Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Algebra 290 (2005) 473–490

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

Locally finite triangulated categories ☆

Jie Xiao, Bin Zhu *

Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, PR China

Received 30 January 2004

Available online 13 June 2005

Communicated by Leonard L. Scott, Jr.

Dedicated to Claus M. Ringel on the occasion of his sixtieth birthday

Abstract

A k-linear triangulated category $\mathcal{A}$ is called locally finite provided $\sum_{X\in \operatorname{ind}\mathcal{A}} \dim_k \operatorname{Hom}_{\mathcal{A}}(X,Y) < \infty$ for any indecomposable object Y in $\mathcal{A}$. It has Auslander–Reiten triangles. In this paper, we show that if a (connected) triangulated category has Auslander–Reiten triangles and contains loops, then its Auslander–Reiten quiver is of the form $\hat{L}_n$:

$$\begin{tikzcd} \underset{n}{\circ} \arrow[r, shift left] & \underset{n-1}{\circ} \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] & \cdots & \underset{2}{\circ} \arrow[r, shift left] & \underset{1}{\circ} \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

By using this, we prove that the Auslander–Reiten quiver of any locally finite triangulated category $\mathcal{A}$ is of the form $\mathbf{Z}\vec{\Delta}/G$, where Δ is a Dynkin diagram and G is an automorphism group of $\mathbf{Z}\vec{\Delta}$. For most automorphism groups G, the triangulated categories with $\mathbf{Z}\vec{\Delta}/G$ as their Auslander–Reiten quivers are constructed. In particular, a triangulated category with $\hat{L}_n$ as its Auslander–Reiten quiver is constructed.

Keywords: Locally finite triangulated category; Triangulated category with loops; Auslander–Reiten quiver; Dynkin diagram

☆ Supported in part by the NSFC (Grants 10471071) and by the Doctoral Program Foundation of Institute of Higher Education (2003).

* Corresponding author.
E-mail addresses: jxiao@math.tsinghua.edu.cn (J. Xiao), bzhu@math.tsinghua.edu.cn (B. Zhu).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.05.011

0. Introduction

The Gabriel's well-known theorem [6] tells us that the path algebra of a quiver is of finite representation type (that is, has only finitely many indecomposable modules up to isomorphisms) if and only if the underlying diagram of the quiver is a disjoint union of Dynkin diagrams. Furthermore Riedtmann's work on self-injective algebras [19] and Happel's work on derived categories [8,9] show that Dynkin diagrams also appear in the stable module categories of self-injective algebras of finite type and in the derived categories of hereditary algebras of finite representation type. As important examples of triangulated categories, derived categories of algebras and stable categories of self-injective algebras inherit some property from their module categories, such as Auslander–Reiten theory. This motivates Happel to introduce the notion of Auslander–Reiten triangles for triangulated categories in [8,9]. In contrast to module categories of Artin algebras [3], not all triangulated categories (even the derived categories of finite dimensional algebras) have Auslander–Reiten triangles. It was proved in [8–10] that the derived category of a finite dimensional algebra has Auslander–Reiten triangles if and only if the global dimension of the algebra is finite. Recently, Reiten and Van den Bergh [18] proved that the existence of Auslander–Reiten triangles is equivalent to the existence of Serre duality in this triangulated category. A class of triangulated categories satisfying a finite condition was introduced in [23], which includes stable categories of self-injective finite dimensional algebras of finite type and derived categories of finite dimensional hereditary algebras of finite type as special examples. These triangulated categories have Auslander–Reiten triangles and were called of finite type in [23]. We rename them now locally finite triangulated categories in the sense of Gabriel.[1]

A triangulated category $\mathcal{A}$ is called locally finite if one of the following two equivalent conditions holds:

(1) $\sum_{X\in \operatorname{ind}\mathcal{A}} \dim_k \operatorname{Hom}_{\mathcal{A}}(X,Y) < \infty$ for any indecomposable object Y in $\mathcal{A}$;
(2) $\sum_{X\in \operatorname{ind}\mathcal{A}} \dim_k \operatorname{Hom}_{\mathcal{A}}(Y,X) < \infty$ for any indecomposable object Y in $\mathcal{A}$ (compare with [4] and [23]).

The main aim of this paper is to provide a case of Gabriel's theorem in triangulated categories, namely, the Auslander–Reiten quiver of any locally finite triangulated category is shaped by Dynkin diagrams. Toward this aim, we should distinguish two classes of triangulated categories those with loops and those without loops in their Auslander–Reiten quivers; and we give a characterization of triangulated categories with loops in their Auslander–Reiten quivers. After that, by using triangulated orbit categories [14] and stable categories, we construct locally finite triangulated categories for most possible cases.

The contents of the paper are as follows: In Section 1, some notions which will be needed in the paper are recalled, locally finite triangulated categories are defined. Some properties of locally finite triangulated categories are given. In Section 2, a general result about the Auslander–Reiten quivers of triangulated categories with loops is proved,

[1] This was suggested to us by B.M. Deng.

that is, the Auslander–Reiten quiver of a connected triangulated category with loops is of the form $\hat{L}_n$, where $\hat{L}_n$ is the following quiver:

$$\begin{tikzcd} \underset{n}{\circ} \arrow[r, shift left] & \underset{n-1}{\circ} \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] \cdots\cdots & \underset{2}{\circ} \arrow[r, shift left] & \underset{1}{\circ} \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

By using this result, Auslander–Reiten quivers of connected locally finite triangulated categories are proved to be $\mathbf{Z}\vec{\Delta}/G$ where Δ is a diagram of Dynkin type and G an automorphism group of $\mathbf{Z}\vec{\Delta}$. In Section 3, we apply covering theory to triangulated categories over an algebraically closed field. The automorphism groups of $D^b(k\vec{\Delta})$, for Dynkin diagrams Δ, are determined, and for most automorphism groups G, we construct triangulated categories whose Auslander–Reiten quivers are $\mathbf{Z}\vec{\Delta}/G$.

1. Preliminaries

1.1. Triangulated categories over a field

We fix some notation and recall some definitions which will be used throughout the paper. Let k be a field. Any category $\mathcal{A}$ in the paper is assumed to be a k-bilinear Hom-finite additive category with Krull–Schmidt property, i.e., any object can be decomposed into a direct sum of indecomposable objects, and such decomposition is unique up to isomorphisms [2,20]. For any category $\mathcal{A}$, we will denote by $\operatorname{ind}\mathcal{A}$ the subcategory of isomorphism classes of indecomposable objects in $\mathcal{A}$; depending on the context, we shall use the same notation to denote the set of isomorphism classes of indecomposable objects in $\mathcal{A}$. The composition of two maps $f: M \to N$ and $g: N \to L$ in $\mathcal{A}$ is denoted by fg. $\mathcal{A}$ is called connected provided for any pair of objects X, Y in $\mathcal{A}$, there are finitely many objects $X_1 = X, X_2, \ldots, X_n = Y$ such that $\operatorname{Hom}_{\mathcal{A}}(X_i, X_{i+1}) \neq 0$ or $\operatorname{Hom}_{\mathcal{A}}(X_{i+1}, X_i) \neq 0$ for each i. A category $\mathcal{A}$ is called a k-linear triangulated category if $\mathcal{A}$ is a k-linear Krull–Schmidt category and a triangulated category. We refer to [9] and [15] for the definition of triangulated categories and to [9] for Auslander–Reiten theory for triangulated categories. The main examples for k-linear triangulated categories are the derived categories of finite dimensional algebras; the stable categories of self-injective finite dimensional algebras.

1.2. Auslander–Reiten triangles

The notation of Auslander–Reiten triangles in a triangulated category were introduced by Happel [8,9].

