Coils for Vectorspace Categories

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Abstract. Coils as components of Auslander-Reiten quivers of algebras and coil algebras are introduced by Assem and Skowroński. This concept is applied in the present paper to vectorspace categories. The four admissible operations on an Auslander-Reiten component of a vectorspace category, and the notions of v-coils and of vcoil vectorspace categories are introduced. A detailed study on the indecomposable objects of factorspace category of a vcoil vectorspace category is carried out.

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0. Introduction.

Vectorspace categories have been an effective tool for solving problems from the theory of finite dimensional algebras, see for example [8, 9, 11, 12]. However the theory of vectorspace categories is of interest on its own. Representations of posets (i.e. partially ordered sets), for instance, have been studied for a long time, see for example [14].

The concept of factorspace categories was introduced by Nazarova and Roiter. It plays an important role in the representation theory of finite dimensional algebras [11, 12]. In [11], Ringel investigated tubular vectorspace categories by using the knowledge on tubular algebras which were well studied by him. He introduced the tubular (co-)extensions of vectorspace categories, and gave complete descriptions of representations of tubular vectorspace categories via tilting modules arising naturally in the categories of modules over the corresponding algebras. Recently, Bauch described the changes of components of one-point (co-)extensions of vectorspace categories on the level of socle-projective modules, and using them, he described the categories of socle-projective modules over multitubular vectorspace categories [3].

Let $\mathcal{K}$ be a Krull-Schmidt category over the field k. A pair $(\mathcal{K}, |-|)$ is a vectorspace category provided $|-|: \mathcal{K} \to k-mod$ is an additive covariant functor (which is usually assumed to be faithful). Given a vectorspace category $(\mathcal{K}, |-|)$, one may form the corresponding factorspace category $\overset{\vee}{\mathcal{V}}(\mathcal{K}, |-|)$ and the corresponding subspace category $\overset{\vee}{\mathcal{U}}(\mathcal{K}, |-|)$. The objects of $\overset{\vee}{\mathcal{V}}(\mathcal{K}, |-|)$ are triples $X = (X_0, X_\omega, \gamma_X)$ with $X_0 \in \mathrm{Obj}\mathcal{K},\ X_\omega \in \mathrm{Obj}(k - mod)$, and $\gamma_X : |X_0| \to X_\omega$ a linear map. For $X,\ Y \in \mathrm{Obj}\ \overset{\vee}{\mathcal{V}}(\mathcal{K}, |-|)$, a morphism $f : X \to Y$ is a pair (f_0, f_ω) with $f_0 \in \mathcal{K}(X_0, Y_0)$ and $f_\omega \in \mathrm{Hom}_k(X_\omega, Y_\omega)$ such that $\gamma_X f_\omega = |f_0|\gamma_Y$, i.e. the following diagram is commutative

$$\begin{array}{ccc} |X_0| & \xrightarrow{\gamma_X} & X_\omega \\ \downarrow\ |f_0| & & \downarrow\ f_\omega \\ |Y_0| & \xrightarrow{\gamma_Y} & Y_\omega \end{array}$$

The objects of $\overset{\vee}{\mathcal{U}}(\mathcal{K}, |-|)$ are defined dually. For simplicity, we denote $\overset{\vee}{\mathcal{V}}(\mathcal{K}, |-|)$, $\overset{\vee}{\mathcal{U}}(\mathcal{K}, |-|)$ by $\overset{\vee}{\mathcal{V}}(\mathcal{K})$, $\overset{\vee}{\mathcal{U}}(\mathcal{K})$ respectively. $\overset{\vee}{\mathcal{U}}(\mathcal{K})$ and $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ are Krull-Schmidt categories with Auslander-Reiten sequences with respect to the $\mathcal{K}-$split exact sequences provided $\mathcal{K}$ is finite, i.e., there are finitely many non-isomorphic indecomposable objects [11, 2.5 (9)]. Let $\mathcal{V}(\mathcal{K})$ (resp. $\mathcal{U}(\mathcal{K})$) denote the full subcategory of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ (resp. $\overset{\vee}{\mathcal{U}}(\mathcal{K})$) defined by all objects $(X_0, X_\omega, \gamma_X)$ (resp. $(X_\omega, X_0, \gamma_X)$) with γ_X an epimorphism (resp. monomorphism). Then there is an equivalence $\mathcal{V}(\mathcal{K}) \cong \mathcal{U}(\mathcal{K})$ [3, 1.1]. Let $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$ be the Auslander-Reiten-quiver of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$. In the following, we will focus on factorspace categories.

We denote by $P(\omega)$, resp. $I(\omega)$ the projective, resp. injective object in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ corresponding to the extension vertex ω. If α is an indecomposable object in a vectorspace category $(\mathcal{K}, |-|)$, then we denote by $\overline{\alpha}$ the projective representation in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ corresponding to α, and α (when viewed as an object in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$) is the injective object in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ corresponding to α (see [11, 2.5] or [3, 2.4]).

Analogously to coils and multicoil algebras which were introduced by Assem and Skowroński in [1], we introduce notions of v-coils and vcoil vectorspace categories. These generalize the notion of tubular vectorspace categories [11] and multitubular vectorspace categories [3]. We introduce four admissible operations on an Auslander-Reiten component of a vectorspace category, and then a component obtained from a stable tube by a sequence of admissible operations is called a v-coil. In general, a v-coil is not a tube, but the part remaining after removing all injectives is a tube. This leads to an axiomatic description of v-coils. A vectorspace category is called a vcoil vectorspace category provided any cycle in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ (that is, any oriented cycle of non-isomorphisms between indecomposable objects in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$) belongs to a v-standard v-coil. We study the indecomposable objects over a v-coil vectorspace category. The study of vcoil vectorspace categories seems to be helpful to understand the work of Zavadskij about posets of polynomial growth [15]. There is a close relation between strongly simply connected algebras of polynomial growth and vcoil vectorspace categories which will be given in a forthcoming paper.

The paper is organized as follows. In section 1 we recall and give some basic notions and basic results, which will be needed later on. Section 2 presents the definitions of four admissible operations and of v-coils. In section 3 we give an axiomatic description of v-coils. The final section contains the study of vcoil vectorspace categories.

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1. Basic notions and facts.

For a finite vectorspace category $(\mathcal{K}, |-|)$, we denote by $G(\mathcal{K})$ a finite rank torsion free abelian group with basis the set of isomorphism classes, say $[X_1], \cdots, [X_n]$, of indecomposable objects in $\mathcal{K}$. Given a $V = (V_0, V_\omega, \gamma_V)$ in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$, let us denote its dimension vector by $\underline{\dim}_{\mathcal{K}} V := ((d_Y)_{Y \in ind\mathcal{K}}, d_\omega) \in G(\mathcal{K}) \times \mathbb{Z}$, where $d_\omega = \dim_k V_\omega$, and d_Y denotes the multiplicity of Y as a direct summands of V_0. An indecomposable representation $V = (V_0, V_\omega, \gamma_V)$ of dimension vector $\vec{d}$ is called sincere provided $d_\omega \neq 0$ and $d_Y \neq 0$, for all $Y \in \mathrm{ind}\mathcal{K}$. The vectorspace category $\mathcal{K}$ is called sincere (or infinitely sincere) if there exists a sincere indecomposable representation of $\mathcal{K}$ (or infinitely many pairwise nonisomorphic sincere indecomposable representations of $\mathcal{K}$ in some dimension $\vec{d}$, respectively). Given a representation $V = (V_0, V_\omega, \gamma_V)$ with dimension vector $\vec{d}$, we define its support to be $\mathrm{Supp}V = \mathrm{add}(\{Y \in \mathrm{ind}\mathcal{K} \mid d_Y \neq 0\})$, the full additive subcategory of $\mathcal{K}$ generated by $\{Y \in$

$\mathrm{ind}\mathcal{K} \mid d_Y \neq 0\}$. Then for a connected component Γ of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$, we define the support of it to be $\mathrm{Supp}\Gamma = \bigcup_{V\in\Gamma} \mathrm{Supp}V$. If $\mathrm{Supp}\Gamma = \mathcal{K}$, then we say that $\mathcal{K}$ has a sincere component Γ, in addition, if Γ is stable tube, then $\mathcal{K}$ has a sincere stable tube.

