morphisms between indecomposable objects

$$\cdots \longrightarrow X_i \xrightarrow{f_i} \cdots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X \qquad (*)$$

with $f_n \cdots f_2 f_1 \neq 0$, for any n. By the condition $\sum_{Y \in \operatorname{Obj}\mathcal{A}} \dim_k \operatorname{Hom}(Y, X) < \infty$, one has that there are finitely many indecomposable objects M_j, $j = 1, 2, \ldots, n'$, in $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(M_j, X) \neq 0$, for any j. Therefore there exists an object M_t (denoted by M, for simplicity), such that M appears infinitely many times in the chain $(*)$. It follows that there exists an infinite chain of non-zero maps

$$\cdots \longrightarrow M \xrightarrow{g_i} \cdots \longrightarrow M \xrightarrow{g_2} M \xrightarrow{g_1} M, \qquad (**)$$

with $g_i \in \operatorname{rad}(M, M)$ and $g_i \cdots g_1 \neq 0$, for any i. This is a contradiction. Then there is a natural number n such that $\operatorname{rad}^n(-, X) = 0$. Dually, one can get the proof for the existence of m such that $\operatorname{rad}^m(X, -) = 0$. □

Before we state the next proposition, let us recall some generalities from Auslander–Reiten theory. A map α: $X \to Y$ is called left almost split if α is not a section and any map $X \to Y'$ which is not a section factors through α. Dually, $\beta: Y \to Z$ is right almost split if β is not a retraction and any map $X \to Y'$ which is not a retraction factors through β.

Proposition 1.3. *Let $\mathcal{A}$ be a connected triangulated category of finite type. Then $\mathcal{A}$ has Auslander–Reiten triangles.*

Proof. We prove that there is an AR-triangle ending at any indecomposable object in $\mathcal{A}$. Dually, one can get the proof for the existence of AR-triangle starting at any indecomposable object in $\mathcal{A}$. Let $M \in \operatorname{ind}\mathcal{A}$. It is easy to see that there is a right almost split map $\beta' : Y' \to M$ (compare [11]). Therefore there is a minimal right almost split map $\beta : Y \to M$. It is easy to verify that the triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} M \xrightarrow{w} TX$ determined by β is an AR-triangle. The proof is finished. □

2. Relations for Grothendieck groups

We recall some basics on the Grothendieck groups of triangulated categories from [6,11]. Let F be the free abelian group generated by representatives of the isomorphism classes of objects in $\mathcal{A}$ and $K_0(\mathcal{A}, 0)$ the factor groups of F by the subgroup generated by the elements of the form $[A] + [C] - [A \sqcup C]$. We have the following well-known facts:

(1) The set $\{[M] \mid M \in \operatorname{ind}\mathcal{A}\}$ is a free basis for $K_0(\mathcal{A}, 0)$.
(2) The Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is the factor group of $K_0(\mathcal{A}, 0)$ modulo the subgroup generated by elements of the forms: $[A] + [C] - [B]$, for all triangles $\delta : A \to B \to C \to TA$. For simplicity, we denote by $[\delta]$ the element $[A] + [C] - [B]$ in $K_0(\mathcal{A}, 0)$.
(3) There is a canonical epimorphism $\phi : K_0(\mathcal{A}, 0) \to K_0(\mathcal{A})$.

Now we state the main theorem in this section.

Theorem 2.1. *Let $\mathcal{A}$ be a triangulated category of finite type. Then* $\operatorname{Ker}\phi$ *is generated by the elements $[\delta]$ in $K_0(\mathcal{A}, 0)$, where $\delta : A \to B \to C \to TA$ runs through all Auslander–Reiten triangles in $\mathcal{A}$.*

To prove it, we need some lemmas.

Lemma 2.2. *Suppose there is a commutative diagram whose rows are triangles in a triangulated category $\mathcal{A}$:*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \downarrow{\scriptstyle Tf} \\
X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z & \xrightarrow{w_1} & TX_1.
\end{array}$$

Then $X \to X_1 \sqcup Y \to Y_1 \xrightarrow{v_1 w} TX$ is a triangle.