Definition 1.2.1. Let $\mathcal{A}$ be a triangulated category. A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\mathcal{A}$ is called an Auslander–Reiten triangle if the following conditions are satisfied:

(AR1) X and Z are indecomposable.
(AR2) $w \neq 0$.
(AR3) If $f: W \to Z$ is not a retraction (i.e., there is no $g: Z \to W$ such that $gf = 1_Z$), then there exists $f': W \to Y$ such that $f'v = f$.

Remark 1.2.2. The maps u and v in an Auslander–Reiten triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ are irreducible maps. The object X is denoted by τZ, Z is denoted by $\tau^{-1}X$ and they are uniquely determined each other (up to isomorphism), where τ is the Auslander–Reiten translation [8,9].

One says that a triangulated category $\mathcal{A}$ has Auslander–Reiten triangles if for any indecomposable object $Z \in \operatorname{Obj}\mathcal{A}$, there exist an AR-triangle ending at Z: $X \to Y \to Z \to TX$, and an AR-triangle starting at Z: $Z \to M \to N \to TZ$. In this case, $\mathcal{A}$ is also called a triangulated category with Auslander–Reiten triangles.

1.3. Serre functors

Now we recall the notation of Serre functor from [18]. Let $\mathcal{A}$ be a k-linear triangulated category. A right Serre functor is an additive functor $H:\mathcal{A} \to \mathcal{A}$ together with isomorphisms

$$\eta_{A,B}:\operatorname{Hom}(A,B) \to \operatorname{Hom}(B,HA)^*$$

for any $A, B \in \mathcal{A}$ which is natural in A and B, where $(-)^* = \operatorname{Hom}_k(-.k)$. A left Serre functor can be defined dually. We call a triangulated category has Serre functor if it has right and left Serre functor. Any right Serre functor is fully faithful, and Serre functor is an equivalence of triangulated categories.

The precise relation between the existences of Auslander–Reiten triangles and of Serre functor was given in [18]: For a k-linear triangulated category $\mathcal{A}$, $\mathcal{A}$ has a Serre functor if and only if it has Auslander–Reiten triangles. Moreover, the action of the Serre functor on objects coincides with τT.

1.4. Locally finite triangulated categories

Now we define the notation of locally finite triangulated categories (compare [23]).

Definition 1.4.1. A k-linear triangulated category $\mathcal{A}$ is called locally finite provided $\operatorname{SuppHom}(X,-)$ contains only finitely many indecomposable objects (denoted by $|\operatorname{SuppHom}(X,-)| < \infty$ for simplicity) for any object X in $\operatorname{ind}\mathcal{A}$, where $\operatorname{SuppHom}(X,-)$ denotes the subcategory generated by objects Y in $\operatorname{ind}\mathcal{A}$ with $\operatorname{Hom}_{\mathcal{A}}(X,Y) \neq 0$.

The derived categories of finite dimensional hereditary algebras of finite type and the stable module categories of finite dimensional self-injective algebras of finite type are examples of locally finite triangulated categories.

The following two lemmas were proved in [23] which are needed later on. We refer the reader to [23] for their proofs.

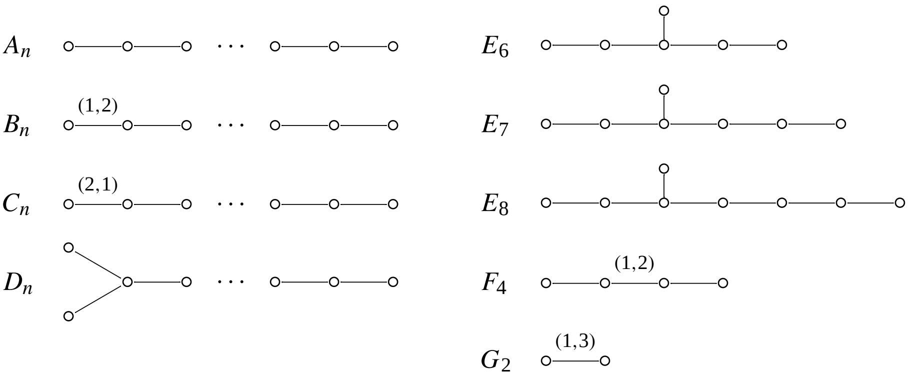
Lemma 1.4.2. *A k-linear triangulated category $\mathcal{A}$ is locally finite if and only if* $|\operatorname{SuppHom}(-,X)| < \infty$ *for any object X in* $\operatorname{ind}\mathcal{A}$.

Lemma 1.4.3. *Let $\mathcal{A}$ be a locally finite k-linear triangulated category. Then $\mathcal{A}$ has Auslander–Reiten triangles.*

1.5. Dynkin diagrams

We recall the Dynkin diagrams and the diagram L_n from [11,12].

The Dynkin diagrams are the following:



and

For any diagram Γ in the list above, there is a subadditive function $f : \Gamma_0 \to \mathbf{N}$ which is not additive. The existence of non-additive subadditive function characterizes these diagrams [11,12].

2. Gabriel's theorem for triangulated categories

2.1. Auslander–Reiten quivers

The Auslander–Reiten quivers of derived categories of some finite dimensional algebras are displayed in [8,9]. In this section, we will give a description of Auslander–Reiten quivers of locally finite triangulated categories.

We recall the notation of Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ of a triangulated category from [9]. Let $\mathcal{A}$ be a triangulated category with Auslander–Reiten triangles. The Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ is a valued stable translation quiver, its vertices are the isomorphism classes $[X]$ of indecomposable objects in $\mathcal{A}$; for two vertices $[X]$ and $[Y]$, there is an arrow with valuation (a, a') as follows:

$$X \xrightarrow{(a,a')} Y$$

provided that there are AR-triangles in $\mathcal{A}$

$$\begin{aligned} &Z \to X^{a} \oplus M \to Y \to TZ, \\ &X \to Y^{a'} \oplus N \to Z \to TX, \end{aligned}$$

with X not being isomorphic to a direct summand of M and Y not being isomorphic to a direct summand of N respectively.

2.2. Triangulated categories with loops in their Auslander–Reiten quivers

An arrow starting and ending at the same vertex in a valued translation quiver is called a loop. It is well known that the Auslander–Reiten quiver of an Artin algebra contains no loop [2,3,20]. But for representations of orders and isolated singularity, loops appear in their stable Auslander–Reiten quivers [5,22]. Therefore there are stable categories of Bass orders with loops in their Auslander–Reiten quivers [13,22]. We will prove a general result on the shape of Auslander–Reiten quivers of k-linear triangulated categories with loops. The result indicates that if the Auslander–Reiten quiver of a triangulated category contains loops, then the triangulated category is very special (as follows).

Theorem 2.2.1. *Let $\mathcal{A}$ be a connected k-linear triangulated category with Auslander–Reiten triangles. Suppose its Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ contains loops. Then $\Gamma_{\mathcal{A}}$ is of the form*

$$\begin{tikzcd} \underset{n}{\circ} \arrow[r, shift left] & \underset{n-1}{\circ} \arrow[l, shift left] \arrow[r, shift left] & \circ \arrow[l, shift left] & \cdots\cdots & \underset{2}{\circ} \arrow[r, shift left] & \underset{1}{\circ} \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

where n is a positive integer. Such quiver is denoted by $\hat{L}_n$.

To prove the theorem, we need to prove a technical lemma. First of all, we set $l_P(-) = \dim_k \mathrm{Hom}(P, -)$ which is a function on $\operatorname{ind}\mathcal{A}$.