Let $(\mathcal{K}, |-|)$ be a finite vectorspace category. There exists a projective realization of $\mathcal{K}$, namely, a finite dimensional $k-$ algebra A, and a $k-A-$bimodule M such that $(\mathcal{K}, |-|) = (\mathrm{Proj}(A), M \otimes -)$ (compare [11, 2.5] or [3, 2.1]). If we denote by Λ the one-point coextension algebra of A by M, then $\overset{\vee}{\mathcal{V}}(\mathcal{K}) \approx \mathrm{Prin}(\Lambda)$, the latter is by definition the full subcategory of $\mathrm{mod}\Lambda$ consisting of $(X_0, X_\omega, \gamma_X)$ with X_0 a projective $A-$module (we sometimes write X_0 as $X|_A$). A vectorspace category $(\mathcal{K}, |-|)$ is called schurian provided the endomorphism ring of each indecomposable object in $\mathcal{K}$ is k. $\mathcal{K}$ is said to be linear (i.e., of poset type) if $\dim_k \mathcal{K}(x, y) \leq 1,\ \forall x, y \in \mathrm{ind}\mathcal{K}$. A vectorspace category is called triangular provided its projective realization algebra is triangular. From these definitions, the triangular vectorspace categories are schurian.

Throughout the following, we assume that vectorspace categories are always schurian and finite.

Let Γ be a connected component of the Auslander-Reiten quiver of a vectorspace category $\mathcal{K}$. We consider its mesh category $k(\Gamma)$ as in [11, 2.1(6)]. Let $I(\Gamma)$ be the ideal of $k(\Gamma)$ generated by the elements $\sigma_{n+1}\cdots\sigma_1$, with

$$x \xrightarrow{\sigma_1} y_1 \xrightarrow{\sigma_2} y_2 \cdots \xrightarrow{\sigma_n} y_n \xrightarrow{\sigma_{n+1}} z,$$

where x is injective, z is projective. Let $k(\Gamma)_0$ be the quotient category of the mesh category $k(\Gamma)$ modulo the ideal $I(\Gamma)$.

Definition 1.1. *Let Γ be a connected component of the Auslander-Reiten quiver of a vectorspace category $\mathcal{K}$. Γ is said to be v-standard provided the subcategory of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ consisting of representations in Γ and $k(\Gamma)_0$ are equivalent.*

From the definition, we have that for any stable or semi-stable tube, the v-standardness is equivalent to the standardness. Hence any component of a tubular vectorspace category is v-standard.

We recall the definition of one-point (co-)extension of a vectorspace category in [11, 4.1] or [3, 5.1].

Definition 1.2. *Let $(\mathcal{K}, |-|)$ and $(\mathcal{K}_1, |-|_1)$ be two vectorspace categories. Then $(\mathcal{K}_1, |-|_1)$ is called one-point extension of $(\mathcal{K}, |-|)$ by rad $\overline{\alpha}$ if the following conditions are satisfied,* *i). $\mathcal{K}_1$ has a sink α, and $\mathcal{K}$ is the full subcategory of $\mathcal{K}_1$ given by all indecomposable objects except α.*

ii). $|-|$ is the restriction of $|-|_1$ to $\mathcal{K}$.

iii). rad$\overline{\alpha}$ is indecomposable.

Note 1. Noted as in [3, 5.1], for the general definition of one-point extensions, condition iii) may be skipped. However, we will only consider situations of one-point extensions in which condition iii) is satisfied. Hence we already include this in our definition.

Note 2. Suppose that P is an indecomposable object in some (Krull-Schmidt) category such that there exists a sink map for P, then we denote the source of the sink map by rad$P$.

Note 3. If the conditions i)-iii) are satisfied, we say that α is the extension point of $(\mathcal{K}_1, |-|)$.

Note 4. We try to say more about the definition of one-point extensions. Let $V = (V_0, V_\omega, \gamma_V) \neq (0, k, 0)$ be indecomposable in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$. Then $\gamma_V : |V_0| \longrightarrow V_\omega$ is an epimorphism. We decompose V_0 into a direct sum of indecomposable objects in $\mathcal{K}$, say $V_0 = \bigoplus_{i=1}^n V_i^{m_i}$($n$ is the number of indecomposable objects in $\mathcal{K}$). We define a new vectorspace category $(\mathcal{K}_1, |-|_1)$ as follows: ind$\mathcal{K}_1$ = ind$\mathcal{K} \bigcup \{\alpha\}$, with $|\alpha| = V_\omega$, and $\mathrm{Hom}_{\mathcal{K}_1}(V_i, \alpha) = k^{m_i}$ such that we may take a basis $u_{i1}, \cdots, u_{im_i}$ of $\mathrm{Hom}_{\mathcal{K}_1}(V_i, \alpha)$ for each i, with $\gamma_V = (\cdots, |u_{i1}|, \cdots, |u_{im_i}|, \cdots)$. Then $(\mathcal{K}_1, |-|_1)$ is a vectorspace category which is a one-point extension of $(\mathcal{K}, |-|)$ by V, the extension point is α.

Let $(\mathcal{K}_1, |-|_1)$ be a one-point extension of $(\mathcal{K}, |-|)$ by $V = (V_0, V_\omega, \gamma_V)$, A the projective realization of $(\mathcal{K}, |-|)$. Then there exists a bimodule ${}_kM_A$ such that $\mathrm{Prin}\begin{pmatrix} A & 0 \\ M & k \end{pmatrix} \approx \overset{\vee}{\mathcal{V}}(\mathcal{K})$, and $A_1 = \begin{pmatrix} k & 0 \\ V_0 & A \end{pmatrix}$ is the projective realization of $(\mathcal{K}_1, |-|_1)$, and if we denote

$$\begin{pmatrix} k & 0 & 0 \\ V_0 & A & 0 \\ V_\omega & M & k \end{pmatrix}$$

by C, then $\mathrm{Prin}(C) \approx \overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$.

Proposition 1.1. *Prin* $(C) \subseteq \overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}),\ Hom(V, -))$.

Proof. We have that $C-$mod $= \overset{\vee}{\mathcal{U}}([M]A, \mathrm{Hom}(V, -)) = \overset{\vee}{\mathcal{V}}(A_1, (V_\omega, M) \otimes -)$ and $X \in \mathrm{Prin}(C) \Leftrightarrow X\ |_{A_1}$ is projective. Because V_0 is projective $A-$module, $(X\ |_{A_1})\ |_A$ is projective.

So if $X \in \mathrm{Prin}(C)$, then $X \mid_A$ projective, and then $X \mid_{[M]A} \in \overset{\vee}{\mathcal{V}}(\mathcal{K})$. It implies that $X \in \overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), \mathrm{Hom}(V,-))$. The proof is finished.

Corollary 1.2. *$Prin(C)$ is a full subcategory of $\overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), Hom(V,-))$ which is closed under extensions*

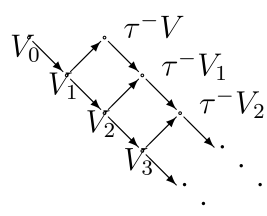
2. Definition of v-coils

In the following, we introduce admissible operations on a v-standard component of Auslander-Reiten quiver of a vectorspace category. Let Γ be a v-standard component of $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$, without $P(\omega)$, $I(\omega)$. For a non-injective indecomposable object $V = (V_\theta, V_\omega, \gamma_V) \in \Gamma$, which is called a pivot, we shall define admissible operations depending on the shape of the support of the functor $\mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(V,-) \mid_{\mathrm{ind}\Gamma}$.