Proof. We assume that the morphism $v_1 w$ is embedded into the triangle $X \stackrel{u'}{\to} M \stackrel{v'}{\to} Y_1 \stackrel{v_1 w}{\longrightarrow} TX$. It suffices to prove $M \cong X_1 \sqcup Y$. Since we have a commutative diagram whose rows and columns are triangles:

$$\begin{array}{ccccccc}
 & & X_1 & = & X_1 & & \\
 & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_1} & & \\
X & \xrightarrow{u'} & M & \xrightarrow{v'} & Y_1 & \xrightarrow{v_1 w} & TX \\
\| & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle v_1} & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX, \\
 & & \downarrow{\scriptstyle w_2} & & \downarrow{\scriptstyle w_1} & & \\
 & & TX_1 & = & TX_1 & &
\end{array}$$

$w_2 = v w_1 = v w T f = 0$. This means that the triangle

$$X_1 \xrightarrow{u_2} M \xrightarrow{v_2} Y \xrightarrow{w_2} TX_1$$

splits, i.e., $M \cong X \sqcup Y$. □

The next lemma was proved in [9], we present different and simple proof here.

Lemma 2.3. *Let* $X \xrightarrow{(f_1, f_2)} Y_1 \sqcup Y_2 \xrightarrow{(g_1, g_2)^t} Z \to TX$ *be a triangle. If* $f_1 = 0$, *then it is isomorphic to the following triangle*:

$$X \xrightarrow{(0, f_2)} Y_1 \sqcup Y_2 \xrightarrow{\begin{pmatrix} g_{11}, & 0 \\ 0, & g_{22} \end{pmatrix}} Z_1 \sqcup Z_2 \longrightarrow TX, \tag{$*$}$$

where $g_{11} : Y_1 \to Z_1$ *is an isomorphism. Moreover, the triangle* $(*)$ *is a direct sum of triangles* $X \xrightarrow{f_2} Y_2 \xrightarrow{g_{22}} Z_2 \to TX$ *and* $0 \to Y_1 \xrightarrow{g_{11}} Z_1 \to 0$.

Proof. We assume that f_2 is embedded into the triangle

$$X \xrightarrow{f_2} Y_2 \xrightarrow{g_{22}} Z_2 \longrightarrow TX,$$

and the isomorphism g_{11} is embedded into the triangle

$$0 \longrightarrow Y_1 \xrightarrow{g_{11}} Z_1 \longrightarrow 0.$$

The direct sum of the two triangles above is again a triangle (compare [6]), i.e.

$$X \xrightarrow{(0, f_2)} Y_1 \sqcup Y_2 \xrightarrow{\begin{pmatrix} g_{11}, & 0 \\ 0, & g_{22} \end{pmatrix}} Z_1 \sqcup Z_2 \longrightarrow TX \tag{$*$}$$

is a triangle. If $f_1 = 0$, then $(f_1, f_2) = (0, f_2)$, and then the triangle

$$X \xrightarrow{(f_1,f_2)} Y_1 \sqcup Y_2 \xrightarrow{\binom{g_1}{g_2}} Z \longrightarrow TX$$

is isomorphic to the triangle $(*)$. The proof is finished. □

Now we are ready to prove the theorem.

Proof of Theorem 2.1. Let $\delta : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{w} TX$ be an arbitrary triangle with $w \neq 0$ and $Z \in \operatorname{ind} \mathcal{A}$. It suffices to prove that the element $[\delta]$ in $K_0(\mathcal{A}, 0)$ can be written as a sum of the elements in $K_0(\mathcal{A}, 0)$ corresponding to some AR-triangles. Suppose $w \in \operatorname{rad}^n(Z, TX)$, and $\delta^* : Z_1 \xrightarrow{u_1} M \xrightarrow{v_1} Z \xrightarrow{w_1} TZ_1$ is an AR-triangle ending at Z. Since g is not a retraction, we have the commutative diagram:

$$\begin{array}{lccccccc}
\delta: & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{w} & TX \\
 & \downarrow & & \downarrow & & \| & & \downarrow \\
\delta^*: & Z_1 & \xrightarrow{u_1} & M & \xrightarrow{v_1} & Z & \xrightarrow{w_1} & TZ_1.
\end{array}$$