Lemma 2.2.2. *Suppose the Auslander–Reiten triangle starting at N is: $N \to M \oplus Y \to N \to TN$ with M indecomposable and $\tau M \cong M$. If $l_P(M) \geqslant l_P(N)$ for any indecomposable object P, then Y is indecomposable or is zero, $\tau Y \cong Y$ and $l_P(N) \geqslant l_P(Y)$.*

Proof. Let $Y = \bigoplus_1^s Y_i$ be the decomposition of Y into indecomposable objects. Then the conditions that $\tau M \cong M$ and $\tau N \cong N$ imply that τ induces a permutation on $\{Y_1, \dots, Y_s\}$. Firstly, we claim that $\tau Y_i \cong Y_i$, for all i. Otherwise we assume that $\tau Y_2 \cong Y_1$ and the Auslander–Reiten triangle starting at Y_1 is $Y_1 \to N \oplus X \to Y_2 \to TY_1$. By applying $\mathrm{Hom}(N, -)$ to this triangle and the Auslander–Reiten triangle starting N respectively, we have the following inequalities:

$$l_N(Y_1) + l_N(Y_2) \geqslant l_N(N) + l_N(X), \tag{1}$$

$$2l_N(N) > l_N(M) + l_N(Y). \tag{2}$$

It follows from (2) and the condition $l_N(M) \geqslant l_N(N)$ that $l_N(N) > l_N(Y)$ which is a contradiction to (1). This shows that $\tau Y_i \cong Y_i$ for all i. Next we prove that the number of indecomposable summands of Y is less than 2. Otherwise we take two indecomposable summands Y_1, Y_2 and assume that the Auslander–Reiten triangle starting at Y_i is of the form: $Y_i \to N \oplus P_i \to Y_i \xrightarrow{g} TY_i$, $i = 1, 2$. By applying $\mathrm{Hom}(N, -)$ to these two triangles respectively, we have that

$$2l_N(Y_i) \geqslant l_N(N) + l_N(P_i) \quad \forall i.$$

Then $2(l_N(Y_1) + l_N(Y_2)) \geqslant 2l_N(N) + l_N(P_1) + l_N(P_2)$. Combining with inequality (2), we have that $2(l_N(Y_1) + l_N(Y_2)) \geqslant 2l_N(N) + l_N(P_1) + l_N(P_2) > 2l_N(Y)$, which is a contradiction. This shows that the number of indecomposables of Y is less than 2. Finally, we prove the last statement in the lemma. By applying $\mathrm{Hom}(P, -)$ to the Auslander–Reiten triangle starting at N, we get that $2l_P(N) \geqslant l_P(M) + l_P(Y)$. It follows from the given condition $l_P(M) \geqslant l_P(N)$ that $l_P(N) \geqslant l_P(Y)$. This finishes the proof. □

Proof of Theorem 2.2.1. Suppose there is a loop at X in $\Gamma_{\mathcal{A}}$. Then we can assume that the Auslander–Reiten triangle starting at X is: $X \xrightarrow{u} X \oplus M \xrightarrow{v} \tau^{-1}X \xrightarrow{w} TX$. We prove that $X \cong \tau^{-1}X$. Otherwise $M \cong \tau^{-1}X \oplus M_1$. Now by applying $\mathrm{Hom}(\tau^{-1}X, -)$ to the Auslander–Reiten triangle above, we have that

$$l_{\tau^{-1}X}(X) + l_{\tau^{-1}X}\big(\tau^{-1}X\big) > l_{\tau^{-1}X}(X) + l_{\tau^{-1}X}\big(\tau^{-1}X\big) + l_{\tau^{-1}X}(M_1).$$

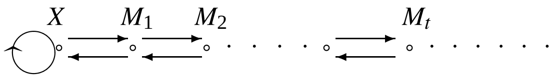
It implies that $l_{\tau^{-1}X}(M_1) < 0$, which is a contradiction. Then we have that $X \cong \tau^{-1}X$ and the Auslander–Reiten triangle starting at X is:

$$X \xrightarrow{u} X \oplus M \xrightarrow{v} X \xrightarrow{w} TX.$$

If $M = 0$, i.e., the Auslander–Reiten triangle starting at X is $X \xrightarrow{u} X \xrightarrow{v} X \xrightarrow{w} TX$. It follows that the space $\mathrm{Irr}(X, X)$ of irreducible map from X to X has dimension 1 as left or right D-space respectively, where $D = \mathrm{End}\,X/\mathrm{rad}(\mathrm{End}\,X)$. Then $u = tv$, for some $t \in D$, and $ut = t'u$ for some $t' \in D$. For any other indecomposable object $Y \in \mathrm{ind}\,\mathcal{A}$ which is not isomorphic to X, if there exists a non-zero morphism $f \in \mathrm{Hom}(X, Y)$, then there exists a morphism $g : X \to Y$ such that $f = ug$, where g is not a section. Also for g, there is a morphism $g_1 \in \mathrm{Hom}(X, Y)$ such that $g = ug_1$. It follows that $f = ug = u(ug_1) = ((utv)g_1) = (t'uv)g_1 = 0$. It is a contradiction. Therefore $\mathrm{Hom}(X, Y) = 0$. Similarly we have that $\mathrm{Hom}(Y, X) = 0$. Then the connectedness of $\mathcal{A}$ makes $\mathrm{ind}\,\mathcal{A} = \{X\}$. Then $\Gamma_{\mathcal{A}}$ is $\hat{L}_1$.

Next we assume $M \neq 0$. Since the Auslander–Reiten triangle starting at X is $X \xrightarrow{u} X \oplus M \xrightarrow{v} X \xrightarrow{w} TX$, it is easy to see $l_P(X) \geqslant l_P(X)$. It follows from Lemma 2.2.2. that M is indecomposable, $\tau M \cong M$ and $l_P(X) \geqslant l_P(M)$.

In the following, we denote M by M_1. Now we consider the Auslander–Reiten triangle starting at M_1: $M_1 \to X \oplus N \to M_1 \to TM_1$. By Lemma 2.2.2. again, we have that the number of indecomposable summands of N is less than 2 and $\tau N \cong N$. If $N = 0$, then the connected component containing X is $\hat{L}_2$. If $N \neq 0$, we continue this process and get a $\hat{L}_n$ (n possibly ∞) in the Auslander–Reiten component Γ:



We now prove n must be finite. From the Auslander–Reiten triangle starting at X: $X \to X \oplus M_1 \to X \to TX$, we have $l_X(X) > l_X(M_1)$. Let $M_1 \to X \oplus M_2 \to M_1 \to TM_1$ be the Auslander–Reiten triangle. By applying $\mathrm{Hom}(X, -)$ on it, we have the inequality: $2l_X(M_1) \geqslant l_X(M_2) + l_X(X)$. It follows that $l_X(M_1) > l_X(M_2)$. An easy induction on t shows that $l_X(M_t) > l_X(M_{t+1})$. Therefore we have the strictly inequality:

$$l_X(X) > l_X(M_1) > \cdots > l_X(M_t) > \cdots.$$

Therefore n must be finite since $l_X(X)$ is a positive integer. For any indecomposable object N, by the connectedness of $\mathcal{A}$, we have that $\mathrm{Hom}(N, X) \neq 0$ or $\mathrm{Hom}(X, N) \neq 0$ for some $X \in \hat{L}_n$. We assume that $N \notin \hat{L}_n$ and $f : X \to N$ is not 0. Then there is a chain of infinitely many irreducible maps between indecomposable objects in $\hat{L}_n$ such that f factors through any maps in the chain. It follows that $f = 0$, contracting to the assumption above. This shows that $N \in \hat{L}_n$. Then $\hat{L}_n$ is the Auslander–Reiten quiver. The proof is finished. $\square$