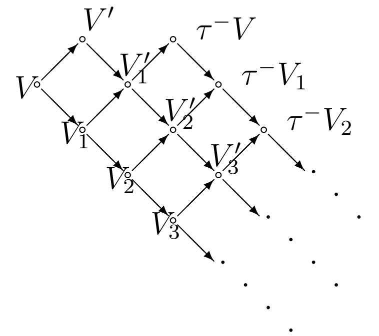
$vad1$). Assume that $\mathrm{SuppHom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(V,-) \mid_{\mathrm{ind}\Gamma}$ consists of a sectional path starting at V :

$$V_0 = V \to V_1 \to V_2 \to \cdots \to V_n \to \cdots ,$$

(and V_i are not injective, for all i) i.e., Γ may look as follows:



We define the modified vectorspace category $(\mathcal{K}_1, |-|)$ to be the one-point extension of $(\mathcal{K}, |-|)$ by V, and insert the component Γ to be of the form (see below), where $V_i' = (k, V_i, 1)$ as an object in $\overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), \mathrm{Hom}(V,-))$ and the morphisms are the obvious ones. The translation τ' is defined as follows: $P = V_0'$ is a projective object in mod$C$, of course, it is a projective object in $\overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), \mathrm{Hom}(V,-))$, $\tau' V_i' = V_{i-1}$ for all $i \geq 1$, and then $\tau'(\tau^- V_i) = V_i'$, for all $i \geq 0$. For remaining vertices, τ' coincides with τ. Then (Γ', τ') is again a translation quiver, which is obtained from Γ by a single ray insertion.



Proposition 2.1. *The component of* $\Gamma_{\overset{\vee}{\mathcal{V}}(K_1)}$ *containing* V *(considered as an object in* $\overset{\vee}{\mathcal{V}}(K_1)$*) is equal to* Γ' *and is v-standard.*

Proof. The proof here is essentially due to Ringel in [11, 4.10] and Bauch in [3, 5.3]. For the first assertion, we point out the steps of a way to prove it. First, we prove that Γ' is a component of $\overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), \mathrm{Hom}(V,-))$ containing V. This can be easily done by using the results in [11, 2.5]. Secondly, we prove that $\Gamma' \subseteq \overset{\vee}{\mathcal{V}}(K_1) = \mathrm{Prin}(C)$, which we only need to verify that $V_i' \in \mathrm{Prin}(C)$, for all i. Because the projective object $P = V'$ is obviously in $\overset{\vee}{\mathcal{V}}(K_1)$, by the exactness of sequence $0 \to V' \to V_1' \to \tau^{-1}V \to 0$, we know that V_1' is in $\overset{\vee}{\mathcal{V}}(K_1)$. An easy induction on i follows that $V_i' \in \overset{\vee}{\mathcal{V}}(K_1)$.

For the proof of the second assertion, we note that if there is an injective I which is the direct predecessor of V_0, then there is a path $I \to V_0 \to P$ in Γ', but we know that $\mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(I,P) = 0$. Hence we may copy the proof of Proposition 4.5.1. in [11] or the proof of Lemma 2.2. in [2]. We will not present here in details. This finishes the proof.

Before we define vad2), we have to fix some notation. For an indecomposable object $X = (X_0, X_\omega, \gamma_X) \in C-\mathrm{mod}$, we set $\overline{X} = (P(X_0), X_\omega, \gamma_X')$ where $P(X_0) \longrightarrow X_0$ is the projective cover of $A-$module X_0, γ_X' denotes the composition of

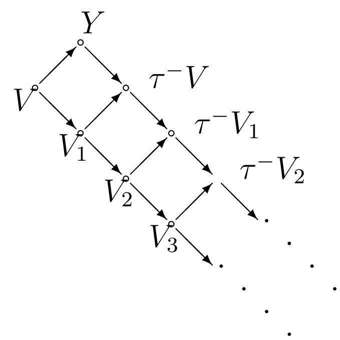
$$M \otimes P(X_0) \quad \xrightarrow{can} \quad M \otimes X_0 \quad \xrightarrow{\gamma_X} X_\omega.$$

Then $\overline{X} \to X$ is the minimal $\mathrm{Prin}(C)-$ approximation of X [3, 2.3].

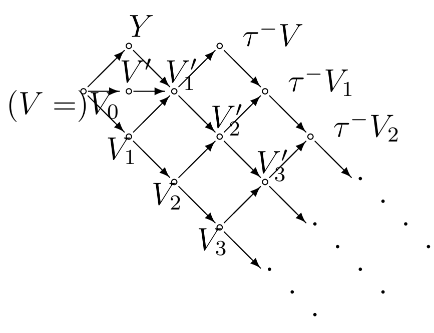
*vad*2). Assume that $\mathrm{SuppHom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(V,-) \mid_{\mathrm{ind}\Gamma}$ consists of two parallel infinite sectional paths starting at V :

$$\begin{array}{ccccccccc}
Y & \to & \tau^{-1}V & \to & \tau^{-1}V_1 & \to & \cdots & \to \tau^{-1}V_{n-1} & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & & \uparrow & & \\
V_0 = V & \to & V_1 & \to & V_2 & \to & \cdots & \to V_n & \to & \cdots ,
\end{array}$$

with V non-injective (of course, V_i is not injective for all i), but Y is injective. i.e Γ may look as follows



We define the modified vectorspace category $(\mathcal{K}_1, |-|_1)$ to be the one-point extension of $(\mathcal{K}, |-|)$ by V, and insert the component Γ to be of the form,



where $V' = (k, V, 1)$, $V_1' = \overline{(k, V_1 \oplus Y, (1,1))} = (k, V_1 \oplus Y, (1,1))$, $V_i' = \overline{(k, V_i, 1)}$, for all $i \geq 2$, and the morphisms are the obvious ones. The translation τ' of Γ' is defined as follows: $P = V'$ is a projective object in $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1, |-|_1)$, $\tau' V_i' = V_{i-1}$ for all $i \geq 1$, and $\tau'(\tau^- V_i) = V_i'$, for all $i \geq 0$. For other vertices, the definition of τ' is same as τ. The modified component (Γ', τ') is again a translation quiver, which is obtained from Γ by a single ray insertion.

Proposition 2.2. *The component of* $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)}$ *containing* V *(considered as an object in* $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$*) is equal to* Γ'*, and is v-standard.*

Proof. The proof is essentially due to Bauch in [3, 5.3], we give here some points. In the category $\overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), \mathrm{Hom}(V, -))$, we firstly have Auslander-Reiten sequence

$$0 \to (k, V, 1) \to (k, V_1 \oplus Y, (1,1)) \to \tau^- V \to 0,$$

with $V' = (k, V, 1)$ projective in Γ', hence $V_1' = (k, V_1 \oplus Y, (1,1))$ is in Γ' and $V = \tau' V_1'$. Secondly, we have the Auslander-Reiten sequence

$$0 \to (k, V_1, 1) \to (k, V_2 \oplus \tau^- V, (1,1)) \to \tau^- V_1 \to 0,$$

in $\overset{\vee}{\mathcal{U}}(\overset{\vee}{\mathcal{V}}(\mathcal{K}), \mathrm{Hom}(V, -))$. Hence by [B, 3.4], the sequence

$$0 \to \overline{(k, V_1, 1)} \to \overline{(k, V_2 \oplus \tau^- V, (1,1))} \to \tau^- V_1 \to 0$$

is an Auslander-Reiten sequence in $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$, and $\overline{(k, V_2 \oplus \tau^- V, (1,1))} = V_2' \oplus \tau^- V$. An easy induction on n shows that there is an Auslander-Reiten sequence

$$0 \to V_{n-1}' \to V_n' \oplus \tau^- V_{n-2} \to \tau^- V_{n-1} \to 0.$$

For the remaining assertion, one may copy the proof of Proposition 1 in section 4.5 in [11] or the proof of Lemma 2.2 in [2], we omit it.