By Lemma 2.2, one has a new triangle $\delta_1 : X \to Y \sqcup Z_1 \to M \xrightarrow{v_1 w} TX$ with $v_1 w \in \operatorname{rad}^{n+1}(-, TX)$, and that $[\delta] = [\delta^*] + [\delta_1]$ in $K_0(\mathcal{A}, 0)$. We decompose M as a direct sum of indecomposable objects: $M = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_k$. Without loss of generality, we assume $k = 2$. Let Y_1 denote $Y \sqcup Z_1$. Then the triangle δ_1 can be written as

$$X \longrightarrow Y_1 \xrightarrow{(f_1,f_2)} M_1 \sqcup M_2 \xrightarrow{\binom{w_1}{w_2}} TX$$

with $w' = \binom{w_1}{w_2} = v_1 w \in \operatorname{rad}^{n+1}(-, TX)$. If f_1 is a retraction, then the triangle δ_1 is isomorphic to the following triangle:

$$X \xrightarrow{(g_1,g_2)} M_1 \sqcup Y_1' \xrightarrow{\begin{pmatrix}1, & *\\ 0, & *\end{pmatrix}} M_1 \sqcup M_2 \xrightarrow{\binom{w_1}{w_2}} TX.$$

It follows that $g_1 = 0$. By Lemma 2.3, we have that the triangle δ_1 is isomorphic to the triangle:

$$X \xrightarrow{(0,g_2)} M_1 \sqcup Y_1' \xrightarrow{\begin{pmatrix}1, & 0\\ 0, & f_{22}\end{pmatrix}} M_1 \sqcup M_2 \xrightarrow{\binom{0}{w_2}} TX.$$

Hence $w_2 \in \operatorname{rad}^{n+1}(-, TX)$ and $[\delta_1] = [\delta_2]$, where δ_2 is the following triangle:

$$\delta_2: \quad X \xrightarrow{g_1} Y_1 \xrightarrow{f_{22}} M_2 \xrightarrow{w_2} TX.$$

In addition, if f_2 is also a retraction, i.e. $w_2 = 0$, then $[\delta] = [\delta^*]$, we have proved the assertion. Now we return to the triangle δ_1 and assume that f_1 and f_2 are not retractions in the following. Assume that the AR-triangles ending at M_1, M_2 are respectively δ_1^*, δ_2^*:

$$\delta_i^*: \quad M_i' \xrightarrow{u_1'} N_i \xrightarrow{v_i'} M_i \longrightarrow TM_i', \quad i = 1, 2.$$

We form the direct sum of them: $\delta_1^* \oplus \delta_2^*$

$$M_1' \sqcup M_2' \xrightarrow{\begin{pmatrix} u_1', & 0 \\ 0, & u_2' \end{pmatrix}} N_1 \sqcup N_2 \xrightarrow{\begin{pmatrix} v_1', & 0 \\ 0, & v_2' \end{pmatrix}} M_1 \sqcup M_2 \longrightarrow T(M_1' \sqcup M_2').$$

It follows from definition of AR-triangles that there is morphism $h : Y_1 \to N_1 \sqcup N_2$ such that the following is a commutative diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y_1 & \xrightarrow{(f_1, f_2)} & M_1 \sqcup M_2 & \xrightarrow{w'} & TX \\ \downarrow & & \downarrow h & & \| & & \downarrow \\ M_1' \sqcup M_2' & \xrightarrow{\begin{pmatrix} u_1' & 0 \\ 0 & u_2' \end{pmatrix}} & N_1 \sqcup N_2 & \xrightarrow{\begin{pmatrix} v_1' & 0 \\ 0 & v_2' \end{pmatrix}} & M_1 \sqcup M_2 & \longrightarrow & T(M_1' \sqcup M_2') \end{array}$$