Corollary 2.2.3. *Assume that $\mathcal{A}$ is a k-linear connected triangulated category with Auslander–Reiten triangles, and $\Gamma_{\mathcal{A}}$ contains loops. Then $TX \cong X$ for any $X \in \mathcal{A}$.*

Proof. By Theorem 2.2.1, the Auslander–Reiten quiver $\Gamma_{\mathcal{A}} = \hat{L}_n$. It is necessary to prove that $TY \cong Y$ for any $Y \in L_n$. If $Y = X$, and assume that

$$X \to X \oplus M_1 \to X \to TX \tag{1}$$

is the Auslander–Reiten triangle starting at X, then

$$TX \to TX \oplus TM_1 \to TX \to T^2X \tag{2}$$

is the Auslander–Reiten triangle starting TX, and with a loop at it. Then $TX \cong X$ since the loop in the Auslander–Reiten component does exist at only X in $\hat{L}_n$. The Auslander–Reiten triangles (1) and (2) at X shows that $TM_1 \cong M_1$. An induction on t proves that $TM_t \cong M_t$ for $M_t \in \hat{L}_n$. This shows that for any indecomposable object $Y \in \hat{L}_n$, $TY \cong Y$. $\square$

2.3. Gabriel's theorem for triangulated categories

In this subsection, the k-linear triangulated category $\mathcal{A}$ is assumed to be locally finite. We recall from [11] the definition of subadditive function on a valued translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau, a)$. A function f from a translation quiver Γ to $\mathbf{N}$ is called subadditive if $f(x) + f(\tau x) \geqslant \sum_{y \to x} f(y)a'_{yx}$ for all non-projective vertices x. A subadditive function is additive if $f(x) + f(\tau x) = \sum_{y \to x} f(y)a'_{yx}$ for all non-projective vertices x.

Remark 2.3.1. For any generalized Dynkin diagram Δ, there is a non-additive subadditive function on the translation quiver $\mathbf{Z}\vec{\Delta}$ (see [11]).

Let $M^* = \bigoplus_{X \in \operatorname{ind} \mathcal{A}} X$. We define a function f from $\operatorname{ind} \mathcal{A}$ to $\mathbf{N}$ as follows:

$$f(X) = \dim_k \operatorname{Hom}_{\mathcal{A}}(M^*, X),$$

where $X \in \operatorname{ind} \mathcal{A}$.

Remark 2.3.2. From the definition of locally finiteness of triangulated categories, one knows that the function f is well defined.

The following lemmas are proved in [23], we give a simpler proof for the completeness.

Lemma 2.3.3. *f is a non-additive subadditive function on $\Gamma_{\mathcal{A}}$.*

Proof. It is necessary to prove that for any Auslander–Reiten triangle $\tau X \to Y \to X \to T\tau X$, $f(\tau X) + f(X) > f(Y)$. For any indecomposable object N, by applying $\operatorname{Hom}(M, -)$ to the Auslander–Reiten triangle above, we have that $l_N(\tau X) + l_N(X) \geqslant l_N(Y)$. The inequality is strict if $N \cong X$. Then by summing all inequalities where N runs all indecomposable objects, we have the required inequality: $f(\tau X) + f(X) > f(Y)$. The proof is finished. $\square$

Lemma 2.3.4. *$f(\tau X) = f(X)$ for any $X \in \Gamma_{\mathcal{A}}$.*

Proof. By Theorem I.2.4. in [18] and Proposition 1.4.3, there is a Serre functor H and $H = \tau T$ on objects of $\mathcal{A}$. Then $\tau X = HT^{-1}(X)$ for any $X \in \operatorname{ind} \mathcal{A}$. Thus $\operatorname{Hom}(\tau X, \tau Y) = \operatorname{Hom}(HT^{-1}(X), HT^{-1}(Y)) \cong \operatorname{Hom}(T^{-1}(X), T^{-1}(Y)) \cong \operatorname{Hom}(X, Y)$. Then $f(\tau X) = \dim_k \operatorname{Hom}(M^*, \tau X) = \dim_k \operatorname{Hom}(\tau M^*, \tau X) = \dim_k \operatorname{Hom}(M^*, X) = f(X)$. The proof is finished. $\square$

Now we come to prove the main theorem in this section, which was proved in [23] under the assumption that the Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ contains no loops.

Theorem 2.3.5. *Let $\mathcal{A}$ be a locally finite triangulated category. Then its Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ is isomorphic to $\mathbf{Z}\overrightarrow{\Delta}/G$, where Δ is a diagram of Dynkin type and G an automorphism group of $\mathbf{Z}\overrightarrow{\Delta}$.*

Proof. Let $\mathcal{A}$ be a locally finite triangulated category. Firstly we assume that $\Gamma_{\mathcal{A}}$ contains no loops. By Lemmas 2.3.3, 2.3.4, we have a subadditive function f on $\Gamma_{\mathcal{A}}$, which is not additive and $f(\tau X) = f(X)$, for any indecomposable object X. It follows from [11] that $\Gamma_{\mathcal{A}} \cong \mathbf{Z}\overrightarrow{\Delta}/G$ and Δ is a Dynkin diagram or A_∞. Since $\mathcal{A}$ is locally finite, A_∞ is not allowed. Next we consider the case that $\Gamma_{\mathcal{A}}$ contains loops. It follows from Theorem 2.1.1. that $\Gamma_{\mathcal{A}}$ is of the form $\hat{L}_n$. In the following, we prove that $\hat{L}_n \cong \mathbf{Z}\overrightarrow{A_{2n}}/\langle T\tau^n \rangle$, where $\overrightarrow{A_{2n}}$ denotes the linear orientation of A_{2n}. Let (i, a_j) denote the vertices of $\mathbf{Z}\overrightarrow{A_{2n}}$, where $i \in \mathbf{Z}$ and a_j are vertices of $\overrightarrow{A_{2n}}$. For any arrow $a_j \to a_{j+1}$ in $\overrightarrow{A_{2n}}$, there are an arrow $(i, a_j) \to (i, a_{j+1})$ and an arrow $(i-1, a_{j-1}) \to (i, a_j)$. It is easy to see $\tau(i, a_j) = (i-1, a_j)$ and $T(i, a_j) = (i+j, a_{2n-j+1})$. Since T and τ are automorphisms of triangulated categories, they are isomorphisms of translation quiver. It is easy to see that $T^{-2} = \tau^{2n+1}$ on $\mathbf{Z}\overrightarrow{A_{2n}}$. So we have $(T\tau^n)^{-2} = \tau$ and then $\tau \in \langle T\tau^n\rangle$. Therefore the orbit quiver $\mathbf{Z}\overrightarrow{A_{2n}}/\langle T\tau^n\rangle$ of $\mathbf{Z}\overrightarrow{A_{2n}}$ contains only vertices $\overline{(0, a_i)}$ where $i = 1, \ldots, n$, and it is of the form $\hat{L}_n$. The proof is finished. $\square$

Remark 2.3.6. There is a derived category of $k\vec{\Delta}$ whose Auslander–Reiten quiver is $\mathbf{Z}\vec{\Delta}$ for any Dynkin diagram Δ. For $\hat{L}_n$, in the next section, we will provide an example of k-linear triangulated categories with $\hat{L}_n$ as its Auslander–Reiten quiver.