Note 1. We may define the one-point co-extensions dually and the dual admissible operations $vad1^*$), $vad2^*$) and these operations also preserve the v-standardness of components.

Note 2. The operations $vad1$), $vad2$), $vad1^*$), $vad2^*$) are called admissible operations on translation quivers.

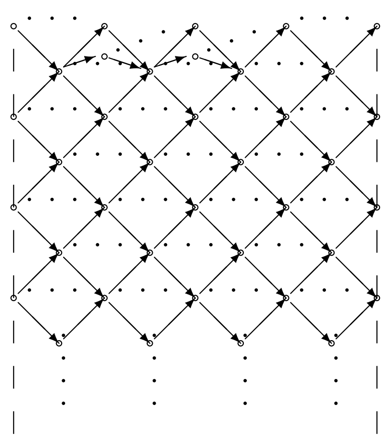
Note 3. The definitions of $vad1$), $vad2$), $vad1^*$), $vad2^*$) may be given on a translation quiver in a similar way.

Definition 2.1. *A translation quiver is called a v-coil of stable rank* n *provided there is a sequence of translation quivers* $\Gamma_{n+1}, \ \Gamma_n, \ \cdots, \Gamma_0$ *such that* Γ_0 *is stable tube of rank* n *and for each* $0 \le i \le n$, Γ_{i+1} *is obtained from* Γ_i *by an admissible operation.*

From the definition, we see that the stable tubes are v-coils, ray tubes and coray tubes without projective-injective vertices and without direct vertices are v-coils having the property in which the defining sequence of operations is $vad1$) and $vad1^*$), respectively. Ray tubes and coray tubes as components of

Auslander-Reiten quiver of poset appeared first in [6], and then the systematic study of them appeared in [11]. In [3], Bauch gave a systematic study on multitubular vectorspace categories, in which the v-coils appeared frequently as a component of its AR-quiver of socle projective modules category. The following example shows that there are some v-coils, which are not tubes.

Example.



Proposition 2.3. *For each v-coil* Γ *of stable rank* n, $n \leq 6$, *there exists a triangular vectorspace category with* Γ *as a component of its Auslander-Reiten quiver.*

Proof. For a stable tube of rank $n(\leq 6)$, there is a critical or tubular vectorspace category admitting such a stable tube. It is easy to get a vectorspace category from this critical or tubular one by one-point extensions or coextensions corresponding to the operations in defining this v-coil. It is a triangular vectorspace category because this critical or tubular category is triangular. The proof is finished.

3. A description of v-coils

Our aim here is to characterize v-coils by means of axioms rather than the inductive construction.

Definition 3.1. *A function* $f: \Gamma_0 \to \mathbb{N}_0$ *is called a* $v-$*length function, if* f *satisfies the following conditions:*

(1). x *is projective vertex, then* $f(x) = \sum_{y \to x} f(y) \neq 0$.

(2). x *is injective vertex iff* $f(x) = 0$.

(3). If x *is not projective vertex, then* $f(x) + f(\tau x) = \sum_{y \to x} f(y)$.

Remark. If we endow a v-length function for any stable tube. then, by the definition of v-coils, one has that any v-coil admits a v-length function.

Let us state our main theorem of this section, which gives an axiomatic description of v-coils.

Theorem 3.1. *Let Γ be a translation quiver, without multiple arrows, and there are no nonzero nonisomorphic morphisms from a projective to itself in $k\Gamma$. Then Γ is a v-coil if and only if Γ satisfies the following conditions:*

(a).If we let Γ' be the full subquiver of Γ consisting of vertices except injectives, then Γ' is a tube.

(b). For each projective vertex p, there exists a ray $[p,\ \infty[$, and p has exactly one direct predecessor. For each injective q, there exists a coray $]\infty,\ q]$, and q has exactly one successor.

(c). The $\tau-$orbit of any projective or injective contains a vertex which belongs to an oriented cycle in Γ.

(d). there exists a $v-$length function on Γ.

Remark. Because there is an equivalence between $\mathcal{V}(\mathcal{K})$ and $\mathcal{U}(\mathcal{K})$, if we replace projectives in the definition of v-length function and Theorem 3.1 by injectives and vice versa, we get a symmetric description of v-coils.

We need some lemmas to prove the theorem. We assume that Γ satisfies conditions (a)-(d).

Lemma 3.1. *Let p (resp. q) be projective (resp. injective). The ray $[p,\infty[$ (resp. coray $]\infty,q]$,) starting at p (ending at q, resp.) contains no injectives (resp. projectives).*

Proof. We only prove the assertion for projective, because the other is similar. Let $p = y_0 \to y_1 \to \cdots$ be the ray starting at p. Suppose t is the minimal index with y_t injective, it follows that $t \neq 0$. Hence $\tau^{-1}y_{t-1}$ and y_{t+1} are two direct successors of the injective y_t, a contradiction.

Lemma 3.2. *Let p be projective, $p = y_0 \to y_1 \to \cdots$ be the ray starting at p. Then for each i, y_i is not projective. Dually the coray ending at an injective: $\cdots \to y_1 \to y_0 = q$ does not contain any injective except q.*

Proof. Otherwise, we assume that y_k is projective, then for each i, y_{k+i} is projective. If it is not, we may assume that y_{k+1} is not projective, it follows that the direct predecessors of projective are $y_{k-1},\ \tau y_{k+1}$, contradicting to condition (a). By Lemma 3.1, the ray $[p,\infty[$ is also a ray in Γ', then the ray starting at y_k is a ray in Γ' consisting of projectives, which is a contradiction to the fact that Γ' is a tube.

The proof for the other assertion is similar to the above.

Lemma 3.3. *Each mesh in Γ contains at most three middle terms.*

Proof. Otherwise we may assume that there are four middle terms in a mesh, the four middle terms are assumed to be $y_1,\ y_2,\ y_3,\ y_4$. Because Γ' is a tube, at least two of the middle terms, say $y_2,\ y_3$, are injectives in Γ. We prove that one of $y_1,\ y_4$ is not projective. Otherwise, we have the equalities $f(\tau x)+f(x) = f(y_1)+f(y_4)$ and $f(y_1) = f(y_4) = f(\tau x)$. Hence $f(x) = f(\tau x)$. We assume that the ray starting at y_4 is $p = y_4 \to z_1 \to \cdots$. If $z_1 \neq x$, then z_1 must be projective, a contradiction. Then $z_1 = x$, and $\tau^{-1}y_4$ is injective. Similarly the ray starting at y_1 goes through x and $\tau^{-1}y_1$ is injective. Then x has two injective direct predecessors. Therefore, the corays ending at those injectives must go through a projective or an injective, a contradiction to Lemma 3.1 and Lemma 3.2. So we assume that y_4 is not projective, it follows that τx has at least three direct predecessors $\tau y_2,\ \tau y_3,\ \tau y_4$ in Γ', a contradiction.

Lemma 3.4. *Assume that Γ contains a mesh with three middle terms , then there is exactly one projective middle term and exact one injective middle term.*

Proof. Let
$$\begin{tikzcd}[row sep=tiny, column sep=tiny] & z_3 \arrow[dr] & \\ \tau x \arrow[ur] \arrow[r] \arrow[dr] & z_2 \arrow[r] & x \\ & z_1 \arrow[ur] & \end{tikzcd}$$
be a mesh with three middle terms in Γ. Then there is at least one injective middle term. Otherwise, x has three direct successors in Γ', a contradiction. Suppose $z_2,\ z_3$ are injectives, then there are two corays ending at $z_2,\ z_3$, respectively. Those corays are corays ending at τx in Γ'. Since Γ' is tube, the corays ending at x are the same, which is assumed: $\cdots \to y_2 \to y_1 \to y_0 = \tau x$. It follows that $y_1 \neq \tau z_2,\ \tau z_3$. Then τx has at least three direct predecessors, a contradiction. We proved that the mesh ending at x with three middle terms contains exactly one injective middle term. We assume that z_2 is injective. Similarly the middle terms of the mesh contain at least one projective. Otherwise there are three direct predecessors of τx in Γ', a contradiction. If the remaining two middle terms $z_1,\ z_3$ are projective, similar discussion as above gives a contradiction. The proof is finished.