Hence, by Lemma 2.2 again, one has a triangle

$$X \longrightarrow M_1' \sqcup M_2' \sqcup Y_1 \longrightarrow N_1 \sqcup N_2 \xrightarrow{\begin{pmatrix} v_1', & 0 \\ 0, & v_2' \end{pmatrix} w'} TX,$$

with $\begin{pmatrix} v_1', & 0 \\ 0, & v_2' \end{pmatrix} w' \in \mathrm{rad}^{n+2}(-, TX)$. We continue this process that we have done for δ_1. By Lemma 1.2, one knows that this process must stop at a finite steps, i.e., up to some finite step, we can get a splitting triangle. Then there are finitely many AR-triangles $\delta_1, \ldots, \delta_n$, such that $[\delta] = [\delta_1] + \cdots + [\delta_n]$. The proof is finished. □

3. Auslander–Reiten quivers

The Auslander–Reiten quivers of Derived categories of some finite dimensional algebras are displayed in [6]. In this section, we will give a description of Auslander–Reiten quivers of some triangulated categories of finite type.

We recall the notion of the Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ of a triangulated category from [6]. Let $\mathcal{A}$ be a triangulated category with Auslander–Reiten triangles. The Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ of it is a valued stable translation quiver, its vertices are the isomorphism classes $[X]$ of indecomposable objects in $\mathcal{A}$; for two vertices $[X]$ and $[Y]$, there is an arrow with valuation $(a_{X,Y}, a'_{X,Y})$ as follows:

$$Y \xrightarrow{(a_{X,Y}, a'_{X,Y})} Y$$

provided there are AR-triangles in $\mathcal{A}$

$$Z \to X^{a_{X,Y}} \sqcup M \to Y \to TZ,$$
$$X \to Y^{a'_{X,Y}} \sqcup N \to Z \to TX,$$

with X (respectively Y) is not the direct summand of M (respectively N).

An arrow in a valued translation quiver $\Gamma_{\mathcal{A}}$ with the same starting and ending vertex is called a loop. It is well known that the Auslander–Reiten quiver Γ_A of an Artin algebra A contains no loops. We conjecture it also true for a triangulated category with Auslander–Reiten sequences, i.e. the Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ of a triangulated category $\mathcal{A}$ contains no loops. We prove the conjecture in a special case in the following.

Theorem 3.1. *Let $\mathcal{A}$ be a connected triangulated category with Auslander–Reiten triangles and* $\operatorname{ind}\mathcal{A}$ *contains at least three objects. Suppose* $\dim_k \operatorname{End}(X) \leqslant 2$, *for any $X \in \operatorname{ind}\mathcal{A}$. Then there does not exist loop in $\Gamma_{\mathcal{A}}$.*

Proof. Suppose there is a loop at X in $\Gamma_{\mathcal{A}}$. Then we have the following AR-triangle starting at $X: X \overset{u}{\to} X \sqcup M \overset{v}{\to} \tau^{-1}X \overset{w}{\to} TX$. If $X \ncong \tau^{-1}X$, then $M \cong \tau^{-1}X \sqcup M_1$. Now apply $\operatorname{Hom}_{\mathcal{A}}(\tau^{-1}X, -)$ to the AR-triangle above. This gives a long exact sequence, which implies the following equality:

$$\begin{aligned}&\dim_k S_1 - \dim_k \operatorname{Hom}\big(\tau^{-1}X, X\big) + \dim_k \operatorname{Hom}\big(\tau^{-1}X, X\big) + \dim_k S_2\\ &\quad + \dim_k \operatorname{Hom}\big(\tau^{-1}X, \tau^{-1}X\big) + \dim_k \operatorname{Hom}\big(\tau^{-1}X, M_1\big)\\ &\quad - \dim_k \operatorname{Hom}\big(\tau^{-1}, \tau^{-1}X\big) = 0,\end{aligned}$$