3. Covering theory for triangulated categories

Throughout this section, we assume that k is an algebraically closed field. As we have seen in Section 2, Auslander–Reiten quivers of locally finite triangulated categories are of the form $\mathbf{Z}\vec{\Delta}/G$, where G is automorphism group of $\mathbf{Z}\vec{\Delta}$ for some Dynkin diagram Δ. An interesting problem is how to construct triangulated categories whose Auslander–Reiten quivers are of the form $\mathbf{Z}\vec{\Delta}/G$. Thanks to the result in [14], we will construct triangulated categories for most G, in particular, a k-linear triangulated category with Auslander–Reiten quiver of the form $\hat{L}_n$. The construction will involve covering technique in representation theory of algebras (see [4] and [7]).

3.1. Covering for triangulated categories

Let $\mathcal{A}$, $\mathcal{A}_0$ be k-linear triangulated categories and $F : \mathcal{A} \to \mathcal{A}_0$ the triangle functor with the property: $F(X) \in \operatorname{ind}\mathcal{A}_0$ for any $X \in \operatorname{ind}\mathcal{A}$. Then functor F induces a functor from $\operatorname{ind}\mathcal{A}$ to $\operatorname{ind}\mathcal{A}_0$ which is also denoted by F. A triangle functor $F : \mathcal{A} \to \mathcal{A}_0$ is called a covering functor of triangulated categories provided the induced functor $F : \operatorname{ind}\mathcal{A} \to \operatorname{ind}\mathcal{A}_0$ is a covering functor in the sense of Bongartz–Gabriel [4,7], i.e., the maps

$$\bigoplus_{F(z)=b} \operatorname{Hom}_{\mathcal{A}}(x, z) \to \operatorname{Hom}_{\mathcal{A}_0}\bigl(F(x), b\bigr) \quad \text{and}$$

$$\bigoplus_{F(t)=a} \operatorname{Hom}_{\mathcal{A}}(t, y) \to \operatorname{Hom}_{\mathcal{A}_0}\bigl(a, F(y)\bigr)$$

which are induced by F, are bijective for any two objects a and b of $\operatorname{ind} F(\mathcal{A}) \subseteq \operatorname{ind}\mathcal{A}_0$. The triangle functor $F : \mathcal{A} \to \mathcal{A}_0$ is called a Galois covering functor with a Galois group G provided the induced functor $F : \operatorname{ind}\mathcal{A} \to \operatorname{ind}\mathcal{A}_0$ is Galois with a Galois group G [4,7].

Covering functor between homotopy categories over algebras was considered in [1]. It was proved that for a Galois covering $F : A \to A/G$ with a Galois group G in the sense of [4,7], there is a Galois covering functor $H^b(\operatorname{pro} A) \to H^b(\operatorname{pro} A/G)$ with Galois group G.

Let A be a finite dimensional k-algebra and n a positive even integer. It was proved in [16] that there are a triangulated category denoted by $D^b(A)/\langle T^n\rangle$ and a functor

$F : D^b(A) \to D^b(A)/\langle T^n \rangle$ which is a Galois covering functor with Galois group $\langle T^n \rangle$. The results there were used to construct Lie algebras in [16,17].

Lemma 3.1.1. *Let $F : \mathcal{A} \to \mathcal{A}_0$ be an additive functor of triangulated categories such that the induced functor $F : \operatorname{ind} \mathcal{A} \to \operatorname{ind} \mathcal{A}_0$ is a covering functor in the sense of Bongartz–Gabriel. If F is dense, then $\mathcal{A}$ is locally finite if and only if so is $\mathcal{A}_0$.*

Proof. Assume $\mathcal{A}$ is locally finite. Let a be an indecomposable object in $\mathcal{A}_0$ and x one of indecomposable objects in $F^{-1}(a)$. Then for any $b \in \operatorname{ind} \mathcal{A}_0$, we have a bijection

$$\bigoplus_{F(z)=b} \operatorname{Hom}_{\mathcal{A}}(x, z) \to \operatorname{Hom}_{\mathcal{A}_0}(a, b).$$

It follows from the finiteness of $\dim_k \operatorname{Hom}_{\mathcal{A}_0}(a, b)$ that $\operatorname{ind} F^{-1}(b)$ contains only finitely many indecomposable objects with non-zero morphism starting from x. Therefore $\sum_{b \in \operatorname{ind} \mathcal{A}_0} \dim_k \operatorname{Hom}_{\mathcal{A}_0}(a, b) = \sum_{b \in \operatorname{ind} \mathcal{A}_0} \sum_{F(y)=b} \dim_k \operatorname{Hom}_{\mathcal{A}}(x, y) < \infty$. Hence $\mathcal{A}_0$ is locally finite. Now we prove the converse. Let x be an indecomposable object of $\mathcal{A}$, and $a = F(x)$. The bijection

$$\bigoplus_{F(z)=b} \operatorname{Hom}_{\mathcal{A}}(x, z) \to \operatorname{Hom}_{\mathcal{A}_0}(a, b)$$

implies that there are finitely many $z \in \operatorname{ind} \mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(x, z) \neq 0$. It follows from the finiteness of indecomposable objects in $\mathcal{A}_0$ with non-zero morphisms from a that there are only finitely many indecomposable objects in $\mathcal{A}$ with non-zero morphisms from x. Then $\mathcal{A}$ is locally finite. The proof is finished. $\square$

Remark 3.1.2. From the lemma above, we know if $F : \mathcal{A} \to \mathcal{A}_0$ is a covering functor of triangulated categories which is dense, then $\mathcal{A}$ is locally finite if and only if so is $\mathcal{A}_0$.

Proposition 3.1.3. *Let $\mathcal{A}$ be a locally finite connected k-linear triangulated category. Then $\Gamma_{\mathcal{A}}$ is isomorphic to $\mathbf{Z}\vec{\Delta}/G$, where Δ is a diagram of classical Dynkin type A, D, E.*

Proof. From Theorem 2.3.2 and Lemma 3.1.1, without loss of the generality, we assume $\mathcal{A}$ is a k-linear triangulated category with $\mathbf{Z}\vec{\Delta}$ as its Auslander–Reiten quiver. There are no cycles (paths starting and ending at the same vertex) in $\mathbf{Z}\vec{\Delta}$. It follows that $\operatorname{End}_{\mathcal{A}} X$ is k. For the valuation $(a_{X,Y}, a'_{X,Y})$ on the arrow from X to Y in $\Gamma_{\mathcal{A}}$, we have $a_{X,Y} = \dim_k \operatorname{Irr}(X, Y) = a'_{X,Y}$ [2,9]. From the listing of Dynkin diagrams in Section 1, one has that $a_{X,Y} = a'_{X,Y} = 1$. Then Δ is a Dynkin diagram of A, D, E type. The proof is finished. $\square$

Proposition 3.1.4. *Let $\mathcal{A}$ be a locally finite connected k-linear triangulated category. Then there is an additive functor $F : D^b(k\vec{\Delta}) \to \mathcal{A}$ such that the induced functor $F : \operatorname{ind} D^b(k\vec{\Delta}) \to \operatorname{ind} \mathcal{A}$ is a covering functor in the sense of Bongartz–Gabriel and $FT \cong TF$, where Δ is a diagram of classical Dynkin type A, D, E.*