Lemma 3.5. *There are two middle terms of a mesh with three middle terms being on the mouth of Γ (i.e this vertex is a starting or ending term of a mesh with either only one middle term in Γ or two middle terms, but one of which is projective or injective).*

Proof. We assume that
$$\begin{tikzcd}[row sep=tiny, column sep=tiny] & z_3 \arrow[dr] & \\ \tau x \arrow[ur] \arrow[r] \arrow[dr] & z_2 \arrow[r] & x \\ & z_1 \arrow[ur] & \end{tikzcd}$$
is a mesh with three middle terms in Γ, and by Lemma 3.4, we assume that z_3 is injective, z_2 is projective. We will prove that z_2, z_3 are on the mouth of Γ. Because of z_3 being injective, the direct predecessor, except τx, of z_3 must be injective if it exists, otherwise we will get two direct successors of the injective z_1, contradicting to (a). It

is similar for the projective z_2. This finishes the proof.

Lemma 3.6. *The number of projective vertices in* Γ *is finite. Dually the number of injective vertices in* Γ *is finite.*

Proof. Let p be an arbitrary projective in Γ, and the oriented cycle in Γ starting p be as follows:

$$p \to x_1 \to x_2 \to \cdots \to x_s \to p.$$

Then arbitrary ray in Γ must intersect with the set of all x_i. By Lemma 3.2 and condition a) in the theorem, we know that the number of projective vertices must be finite. The proof is finished.

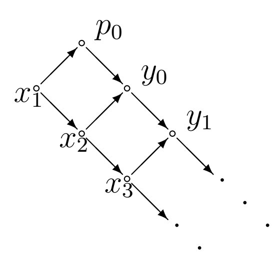
Before we complete the proof of Theorem 3.1, we need some notions used in [2]. A mesh with three middle terms is called an exceptional mesh, and the projectives (injectives) in exceptional meshes are called exceptional projectives (injectives).

Proof of Theorem 3.1: The necessity follows by an easy induction on the number of admissible operations and the fact that all conditions are trivially satisfied for stable tubes. So we only need to prove the sufficiency. We assume that Γ satisfies conditions in the theorem.

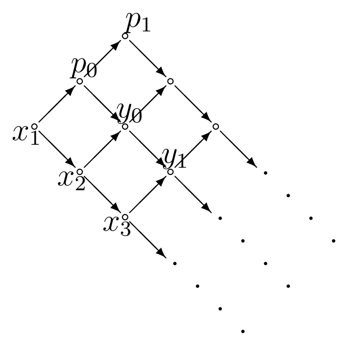
If Γ does not contain any exceptional projective, we will prove that Γ is obtained from a stable tube by a sequence of admissible operations $vad1), vad1^*)$.

Case 1). If Γ contains neither projective nor injective, then Γ is a stable translation quiver and $\Gamma = \Gamma'$, it follows that Γ is a stable tube.

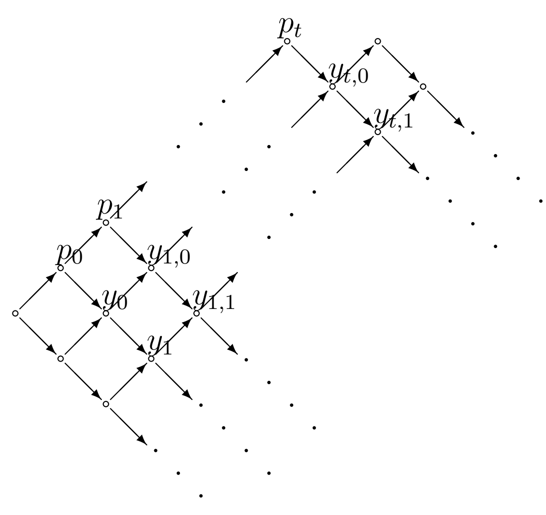
Case 2). By duality, we may assume that Γ contains a projective p_0. We know, by (b) and Lemmas 3.1, 3.2), that Γ contains a full subquiver of the form:



If the successor of p_0, except y_0, is p_1, then p_1 must be projective. Otherwise the predecessors of p are $x_1,\ \tau p_1$, a contradiction. Then Γ contains a full subquiver of the form:

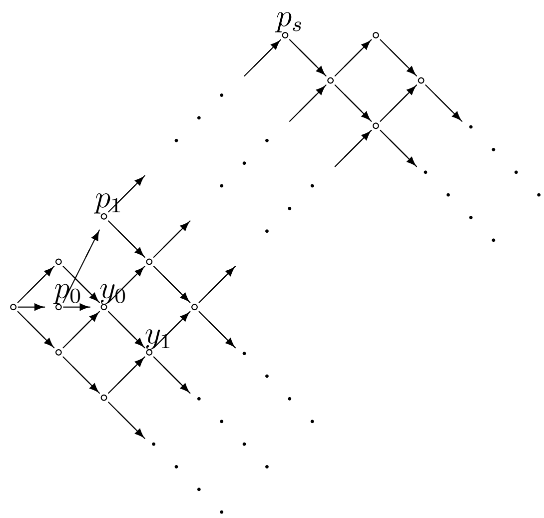


and we consider p_1. Because the number of projective vertices is finite, there is a projective p_t, such that p_t has only one successor. Then Γ contains a full subquiver of the form (see below). In all cases above, we may delete the ray starting at p_t : $p_t \to y_{t,0} \to y_{t,1} \to \cdots$, from Γ. Then the remaining translation quiver Γ_1 satisfies the conditions in the theorem, but the number of projectives in Γ_1 is one less than Γ. Repeating this process, we obtain a translation quiver Γ_t containing no projectives. By duality, we do this for Γ_t to get a translation quiver Γ_{t+s}, which contains neither projective nor injective, and satisfies the conditions in the theorem. Then Γ_{t+s} is a stable tube, and Γ is a v-coil in which the admissible operations defining it are $vad1$), or $vad1^*$).



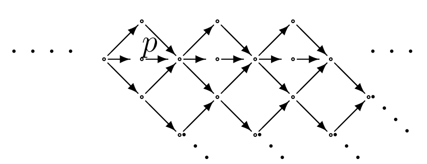
For the remaining cases, we assume that Γ contains an exceptional projective p_0, and $p_0 \to y_0 \to y_1 \to \cdots$ is the ray starting at p_0. If p_0 has two successors, say y_0, p_1, we know that p_1 must be projective as before. We assume that $p_1 \to p_2 \to \cdots \to p_s$ is the maximal sectional path from p_1

to mouth. Then all p_i are projective and the p_s has only one successor. It follows that Γ contains a full subquiver of the form:

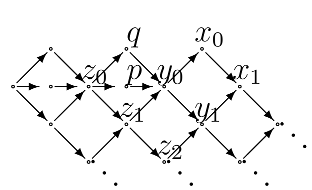


Then after deleting the rays starting at $p_1, \cdots, p_s$, we get a translation quiver Γ_1 satisfying the conditions in the theorem, in which p_0 has only one successor. We repeat this process and its dual. The remaining translation quiver Γ_* contains only exceptional projectives, and satisfies conditions in the theorem.