where $S_1 = \operatorname{Im}(\operatorname{Hom}(\tau^{-1}X, T^{-1}w))$, $S_2 = \operatorname{Im}(\operatorname{Hom}(\tau^{-1}X, w))$. After a direct calculation, the equality above becomes the equality: $\dim_k S_1 + \dim_k S_2 + \dim_k \operatorname{Hom}(\tau^{-1}X, M_1) = 0$. Therefore $S_2 = 0$ which is a contradiction. Then we have that $X \cong \tau^{-1}X$ and the AR-triangle starting at X is

$$X \overset{u}{\to} X \sqcup M \overset{v}{\to} X \overset{w}{\to} TX.$$

If $M = 0$, then the AR-triangle starting at X is $X \overset{u}{\to} X \overset{v}{\to} X \overset{w}{\to} TX$. It follows that $u = tv$, for some $t \in k$ and w is not a retraction. For any $Y \in \operatorname{ind}\mathcal{A}$ with $Y \ncong X$, if there exists a non-zero morphism $f \in \operatorname{Hom}(X, Y)$, then there exists a morphism $g : X \to Y$ such that $f = ug$, where g is not a section. Also for g, there is a morphism $g_1 \in \operatorname{Hom}(X, Y)$ such that $g = ug_1$. It follows that $f = ug = u(ug_1) = t^{-1}((uv)g_1) = 0$. It is a contradiction. Therefore $\operatorname{Hom}(X, Y) = 0$. Similarly we have that $\operatorname{Hom}(Y, X) = 0$. The conectness of $\mathcal{A}$ makes $\operatorname{ind}\mathcal{A} = \{X\}$, contradicting to the condition on the number of $|\operatorname{ind}\mathcal{A}|$. This proves that $M \neq 0$. If M contains X as a direct summand, then by applying $\operatorname{Hom}(X, -)$ to the AR-triangle starting at X as above, one can also get a contradiction as above. Therefore the AR-triangle starting $X : X \to X \sqcup M \to X \to TX$ has properties that $M \neq 0$ and M does not contain X as a direct summand. By applying $\operatorname{Hom}(X, -)$ to this AR-triangle, one gets the inequality

$$\dim_k \operatorname{Hom}(X, X) \geqslant 1 + \dim_k \operatorname{Hom}(X, M).$$

It follows from the condition $\dim_k \operatorname{End}(X) \leqslant 2$ that $\dim_k \operatorname{Hom}(X, M) \leqslant \dim_k \operatorname{Hom}(X, X) - 1 \leqslant 1$. By applying $\operatorname{Hom}(-, M)$ to the AR-triangle starting at X, one gets another inequality

$$\dim_k \operatorname{Hom}(X, M) + \dim_k \operatorname{Hom}(X, M) \geqslant \dim_k \operatorname{Hom}(X, M) + \dim_k \operatorname{Hom}(M, M).$$

It follows that $\dim_k \operatorname{Hom}(M, M) \leqslant \dim_k \operatorname{Hom}(X, M) \leqslant 1$. Hence $\dim_k \operatorname{Hom}(M, M) = 1$ and M is indecomposable. Now let $M \to X \sqcup Z \to M \to TM$ be the AR-triangle starting at M. By applying $\operatorname{Hom}(-, M)$ to this AR-triangle, one gets the inequality $\dim_k \operatorname{Hom}(M, M) + \dim_k \operatorname{Hom}(M, M) \geqslant \dim_k \operatorname{Hom}(X, M) + \dim_k \operatorname{Hom}(Z, M) + 1 = 2 + \dim_k \operatorname{Hom}(Z, M)$. It follows that $\dim \operatorname{Hom}(Z, M) = 0$ and $Z = 0$. It is easy to prove that

$$\operatorname{Hom}(Y, X) = \operatorname{Hom}(Y, M) = 0 = \operatorname{Hom}(X, Y) = \operatorname{Hom}(M, Y),$$

for all $Y \in \operatorname{ind} \mathcal{A} \backslash \{M, X\}$. The connectness of $\mathcal{A}$ makes $\operatorname{ind} \mathcal{A} = \{M, X\}$. It is a contradiction. The proof is finished. □