Proof. From Theorem 2.3.2 and Proposition 3.1.3, we know that when $\Gamma_{\mathcal{A}}$ does not contain loops, there is a covering of translation quivers $\pi : \mathbf{Z}\overrightarrow{\Delta} \to \Gamma_{\mathcal{A}}$, where Δ is a diagram of classical Dynkin type A, D, E [4]; when $\Gamma_{\mathcal{A}}$ contains loops, $\Gamma_{\mathcal{A}} \cong \hat{L}_n$ and then $\Gamma_{\mathcal{A}} \cong \mathbf{Z}\overrightarrow{A_{2n}}/\langle T\tau^n\rangle$, for which there is a covering functor of translation quivers $\pi : \mathbf{Z}\overrightarrow{A_{2n}} \to \Gamma_{\mathcal{A}}$. It follows from (Section 3.1 in [3]) that there is a covering functor $F : k(\mathbf{Z}\overrightarrow{\Delta}) \to \operatorname{ind}\mathcal{A}$, where $k(\mathbf{Z}\overrightarrow{\Delta})$ is the mesh category of $\mathbf{Z}\overrightarrow{\Delta}$. From Proposition 5.6 in [8], we know that $\operatorname{ind} D^b(k\overrightarrow{\Delta})$ is equivalent to $k(\mathbf{Z}\overrightarrow{\Delta})$. Therefore we have the covering functor $F : \operatorname{ind} D^b(k\overrightarrow{\Delta}) \to \operatorname{ind}\mathcal{A}$. By extending the functor F to $D^b(k\overrightarrow{\Delta})$, we have the desired functor from $D^b(k\overrightarrow{\Delta})$ to $\mathcal{A}$ which is also denoted by F. F preserves the Auslander–Reiten triangles since it is a covering of translation quivers if we restrict F to the Auslander–Reiten quiver of $D^b(k\overrightarrow{\Delta})$. Therefore for any indecomposable object X, $FT(X) \cong TF(X)$. Then $FT \cong TF$. The proof is finished. $\square$

Remark 3.1.5. The functor F should be a triangle functor of triangulated categories, but we do not find a proof for it at the present.

Corollary 3.1.6. *Let $\mathcal{A}$ be a k-linear triangulated category having Auslander–Reiten triangles. Then $\mathcal{A}$ is locally finite if and only if the Auslander–Reiten quiver $\Gamma_{\mathcal{A}}$ is isomorphic to $\mathbf{Z}\overrightarrow{\Delta}/G$, where Δ is a diagram of classical Dynkin type.*

Proof. The necessity follows from Theorem 2.3.5. We prove the sufficiency. Using the same proof as for Proposition 3.1.4., we have a covering functor $F : \operatorname{ind} D^b(k\overrightarrow{\Delta}) \to \operatorname{ind}\mathcal{A}$ in the sense of Bongartz–Gabriel which is dense on objects. It is easy to know $D^b(k\overrightarrow{\Delta})$ is of locally finite type for any Dynkin diagram Δ. By Lemma 3.1.1, $\mathcal{A}$ is locally finite. The proof is finished. $\square$

3.2. Covering for Frobenius categories

In the subsection, we study the Galois covering of stable categories of Frobenius categories. The stable category of a Frobenius category $\mathcal{A}$ is denoted by $\operatorname{stab}(\mathcal{A})$. It is a k-linear triangulated category whose translation functor is the suspension functor [9]. Let F be a functor between two Frobenius categories. If F preserves projective objects, then F induces a functor between the corresponding stable categories, which is also denoted by F.

Proposition 3.2.1. *Let $\hat{\mathcal{A}}$ and $\mathcal{A}$ be Frobenius categories, $F : \hat{\mathcal{A}} \to \mathcal{A}$ a Galois covering functor with group G. If F preserves projectives and all projectives in $\mathcal{A}$ are images of projectives in $\tilde{\mathcal{A}}$ under the functor F, then the induced functor $F : \operatorname{stab}(\hat{\mathcal{A}}) \to \operatorname{stab}(\mathcal{A})$ is a Galois covering functor with group G.*

Proof. It is easy to see that $f : a \mapsto b$ factors through an injective object in $\hat{\mathcal{A}}$ if and only if $F(f) : F(a) \mapsto F(b)$ factors through an injective object in $\mathcal{A}$. Then $F : \operatorname{stab}\hat{\mathcal{A}} \to \operatorname{stab}\mathcal{A}$ is a covering functor with Galois group G. It remains to prove that F preserves triangles. Since F is covering functor, F is faithful, and then F preserves injective (surjective) morphisms. Let $X \xrightarrow{f} Y \to Z \to TX$ be a distinguished triangle associated morphism f. Then Z is isomorphic to the push-out of f and the injective cover of $X: X \to I(X)$ (in stab($\tilde{\mathcal{A}}$), we can assume that Z is this push-out), i.e., we have the following push-out diagram:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"'] & I(X) \arrow[r] \arrow[d] & T(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & T(X) \arrow[r] & 0.
\end{tikzcd}$$

By applying F to this diagram, we also get a push-out diagram. This shows that $F(X) \xrightarrow{F(f)} F(Y) \to F(Z) \to TF(X)$ is a distinguished triangle. The proof is finished. $\square$

For a finite dimensional algebra A, we denote by $T(A)$ the trivial extension of A. It is a Z-graded algebra. The category gr-mod($T(A)$) of Z-graded $T(A)$-modules is isomorphic to the module category of the repetitive algebra $\hat{A}$ of A.

Corollary 3.2.2. *The forgetful functor* $F:\mathrm{stab}(\mathrm{mod}\,\hat{A}) \to \mathrm{stab}(\mathrm{mod}\,T(A))$ *is a Galois covering functor with Galois group* $G = \langle T^2\tau \rangle$.

Proof. $T(A)$ and $\hat{\mathcal{A}}$ are self-injective algebras and $F:\mathrm{mod}\,\hat{A} \to \mathrm{mod}\,T(A)$ is a Galois covering with Galois group G. By applying Proposition 3.2.1, we get the desired conclusion. The proof is finished. $\square$

3.3. Automorphisms of Dynkin diagrams

For each classical Dynkin diagram Δ of type A, D, E, we fix an orientation on it as follows:

$$A_n \quad \begin{tikzcd} \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ & \cdots & \circ \arrow[r] & \circ \end{tikzcd}$$

$$D_n \quad \begin{tikzcd} & 2\circ \arrow[d] & & & & & \\ 1\circ \arrow[r] & \circ \arrow[r] & \circ & \cdots & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

$$E_6 \quad \begin{tikzcd} & & \circ \arrow[d] & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

$$E_7 \quad \begin{tikzcd} & & \circ \arrow[d] & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

$$E_8 \quad \begin{tikzcd} & & \circ \arrow[d] & & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

The proof for the following lemma is obvious.

Lemma 3.3.1. *Let Δ be one of classical Dynkin diagrams and $\vec{\Delta}$ the corresponding quiver with the fixed orientation as above. We fix an embedding of $\vec{\Delta}$ in $\mathbf{Z}\vec{\Delta}$. If σ is an automorphism of translation quiver $\mathbf{Z}\vec{\Delta}$, then $\sigma(\vec{\Delta})$ is isomorphic to $\vec{\Delta}$ as subquivers of $\mathbf{Z}\vec{\Delta}$.*

Proposition 3.3.2. *Let Δ be one of classical Dynkin diagrams. Then the automorphism groups* $\mathrm{Aut}(\mathbf{Z}\vec{\Delta})$ *can be described as follows*:

(1) *When Δ is of type A_n,* $\mathrm{Aut}(\mathbf{Z}\overrightarrow{A_n}) \cong \langle T, \tau\rangle/\langle T^2\tau^{n+1}\rangle$;
(2) *When Δ is of type D_n,* $\mathrm{Aut}(\mathbf{Z}\overrightarrow{D_n}) \cong \langle \tau, \sigma\rangle$; *where σ is the automorphism induced by the automorphism of D_n which fixes all vertices in D_n except exchanging vertices* 1 *and* 2;
(3) *When Δ is of type E_6,* $\mathrm{Aut}(\mathbf{Z}\overrightarrow{E_6}) \cong \langle T, \tau\rangle/\langle T^2\tau^{12}\rangle$;
(4) *When Δ is of type E_n, for $n = 7, 8$,* $\mathrm{Aut}(\mathbf{Z}\overrightarrow{E_n}) \cong \langle \tau\rangle$.