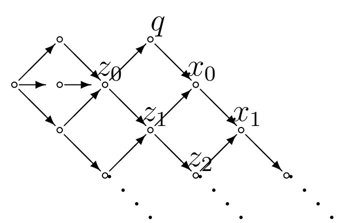
So we may assume that Γ contains only exceptional projectives. We first show that Γ contains an exceptional p with $\tau^- p$ non-injective. Otherwise, Γ looks like:



Then there is no oriented cycle involving p, a contradiction. Let p be such exceptional projective vertex. Then Γ contains a full translation subquiver of the form:



So we delete the ray starting at p : $p \to y_0 \to y_1 \to \cdots$. The remaining quiver contains a full translation subquiver of the form:



Then we delete the coray ending at q. The remaining quiver Γ_{*1} satisfies the conditions in the theorem. In Γ_{*1} the projectives and injectives are exceptional, and the number of such projectives is one less than that in Γ_*. By repeating this process, we will obtain a stable tube. Then Γ is obtained from a stable tube by admissible operations $vad1), vad2)$ and their dual $vad1^*), vad2^*)$. We finish the proof.

4. Vcoil vectorspace categories

Definition 4.1. *A vectorspace category is called a vcoil vectorspace category provided every cycle in* $\overset{\vee}{\mathcal{V}}(\mathcal{K})$ *is contained in a v-standard v-coil in* $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$.

Remark 1. Any representation directed vectorspace category is vcoil.

Remark 2. Vcoil vectorspace categories are triangular. Namely, we have the following property.

Proposition 4.1. *Let* $(\mathcal{K}, |-|)$ *be a vectorspace category of polynomial growth. Then* $(\mathcal{K}, |-|)$ *is schurian if and only if* $(\mathcal{K}, |-|)$ *is triangular.*

Proof. We only need to prove the necessity. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_0$ be a chain of nonzero nonisomorphic morphisms between indecomposable objects in $\mathcal{K}$. If all X_i are of one dimensional, we may get that $f_0 \cdots f_n \neq 0$. It follows that $\mathrm{End}_{\mathcal{K}} X_0 \neq k$, a contradiction. So there must exist two-dimensional objects contained in the cycle above. If the number of two-dimensional objects is one, we assume that the only two-dimensional object is X_i. Then we also get a nonzero nonisomorphic morphism from X_i to X_i, which follows that $\mathcal{K}$ is not schurian, a contradiction. Then there exists a cycle between indecomposable objects of dimension two. We assume that $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_0$ is such one with minimal length. The condition of minimal length of the cycle follows that $\mathrm{Hom}(X_{i+1}, X_i) = 0$. Then by [12], $n = 1$. We assume that $f,\ g \in \mathrm{Hom}_{\mathcal{K}}(X_i, X_{i+1})$, where $X_3 = X_0$, are non zero, then $\mathrm{Ker}(|f|) = \mathrm{Ker}(|g|)$, $\mathrm{Im}(|f|) = \mathrm{Im}(|g|)$, which follows that

$|f|, |g|$ are linear dependent. Then $\mathrm{Hom}_{\mathcal{K}}(X_i, X_{i+1}) = k$. This is a contradiction [5]. This finishes the proof.

Proposition 4.2. *Any vcoil vectorspace category is cycle-finite, and of polynomial growth, of course, it is of tame type.*

Proof. We only need to prove that $\mathcal{K}$ is cycle-finite, the others are deduced from [4]. Let $(\mathcal{K}, |-|)$ be a vcoil. If $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_0$ is a cycle in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$, then the cycle is, by definition, in a v-standard v-coil. It follows that $f_i \notin \mathrm{rad}^\infty(\overset{\vee}{\mathcal{V}}(\mathcal{K}))$, by the v-standardness of this v-coil.

Corollary 4.3. *Let $(\mathcal{K}, |-|)$ be a linear vectorspace category. Then $(\mathcal{K}, |-|)$ is a vcoil if and only if $(\mathcal{K}, |-|)$ is of polynomial growth.*

Proof. The necessity follows from Proposition 3.3.2. The sufficiency is deduced from the descriptions on Auslander-Reiten quivers of posets of polynomial growth [15].

For any full subcategory $\mathcal{K}_1$ of a vectorspace category $\mathcal{K}$, we have a natural embedding functor $F : \overset{\vee}{\mathcal{V}}(\mathcal{K}_1) \longrightarrow \overset{\vee}{\mathcal{V}}(\mathcal{K})$, and a restriction functor $G : \overset{\vee}{\mathcal{V}}(\mathcal{K}) \longrightarrow \overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$, such that $G \circ F = 1_{\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)}$. Similarly to [1], we have that any full subcategory of a cycle-finite vectorspace category is cycle-finite. Now we prove that any full subcategory of a vcoil vectorspace category is a vcoil vectorspace category.

Before we state the next proposition, we have to fix some notation. Let $\mathcal{K}$ be a vcoil vectorspace category, $\mathcal{K}_1$ the full subcategory of $\mathcal{K}$. Let Γ be a component of Auslander-Reiten quiver of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$, $\mathcal{C}$ the full subcategory of $\mathrm{ind}\overset{\vee}{\mathcal{V}}(\mathcal{K})$ consisting of objects of $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$ in Γ, which lie on a cycle in $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$.

Proposition 4.4. *Let $\mathcal{K}_1$ be a full subcategory of a vcoil vector space category $\mathcal{K}$. Then $\mathcal{K}_1$ is a vcoil vectorspace category.*

Proof. Because any cycle in $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$ is also a cycle in $\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$, hence it is contained in a vcoil Γ in the Auslander-Reiten quiver of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$. So it suffices to show that, if $\mathcal{C} \neq \phi$, the quiver of $\mathcal{C}$ (as above) is a v-standard v-coil. If Γ is a stable tube, suppose that there is a vertex $N \notin \mathcal{V}(\mathcal{K}_1)$, then there is a projective object $\overline{a}$ in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$, such that $a \notin \mathcal{K}_1$ and a is a direct summand of $N\mid_{\mathcal{K}}$. It follows that there is a vertex M on the mouth of Γ such that a is a direct summand of $M\mid_{\mathcal{K}}$ by the $\mathcal{K}-$split exact structure of $\overset{\vee}{\mathcal{V}}(\mathcal{K})$. If we denote by $\widehat{M}$ the full translation subquiver of Γ consisting of all vertices X' such that there exist sectional paths $X \longrightarrow \cdots \longrightarrow X'$ for some X on the coray $]\infty,\ M]$ and $X' \longrightarrow \cdots \longrightarrow Y$ for some Y on ray

$[M,\ \infty[$, then $\widehat{M} \bigcap \overset{\vee}{\mathcal{V}}(\mathcal{K}_1) = \phi$. It follows that $\mathcal{C} = \phi$. If Γ is a non-stable v-coil, which is obtained from a stable tube Γ_0 by a sequence of admissible operations, then there is a sequence $\Gamma_0,\ \Gamma_1,\ \cdots, \Gamma_m = \Gamma$ such that Γ_{i+1} is obtained from Γ_i by an admissible operation. The corresponding (co-)extension points are assumed to be a_i. If $\Gamma_0 \subseteq \mathcal{C}$, We suppose that $\Gamma_{t-1} \subseteq \mathcal{C}$ (for which we set $\Gamma'_{t-1} = \Gamma_{t-1}$) but $\Gamma_t \nsubseteq \mathcal{C}$, then $a_t \notin \mathcal{K}_1$. If we denote by Γ'_t the remaining quiver after deleting the ray starting at $\overline{a_t}$, or the coray ending at a_t. Since a_t is an extension point or a coextension point, Γ'_t is a v-coil, and the standardness of Γ'_t is followed from the standardness of Γ_t (Γ). An easy induction on t shows that Γ'_m is a v-coil. Then $\mathcal{C} = \Gamma'_m$ is a v-standard v-coil of $\mathcal{K}_1$. If $\Gamma_0 \nsubseteq \mathcal{C}$, then $\mathcal{C} = \phi$. We finished the proof.

Let $< -, - >_{\mathcal{K}}$ be the bilinear form on $G(\mathcal{K}) \times \mathbb{Z}$ defined in [11, 2.5], i.e

$$< \vec{x}, \vec{y} >= x_0 y_0 + \sum_{X,Y \in \mathrm{ind}\mathcal{K}} x_X y_Y \dim_k \mathcal{K}(X,Y) - y_0 \sum_{X \in \mathrm{ind}\mathcal{K}} x_X \dim_k |X|,$$

where $G(\mathcal{K})$ is a finite torsion free abelian group with the set of isomorphism classes, say $[X_1],\ \cdots,\ [X_n]$, of indecomposable objects in $\mathcal{K}$.