In the rest of this section, we assume that the trangulated category $\mathcal{A}$ has AR-triangles. A path in $\mathcal{A}$ is a sequence $(X_0, X_1, \ldots, X_s)$ of indecomposable objects in $\mathcal{A}$ such that $\operatorname{rad}(X_{i-1}, X_i) \neq 0$ for all $1 \leqslant i \leqslant s$. The sequence is called sectional if $\tau X_{i+2} \ncong X_i$, for all i. If $s \geqslant 1$ and $X_0 = X_s$, then the path $(X_0, X_1, \ldots, X_s)$ is called a cycle in $\mathcal{A}$. It is well known that there is no sectional cycle in the Auslander–Reiten quiver of an Artin algebra [1]. The next proposition tell us that if our conjecture holds, then there is also no sectional cycle in the Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ of a triangulated category $\mathcal{A}$.

Proposition 3.2. *Let* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$ *be a sectional sequence of irreducible morphisms between indecomposable objects in* $\mathcal{A}$*. Then the composition* $f = f_1 f_2 \cdots f_n$ *is nonzero.*

Proof. It is the consequence of the following lemma. □

Lemma 3.3. *Let* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$ *be a sectional sequence of irreducible morphisms between indecomposable objects in* $\mathcal{A}$*. Suppose that the composition* $f_1 \cdots f_n$ *either is* 0 *or factors through a morphism* $g : B \to A_{n+1}$ *with* B *indecomposable such that* $(f_n, g) : A_n \sqcup B \to A_{n+1}$ *is irreducible. Then* $\tau A_i \cong A_{i-2}$ *for some* i *with* $3 \leqslant i \leqslant n + 1$.

Proof. The proof for this lemma is similar to the case of algebras (compare [1]), we present it for completeness. We use induction on n. For $n = 2$ it is easy to see the lemma is true. For the inductive step assume the claim holds for $n \geqslant 2$, we want to prove that the claim holds for $n + 1$. Let $\bar{f} = f_1 \cdots f_n$, and assume first $\bar f f_{n+1} = 0$. If $\bar f = 0$ we are done by induction. If $\bar f \neq 0$, we consider the commutative diagram whose rows are triangles:

$$\begin{array}{ccccccc}
\tau A_{n+2} & \xrightarrow{(g_1,s)} & A_{n+1} \sqcup E & \xrightarrow{\binom{f_{n+1}}{t}} & A_{n+2} & \longrightarrow & T\tau A_{n+2} \\
\uparrow{\scriptstyle \exists h_1} & & \uparrow{\scriptstyle (\bar f,0)} & & \uparrow & & \\
A_1 & \xrightarrow[\mathrm{id}_{A_1}]{} & A_1 & \longrightarrow & 0 & \longrightarrow & TA_1.
\end{array}$$

Hence $\bar f = h_1 g_1$. Now if $\tau A_{n+2} \cong A_n$, we are done. Otherwise $(f_n\ g_1)^{\mathrm t} : A_n \sqcup \tau A_{n+2} \to A_{n+1}$ is irreducible and then we are done by induction. If $\bar f f_{n+1} = gh$ for some $h : A_1 \to B$ and $g : B \to A_{n+2}$ with $(f_{n+1}\ g) : A_{n+1} \sqcup B \to A_{n+2}$ being irreducible, then there exists h' such that the diagram commutes:

$$\begin{array}{ccccccc}
\tau A_{n+2} & \xrightarrow{(f'_{n+1},g',t')} & A_{n+1} \sqcup B \sqcup E & \xrightarrow{\begin{pmatrix} f_{n+1} \\ g \\ t \end{pmatrix}} & A_{n+2} & \longrightarrow & T\tau A_{n+2} \\
\uparrow{\scriptstyle \exists h'} & & \uparrow{\scriptstyle (\bar f,-h,0)} & & \uparrow & & \\
A_1 & \xrightarrow[\mathrm{id}_{A_1}]{} & A_1 & \longrightarrow & 0 & \longrightarrow & TA_1,
\end{array}$$

hence $\bar f = h' f'_{n+1}$. If A_n is isomorphic to τA_{n+2}, we are done. If $A_n \ncong \tau A_{n+2}$, then $(f_n\ f'_{n+1})^{\mathrm t} : A_n \sqcup \tau A_{n+2} \to A_{n+1}$ is irreducible, we are done by induction. The proof is finished. $\square$