Proof. Obviously T and τ are automorphisms of $\mathbf{Z}\vec{\Delta}$ for any Dynkin diagram Δ and σ is also an automorphism of $\mathbf{Z}\overrightarrow{D_n}$. Let f be an automorphism of $\mathbf{Z}\vec{\Delta}$. It follows from Lemma 3.3.1 and a routine exercise that f is a power of T or τ. For $\Delta = A_n$, we have that $T^2 = \tau^{-(n+1)}$; for $\Delta = D_n$, we have that $T = \tau^{-n}$; for $\Delta = E_6$, we have that $T^2 = \tau^{-12}$; for $\Delta = E_7$, we have that $T = \tau^{-9}$ and for $\Delta = E_8$, we have that $T = \tau^{-15}$. This finishes the proof. $\square$

Let $\mathcal{A}$ be an arbitrary triangulated category. A sextuple $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is called an anti-triangle if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$ is a triangle in $\mathcal{A}$. The category $\mathcal{A}$, endowed with the family of anti-triangles, is a triangulated category. We denote it by $\mathcal{A}^a$.

Proposition 3.3.3. *Let $\mathcal{A}$ be a triangulated category. Then the translation functor T induces a triangle functor* (*in fact, an isomorphism*) (T, id_{T^2}) *from $\mathcal{A}$ to $\mathcal{A}^a$ and a triangle functor* (*in fact, an isomorphism*) $(T, -\mathrm{id}_{T^2})$ *from $\mathcal{A}$ to itself. Moreover, A has Auslander–Reiten triangles if and only if so has $\mathcal{A}^a$ and their Auslander–Reiten quivers are the same, T induces an automorphism of translation quiver $\Gamma_{\mathcal{A}}$, which is also denoted by T.*

Proof. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ be a triangle in $\mathcal{A}$. Then the image of this triangle under T is a triangle in $\mathcal{A}^a$, since it is isomorphic to the anti-triangle $TX \xrightarrow{-Tu} TY \xrightarrow{-Tv} TZ \xrightarrow{Tw} T^2X$. Then T is triangle functor from $\mathcal{A}$ to $\mathcal{A}^a$. If A has Auslander–Reiten triangles, then $\mathcal{A}^a$ has so. The Auslander–Reiten quivers of them are the same. Therefore T induces an automorphism of this quiver $\Gamma_{\mathcal{A}}$. The proof is finished. $\square$

By Propositions 3.3.2 and 3.3.3, we have all automorphisms of $D^b(k\vec{\Delta})$ for Dynkin diagrams Δ.

Corollary 3.3.4. *Let Δ be one of classical Dynkin diagrams. Then the shift functor T and Auslander–Reiten translation τ are generators of automorphism groups of $D^b(k\vec{\Delta})$ which satisfy the relations in Proposition* 3.3.2.

Now we recall from [14] the construction of triangulated orbit categories. Let $F : D^b(k\vec{\Delta}) \to D^b(k\vec{\Delta})$ be a triangle functor, which is assumed to satisfy the following properties; see [14].

(g1) For each U in $\operatorname{ind} k\vec{\Delta}$, only a finite number of objects $F^n U$, where $n \in \mathbf{Z}$, lie in $\operatorname{ind} k\vec{\Delta}$.
(g2) There is some $N \in \mathbf{N}$ such that $\{U[n] \mid U \in \operatorname{ind} k\Delta, n \in [-N, N]\}$ contains a system of representatives of the orbits of F on $\operatorname{ind} D^b(k\vec{\Delta})$.

We denote by $D^b(k\vec{\Delta})/F$ the corresponding factor category. The objects are by definition the F-orbits of objects in $D^b(k\vec{\Delta})$, and the morphisms are given by

$$\operatorname{Hom}_{D^b(k\vec{\Delta})/F}\big(\tilde{X}, \tilde{Y}\big) = \bigoplus_{n \in \mathbf{Z}} \operatorname{Hom}_{D^b(k\vec{\Delta})}\big(X, F^n Y\big).$$

Here X and Y are objects in $D^b(k\vec{\Delta})$, and $\tilde{X}$ and $\tilde{Y}$ are the corresponding objects in $D^b(k\vec{\Delta})/F$. It is proved by Keller [14] that $D^b(k\vec{\Delta})/F$ is a triangulated category and that the natural functor $\pi : D^b(k\vec{\Delta}) \to D^b(k\vec{\Delta})/F$ is a triangle functor. The shift in $D^b(k\vec{\Delta})/F$ is induced by the shift in $D^b(k\vec{\Delta})$, and is also denoted by T.

Theorem 3.3.5. *Let Δ be a Dynkin diagram, m, n integers. Let G be one of the automorphism groups of $\mathbf{Z}\vec{\Delta}$ of the forms (τ^m), (T^m) for any m, or of the form $(T\tau^m)$, where $m \neq (n+1)/2$ if Δ is of type A_n; or $m \neq 6$ if Δ is of type E_6. Then there are a locally finite triangulated category $D^b(k\vec{\Delta})/G$, and a covering functor $\pi : D^b(k\vec{\Delta}) \to D^b(k\vec{\Delta})/G$ with Galois group G.*

Proof. Let F be one of the automorphisms τ^m, T^m, or $T\tau^m$ satisfying the conditions in the theorem. Then F is a triangle functor satisfying conditions (g1) and (g2) above, and the orbit category $D^b(k\vec{\Delta})/F$ is a Krull–Schmidt triangulated category and the projection functor $\pi : D^b(k\vec{\Delta}) \to D^b(k\vec{\Delta})/F$ is a triangle functor [14]. Given an indecomposable object X in $D^b(k\vec{\Delta})$, its image $F(X)$ is indecomposable in $D^b(k\vec{\Delta})/G$. Therefore π induces a functor between their indecomposables $\pi : \operatorname{ind}(D^b(k\vec{\Delta}) \to \operatorname{ind} D^b(k\vec{\Delta})/F$, it is a covering functor in the sense of Bongartz–Gabriel [4]. This finishes the proof. □

Corollary 3.3.6. *The orbit category $D^b(k\overrightarrow{A_{2n}})/\tau^n T$ is a triangulated category with Auslander–Reiten quiver $\hat{L}_n$.*

Proof. The group $\langle \tau^n T \rangle$ contains τ since $\tau^{2n+1} = T^{-2}$ as automorphisms of $D^b(k\overrightarrow{A_{2n}})$. Then the Auslander–Reiten quiver of the orbit triangulated category $D^b(k\overrightarrow{A_{2n}})/\tau^n T$ is $\mathbf{Z}\overrightarrow{A_{2n}}/\tau^n T$. It is isomorphic to $\hat{L}_n$ as proved in the proof of Theorem 2.3.5. The proof is finished. □

Example 3.3.7. The k-linear triangulated category $D^b(k\overrightarrow{A_2})/\tau T$ has one indecomposable with one loop. Its AR-quiver is •◯.

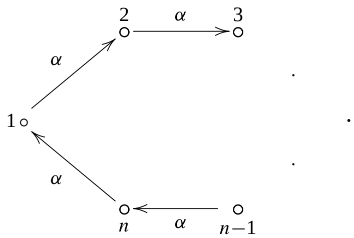
There is an another way by taking covering of stable categories to realize the triangulated categories with $Z\overrightarrow{A_m}/\langle\tau^n\rangle$ as their Auslander–Reiten quivers.