It was proved in [11, 2.5] that

$$< \underline{\dim} M, \underline{\dim} N >= \dim_k \mathrm{Hom}(M,N) - \dim_k \mathrm{Ext}^1(M,N).$$

Let $\chi_{\mathcal{K}}$ be the corresponding quadratic form of the vector space category $\mathcal{K}$.

Lemma 4.5. *Let Γ be a stable tube of rank n in $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$, where $\mathcal{K}$ is a cycle-finite vectorspace category. Then for each $X \in \Gamma$, $\chi_{\mathcal{K}}(\underline{dim} Y) = 0$, where $Y = X \oplus \tau X \oplus \cdots \oplus \tau^{n-1} X$.*

Proof. We have that

$$\chi_{\mathcal{K}}(\underline{\dim}(Y) = \dim_k \mathrm{Hom}(Y,Y) - \dim_k \mathrm{Ext}^1(Y,Y).$$

By [10], $\mathrm{DExt}^1(Y,Y) = \overline{\mathrm{Hom}}(Y, \tau Y)$, where $\overline{\mathrm{Hom}}(Y, \tau Y)$ is the factor space $\mathrm{Hom}(Y, \tau Y)/I(Y, \tau Y)$, where $I(Y, \tau Y)$ is the subset of $\mathrm{Hom}(Y, \tau Y)$ consisting of maps which admits a factorization through an injective object in $\overset{\vee}{\mathcal{V}}(\mathcal{K})$. Because Γ is a stable tube, and $\mathcal{K}$ is of cycle-finite, then $I(Y, \tau Y) = 0$. It follows that $\chi_{\mathcal{K}}(\underline{\dim}(Y) = 0$. The proof is finished.

Theorem 4.6. *Let $\mathcal{K}$ be a cycle-finite vectorspace category. If the Auslander-Reiten quiver $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$ contains a sincere stable tube, then $\mathcal{K}$ is a critical or a tubular vectorspace category.*

Proof. We know from [5] that, if $\mathcal{K}$ is an infinitely sincere non-linear vectorspace category of polynomial growth, $\mathcal{K}$ is critical or tubular. So it suffices to prove that $\mathcal{K}$ is an infinitely sincere vectorspace category of polynomial growth. By Lemma 4.5, our condition follows that there is a sincere dimension vector $\vec{d}$ such that $\chi_{\mathcal{K}}(\vec{d}) = 0$, which follows that there are infinitely many pairwise nonisomorphic indecomposable objects of the dimension $\vec{d}$ [5]. For a linear vectorspace category $\mathcal{K}$, by Lemma 4.5, $\chi_{\mathcal{K}}(\vec{d}) = 0$. It follows that $\mathcal{K}$ is critical or tubular. The proof is finished.

Theorem 4.7. *Let $\mathcal{K}$ be a vcoil vectorspace category, Γ a non-stable v-coil of $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$. Then there exists a critical full convex subcategory $\mathcal{K}'$ of $\mathcal{K}$ and a stable tube Γ' of $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K}')}$ such that Γ is obtained from Γ' by a sequence of admissible operations and $\mathcal{K}_1 = Supp\Gamma$ is obtained from $\mathcal{K}'$ by the corresponding one-point extensions or coextensions.*

Proof. It is easy to prove that, if Γ is obtained from a stable tube Γ' by a sequence of admissible operations, $\mathcal{K}_1$ is obtained from $\mathcal{K}' = \mathrm{Supp}\Gamma'$ by the corresponding one point (co-)extensions (similar to that in [2]). It is sufficient to prove that $\mathcal{K}'$ is a critical convex subcategory in $\mathcal{K}_1$. We first prove the convexness of $\mathcal{K}'$. Otherwise, we assume that $X \to X_1 \to \cdots \to X_n \to Y$ is a chain of nonzero nonisomorphic morphisms between indecomposable objects in $\mathcal{K}_1$ with $X,\ Y \in \mathcal{K}'$, but each $X_i \notin \mathcal{K}'$. Then X_i is a coextension point or an extension point in $\mathcal{K}_1$. By duality, we may assume that X_1 is an extension point, then the projective object $\overline{X_1} \in \Gamma$, and $\overline{X_1} \to \cdots \overline{X_n} \to \overline{Y} \to M$, where $M \in \Gamma'$. Hence there exists a chain of irreducible maps in $\Gamma : M \to \cdots \to \overline{X_1}$, because Γ is a v-coil. Connecting the two chains above, we get a cycle, which lies in Γ. Then projective object $\overline{Y} \in \Gamma$, which is a contradiction. This follows that $\mathcal{K}'$ is convex in $\mathcal{K}_1$. We secondly prove that $\mathcal{K}'$ is a critical category. By Theorem 4.6, it suffices to prove that $\mathcal{K}'$ is not tubular. Otherwise, $\mathcal{K}_1$ is obtained from a tubular vectorspace category by one point extensions or coextensions. For simplicity and by duality, we assume that $\mathcal{K}_1$ is a one point extension of tubular category $\mathcal{K}'$ by $M \in \Gamma'$. Then, in the notation of [11, 5.8], we have that $\mathrm{ind}\overset{\vee}{\mathcal{V}}(\mathcal{K}_1)$ is of the form $\mathcal{P}_0 \bigcup \tau_0 \bigcup (\bigcup_{q \in \mathbb{Q}^+} \tau_q) \bigcup \tau_\infty \bigcup \mathcal{Q}_\infty$, where $\mathcal{P}_0$ is a preprojective component, Q_∞ is a preinjective component and $\tau_q,\ q \in \mathbb{Q}^+ \bigcup \{0, \infty\}$, are $\mathbb{P}(k)$-families of tubes. It is easy to see that M must be in τ_∞, otherwise, $\mathcal{K}_1$ will be not of tame type [2]. This is a contradiction, because τ_∞ does not contain any sincere stable tube. Then $\mathcal{K}'$ is a critical vectorspace category. The proof is finished.

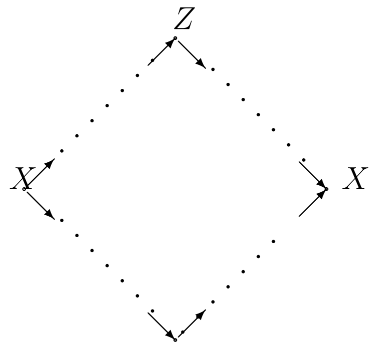
Corollary 4.8. *Let $\mathcal{K}$ be a vcoil vectorspace category, Γ a non-stable v-coil of $\Gamma_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}$, $M \in \Gamma$. If $SuppM$ contains a critical subcategory $\mathcal{K}_0$, then $SuppM$ is obtained from $\mathcal{K}_0$ by at most once one-point extension and at most once one-point coextension.*

Proof. The conclusion is deduced from the position of M in Γ. By Theorem 4.7, there is a critical subcategory $\mathcal{K}_0$ such that $\mathcal{K}$ is obtained from $\mathcal{K}_0$ by the one-point extensions or one-point coextensions which correspond to those operations in the defining Γ. Suppose Supp$M$ contains $\mathcal{K}_0$ as an additive subcategory, If M is on the ray starting at a projective $\overline{\alpha}$, and is on the coray ending at an injective β, then Supp$M = \mathrm{add}(\mathcal{K}_0 \bigcup \{\alpha, \beta\})$ and α is the extension vertex of $\mathcal{K}_0$, β is the coextension vertex of $\mathcal{K}_0$. If M is on the ray starting at a projective $\overline{\alpha}$, and is not on any coray ending at an injective, then Supp$M = \mathrm{add}(\mathcal{K}_0 \bigcup \{\alpha\})$ and α is the extension vertex of $\mathcal{K}_0$. If M is not on any ray starting at a projective but is on a coray ending at an injective β, then Supp$M = \mathrm{add}(\mathcal{K}_0 \bigcup \{\beta\})$ and β is the coextension vertex of $\mathcal{K}_0$. If M is not on any ray starting at a projective and is not on any coray ending at an injective, then Supp$M = \mathcal{K}_0$. The proof is finished.