We recall from [7] the definition of subadditive function on a translation quiver. A function from a translation quiver Γ to $\mathbf{N}$ is called subadditive if $f(x) + f(\tau x) \geqslant \sum_{y \to x} f(y)$ for all non-projective vertices x.

Throughout the rest of this section, the triangulated category $\mathcal{A}$ is assumed of finite type. Let $M_{\mathcal{A}} = \bigsqcup_{X \in \mathrm{ind}\,\mathcal{A}} X$. We define a functor l from $\Gamma_{\mathcal{A}}$ to $\mathbf{N}$ as follows:

$$l(X) = \dim_k \mathrm{Hom}_{\mathcal{A}}(M, X), \quad \text{where } X \in \mathrm{ind}\,\mathcal{A}.$$

Remark. From the definition of finiteness of triangulated categories, one knows that the function l is well-defined.

Proposition 3.4. *l is a subadditive function on $\Gamma_{\mathcal{A}}$.*

Proof. Let $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} TX$ be an AR-triangle in $\mathcal{A}$ and M an arbitrary indecomposable object in $\mathcal{A}$. Now applying $\mathrm{Hom}(M, -)$ to the triangle above. This gives a long exact sequence in k-mod:

$$\cdots \longrightarrow \mathrm{Hom}(M, T^{-1}Z) \xrightarrow{(T^{-1}w)^*_M} \mathrm{Hom}(M, X) \xrightarrow{(u)^*_M} \mathrm{Hom}(M, Y)$$
$$\xrightarrow{(v)^*_M} \mathrm{Hom}(M, Z) \xrightarrow{(w)^*_M} \mathrm{Hom}(M, TX) \longrightarrow \cdots,$$

where $(f)^*_M$ denotes the morphism $\mathrm{Hom}(M, f)$. Let $S_{T^m Z}(M)$ denote the image of morphism $(T^m w)^*$. It follows from the long exact sequence above that the sequence is exact:

$$0 \to S_{T^{-1}Z}(M) \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y) \to \mathrm{Hom}(M, Z) \\ \to S_Z(M) \to 0,$$

where

$$S_Z(M) = \begin{cases} k, & \text{if } M \cong Z, \\ 0, & \text{if } M \ncong Z. \end{cases}$$

Therefore we have that

$$l(X) + l(Z) = \begin{cases} l(Y) + 2, & \text{if } T^- Z \ncong Z, \\ l(Y) + 1, & \text{if } T^- Z \cong Z. \end{cases}$$

It follows that $l(X) + l(Z) > l(Y)$. The proof is finished. □

Now we prove that main theorem in this section, which gives a connection between Dynkin diagrams and the Auslander–Reiten quivers of some triangulated categories of finite type.

Theorem 3.5. *Let $\mathcal{A}$ be a triangulated category of finite type. Suppose that the Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ contains no loops. Then $\Gamma_{\mathcal{A}} \cong \mathbf{Z}\Delta/G$, where Δ is a Dynkin diagram.*

Proof. By proposition 3.4, we have a subadditive function l on $\Gamma_{\mathcal{A}}$. It follows from [7] that $\Gamma_{\mathcal{A}} \cong Z\Delta/G$ and Δ is a Dynkin diagram or A_∞. Since $\mathcal{A}$ is of finite type, Δ is a Dynkin diagram. The proof is finished. □

Acknowledgments

The authors are grateful to the referee for valuable comments and suggestions, which improved the manuscript, and for pointing out some inaccuracies. The revision of the paper was done when B. Zhu was visiting the Free University of Brussels, VUB. He would like to thank S. Caenepeel and VUB for warm hospitality and many helps that he received.

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