Proposition 3.3.8. *For any positive integers n and m, there is a stable category $\mathcal{A}$ with $Z\overrightarrow{A_m}/\langle\tau^n\rangle$ as its Auslander–Reiten quiver, and there is a Galois covering $F: D^b(\overrightarrow{A_m}) \to \mathcal{A}$ with Galois group $\langle\tau^n\rangle$.*

Proof. Let $\hat{A} = k\hat{Q}/I$ be the infinite dimensional algebra, where $\hat{Q}$ is the infinite quiver:

$$\cdots \xrightarrow{\alpha} \underset{0}{\circ} \xrightarrow{\alpha} \underset{1}{\circ} \xrightarrow{\alpha} \circ \quad \cdots \quad \underset{n}{\circ} \xrightarrow{\alpha} \underset{n+1}{\circ} \xrightarrow{\alpha} \cdots$$

and I is generated by α^{m+1}. It is easy to see that $\hat{A} = k\hat{Q}/I$ is a self-injective algebra whose stable Auslander–Reiten quiver is $\mathbf{Z}\overrightarrow{A_m}$. $\hat{A} = k\hat{Q}/I$ is the Galois cover of $A = kQ/I_0$ with Galois group C_n, the cyclic group of order n, where Q is the quiver



and I_0 is generated by α^{m+1}. Then A is also a self-injective algebra. It follows from Bongartz–Gabriel that there is a Galois covering $F: \operatorname{ind}\hat{A} \to \operatorname{ind} A$ with Galois group $\langle\tau^m\rangle$. F preserves the injective modules in $\hat{A}$-mod and all injective A-modules are images of injective $\hat{A}$-modules. Then by Proposition 3.2.1 that $F: \operatorname{stab}(\hat{A}\text{-mod}) \to \operatorname{stab}(A\text{-mod})$ is a Galois covering of triangulated categories with Galois group $\langle\tau^n\rangle$. Then the Auslander–Reiten quiver of $\operatorname{stab}(A\text{-mod})$ is $\mathbf{Z}A_m/\langle\tau^n\rangle$. The proof is finished. □

3.4. Grothendieck groups of locally finite triangulated categories

Now we apply Theorem 2.3.5 to the Grothendieck groups of triangulated categories. We recall some basics on the Grothendieck groups of triangulated categories from [9,10,23]. Let K be the free abelian group generated by representatives of the isomorphism classes of objects in $\mathcal{A}$ and $K_0(\mathcal{A}, 0)$ the factor groups of K by the subgroup generated by the elements of the form $[A] + [C] - [A \oplus C]$. We have the following well-known facts:

(1) The set $\{[M] \mid M \in \operatorname{ind}\mathcal{A}\}$ is a free basis for $K_0(\mathcal{A}, 0)$;
(2) The Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is the factor group of $K_0(\mathcal{A}, 0)$ modulo the subgroup $G(\mathcal{A})$ generated by elements of the forms: $[A] + [C] - [B]$, provided there exists $\delta: A \to B \to C \to TA$.
(3) There is a canonical epimorphism $\phi: K_0(\mathcal{A}, 0) \to K_0(\mathcal{A})$.

Proposition 3.4.1. *Let $\mathcal{A}$ be a connected locally finite triangulated category. Then $K_0(\mathcal{A}) \simeq \mathbf{Z}^n/H$.*

Proof. If $\Gamma_{\mathcal{A}}$ contains loops, then $\Gamma_{\mathcal{A}} = \hat{L}_n$. It is easy to see that $K_0(\mathcal{A}) = 0$. If $\Gamma_{\mathcal{A}}$ contains no loops, from Theorem 2.3.5, Proposition 3.1.4, we have a functor $F : D^b(k\overrightarrow{\Delta}) \to \mathcal{A}$, which induces a surjective group morphism $f : K_0(D^b(k\overrightarrow{\Delta}, 0)) \to K_0(\mathcal{A}, 0) : [X] \mapsto [FX]$. The restriction of F to ind $D^b(k\overrightarrow{\Delta})$ preserves Auslander–Reiten triangles. We know from Theorem 2.1 in [23] that the relation group $G(\mathcal{A})$ is generated by all Auslander–Reiten triangles in $\mathcal{A}$ for any locally finite triangulated category. Therefore the map f maps the relation group $G(D^b(k\overrightarrow{\Delta}))$ of the Grothendieck group of $D^b(k\overrightarrow{\Delta})$ to the relation group $G(\mathcal{A})$ of the Grothendieck group of $\mathcal{A}$. It follows that f induces a surjective morphism from $D^b(k\overrightarrow{\Delta})$ to that of $\mathcal{A}$. It follows from Lemma 1.2 in Chapter III in [9] that $K_0(D^b(k\Delta)) \simeq \mathbf{Z}^n$ where n is the number of vertices of Δ. Then $K_0(\mathcal{A}) \simeq \mathbf{Z}^n/H$. The proof is finished. $\square$

Remark 3.4.2. There are triangulated categories with no loops but their Grothendieck groups are zero. We refer to [21] for examples.

Acknowledgments

The authors are grateful to B.M. Deng, D. Happel, B. Keller and L.G. Peng for their valuable comments and suggestion. The authors would like to thank the referee for suggestions.

References

[1] H. Asashiba, A covering technique for derived equivalences, J. Algebra 1 (1997) 382–415.
[2] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Stud. Adv. Math., vol. 36, Cambridge Univ. Press, 1995.
[3] M. Auslander, I. Reiten, Representation theory of artin algebras III, Comm. Algebra 3 (1975) 239–294.
[4] K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (1982) 331–378.
[5] E. Dieterich, The Auslander–Reiten quiver of an isolated singularity, in: Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 244–264.
[6] P. Gabriel, Unzerlegbare Darstellungen, I, Manuscripta Math. 6 (1972) 71–103.
[7] P. Gabriel, The universal cover of a representation-finite algebra, in: Lecture Notes in Math., vol. 903, Springer, Berlin, 1981, pp. 68–105.
[8] D. Happel, On the derived category of a finite-dimensional algebra, Comment. Math. Helv. 62 (1987) 339–389.
[9] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc. Lecture Note Soc., vol. 119, 1988.
[10] D. Happel, Auslander–Reiten triangles in derived category of a finite-dimensional algebra, Proc. Amer. Math. Soc. 112 (1991) 416–426.
[11] D. Happel, U. Preiser, C.M. Ringel, Verberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules, in: Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 280–294.
[12] D. Happel, U. Preiser, C.M. Ringel, Binary polyhedral groups and Euclidean diagrams, Manuscript Math. 31 (1980) 317–329.

[13] D. Happel, K.W. Roggenkamp, The derived category of a hereditary order, Arch. Math. 65 (1995) 216–225.
[14] B. Keller, Triangulated orbit categories, preprint, 2003.
[15] A. Neeman, Triangulated categories, Ann. of Math. Stud. 148 (2001).
[16] L.G. Peng, J. Xiao, Triangulated categories and simple Lie algebras, J. Algebra 198 (1997) 19–56.
[17] L.G. Peng, J. Xiao, Triangulated categories and Kac–Moody algebras, Invent. Math. 140 (2000) 563–603.
[18] I. Reiten, M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2002) 295–366.
[19] Chr. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, Comment. Math. Helv. 55 (1980) 199–224.
[20] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math., vol. 1099, Springer, Berlin, 1984.
[21] H. Tachikawa, T. Wakamatsu, Cartan matrices and Grothendieck groups of stable categories, J. Algebra 144 (1991) 390–398.
[22] A. Wiedemann, Orders with loops in their Auslander–Reiten graph, Comm. Algebra 9 (1981) 641–656.
[23] J. Xiao, B. Zhu, Relations for the Grothendieck groups of triangulated categories, J. Algebra 257 (2002) 37–50.