Theorem 4.9. *Let $\mathcal{K}$ be a vcoil vectorspace category, $\chi_{\mathcal{K}}$ the quadratic form of $\mathcal{K}$. Then for any $X \in ind \overset{\vee}{\mathcal{V}}(\mathcal{K})$, $\chi_{\mathcal{K}}(\underline{dim}X) \in \{0, 1, 2\}$.*

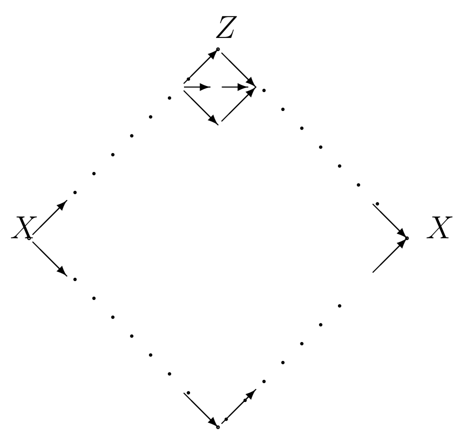
Proof. For $X \in \mathrm{ind} \overset{\vee}{\mathcal{V}}(\mathcal{K})$, by [11, 2.5], $\chi_{\mathcal{K}}(\underline{\mathrm{dim}}X) = \mathrm{dimHom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X)$ $-$ $\mathrm{dimExt}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X)$. Since $\mathcal{K}$ is a vcoil vectorspace category, the indecomposable representations are divided into two non-intersecting classes, one of which consists of directing indecomposable objects, another consists of non-directing ones, which are contained in some v-standard v-coils. For each directing object $X \in \mathrm{ind} \overset{\vee}{\mathcal{V}}(\mathcal{K})$, $\mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) = k$, $\mathrm{Ext}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) = 0$. Hence, $\chi_{\mathcal{K}}(\underline{\mathrm{dim}}X) = 1$ [11, 2.5]. For each non-directing object $X \in \mathrm{ind} \overset{\vee}{\mathcal{V}}(\mathcal{K})$, we suppose that Γ is a v-standard v-coil containing X. Because of the v-standardness of Γ, we have that $\mathrm{dim}_k\mathrm{Ext}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) \leq \mathrm{dim}_k\mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X)$. It suffices to prove that $\mathrm{dim}_k\mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) \leq 3$, and that, if $\mathrm{dim}_k\mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) = 3$, $\mathrm{dimExt}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) \geq 1$. For $X \in \Gamma$, X is on a ray and a coray. We divide our consideration into several cases according to the position of X on Γ.

(1). If the sectional sub-path, from a mouth vertex to X, of the ray containing X intersects the sectional sub-path, from X to a mouth vertex, of the coray containing X at a non-projective, non-injective vertex Z, i.e Γ contains a full subquiver of the form:



then $\dim_k \mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X,X) = 2, \ \dim_k \mathrm{Ext}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X,X) = 1.$

(2). If the sectional sub-path, from a mouth vertex to X, of the ray containing X intersects the sectional sub-path, from X to a mouth vertex, of the coray containing X at a projective, or an injective vertex Z, which is a middle term of a mesh with three middle terms, i.e Γ contains a full subquiver of the form:

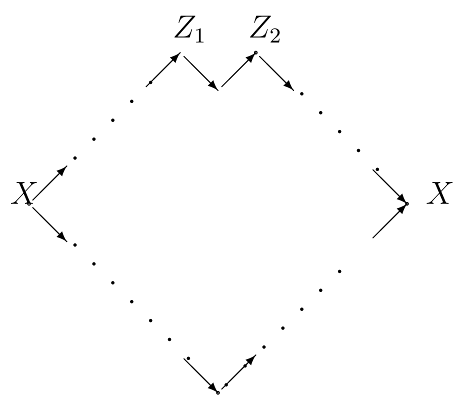


then $\dim_k \mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X,X) = 3, \quad \dim_k \mathrm{Ext}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X,X) = 1.$

(3). If the sectional sub-path, from a mouth vertex to X, of the ray containing X intersects the sectional sub-path, from X to a mouth vertex, of the coray containing X at a projective (or an injective) vertex Z, which is not a middle term of a mesh with three middle terms, but the sectional path from Z to X (from X to Z resp.) contains an injective (a projective resp.) middle term of a mesh with three middle terms, similar as in (2), we have that $\dim_k \mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X,X) = 3, \ \dim_k \mathrm{Ext}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X,X) = 1.$

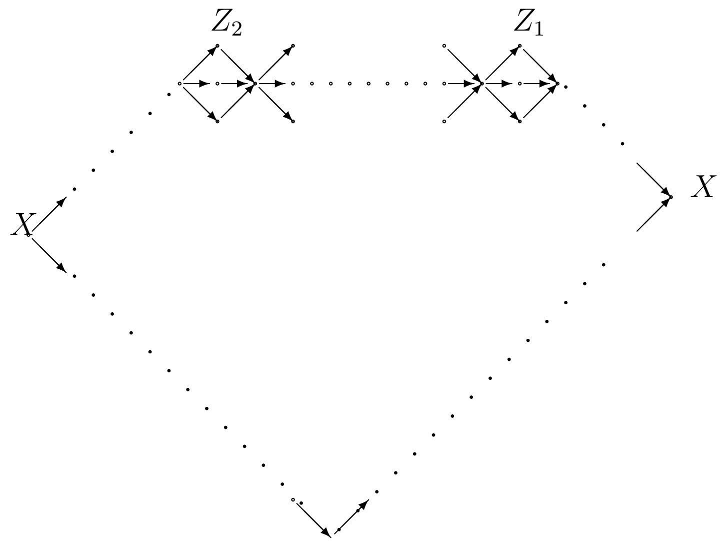
(4). If the sectional sub-path, from a mouth vertex Z_2 to X, of the ray containing X does not intersect the sectional sub-path, from X to a mouth vertex Z_1, of the coray containing X, with Z_1 non-injective, or Z_2 non-projective, then $\dim_k \mathrm{Hom}_{\overset{\vee}{\mathcal{V}(\mathcal{K})}}(X, X) \leq 2,\ \dim_k \mathrm{Ext}_{\overset{\vee}{\mathcal{V}(\mathcal{K})}}(X, X) = 0$ or 1.

(5). If the sectional sub-path, from a mouth vertex Z_1 to X, of the ray containing X does not intersect the sectional sub-path, from X to a mouth vertex Z_2, of the coray containing X, with Z_1 injective, and Z_2 projective, and Z_i are not middle terms of some mesh with three middle terms. i.e Γ contains a full subquiver of the form:



then $\dim_k \mathrm{Hom}_{\overset{\vee}{\mathcal{V}(\mathcal{K})}}(X, X) = 1,\ \dim_k \mathrm{Ext}_{\overset{\vee}{\mathcal{V}(\mathcal{K})}}(X, X) = 1.$

(6). If the sectional sub-path, from a mouth vertex Z_1 to X, of the ray containing X does not intersect the sectional sub-path, from X to a mouth vertex Z_2, of the coray containing X, with Z_1 injective, and Z_2 projective, and Z_i are middle terms of some mesh with three middle terms. i.e Γ contains a full subquiver of the form:



then $\dim_k \mathrm{Hom}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) = 3,\ \dim_k \mathrm{Ext}_{\overset{\vee}{\mathcal{V}}(\mathcal{K})}(X, X) = 1.$
The discussions above finish the proof.

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