Smash Products of Quasi-hereditary Graded Algebras

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Abstract. The Smash product of a finite dimensional quasi-hereditary algebra graded by a finite group with the group is proved to be a quasi-hereditary algebra. Some elementary relations between the good modules of the two quasi-hereditary algebras are given.

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Quasi-hereditary algebras have been defined by Cline, Parshall, and Scott [2] in order to deal with highest weight categories as they arise in the representation theory of semisimple complex Lie-algebras. Many algebras which arise rather natural have been shown to be quasi-hereditary: the Schur algebras[9], the Auslander algebras[4], and the endomorphism algebra of direct sum of all indecomposable $\Delta$-good modules for any $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra[13] , and so on. In this note, we will prove that the smash product $\Lambda \# G^\ast$ of a quasi-hereditary G-graded algebra $\Lambda$ with $G$ is a quasi-hereditary algebra.

Quasi-hereditary algebras depend heavily on the ordering of the simple modules. Let $\mathcal{K}$ be an algebraically closed field. For a finite dimensional algebra $\Lambda$ over $\mathcal{K}$, we fix an ordering on the simple $\Lambda$–modules: $S(1), S(2), \cdots, S(n)$. Let $P(i)$ be the projective cover of $S(i)$, $Q(i)$ the injective envelope of $S(i)$. We denote by $\Delta(i)$ the maximal factor of $P(i)$ with composition factors of the form $S(j)$, where $j \leq i$, and similarly, let $\nabla(i)$ be the maximal submodule of $Q(i)$ with composition factors of form $S(j)$,where $j \leq i$. Let $\Delta = \{\Delta(1), \cdots, \Delta(n)\}$, $\nabla = \{\nabla(1), \cdots, \nabla(n)\}$. We denote by $\mathcal{F}(\Sigma)$ the full subcategory of $\text{mod}\Lambda$ consisting of modules which have a filtration with factors in $\Sigma$, where $\text{mod}\Lambda$ denotes the category of f.g. left modules over $\Lambda$. 
and $\Sigma$ is a set of modules. This modules are said to be $\Sigma$-good. The algebra $\Lambda$ is called quasi-hereditary with respect to the ordering of simple modules (written as q.h. algebra for simplicity) if $\text{End}(\Delta(i))$ is a division ring, for all $i = 1, \cdots, n$, and $P(i) \in \mathcal{F}(\Lambda)$ for all $i$. For a q.h. algebra, $\mathcal{F}(\Lambda)$ has (relative) Auslander-Reiten sequences [10]. In case there are only finitely many isomorphism classes of indecomposable $A$-modules which belong to $\mathcal{F}(\Delta)$, we say that $A$ is $\mathcal{F}(\Delta)$-finite.

**Definition 1.** A $\mathcal{K}$-algebra $\Lambda$ is called a $G$-graded algebra if $\Lambda = \bigoplus_{g \in G} \Lambda_g$ (as $\mathcal{K}$-subspaces) and $\Lambda_g \Lambda_h \subseteq \Lambda_{gh}$, for all $g, h \in G$.

For a $G$-graded algebra, we can construct another $\mathcal{K}$-algebra, which is called the smash product of $\Lambda$ with $G$ (compare [1], [3], [7]) as follows: let

$$\Lambda^* \# G = \bigoplus_{g \in G} \Lambda_p g$$

be the free left $\Lambda$-module on the generators $p_g, g \in G$. Elements $xp_g$, $yp_h$ multiply by

$$(xp_g)(yp_h) = xy_{gh^{-1}} p_h,$$

where $x_g$ denotes the $g$-component of $x$.

With this multiplication, the free module is an associative $\mathcal{K}$-algebra. $\Lambda^* \# G$ has no identity, but it has local identity in general. In case of $G$ being finite, $\Lambda^* \# G$ has an identity. It was proved in [7] that $\Lambda^* \# G \cong (\Lambda_{gh^{-1}})_{G \times G}$, where $(\Lambda_{gh^{-1}})_{G \times G}$ denotes the ring consisting of functions from $G \times G$ to $\Lambda$ with only finitely many non-zero values, and such that $f(g, h)$ must be in $\Lambda_{gh^{-1}}$.

Let $\Lambda = \bigoplus_{g \in G} \Lambda_g$ be a $G$-graded algebra with identity 1 and with a complete set of orthogonal idempotents $e_i, i = 1, \cdots, n$, and with $e_i \in \Lambda_e$. We denote the category of $G$-graded $\Lambda$-module by $\Lambda$-Gr, denote the full subcategory of $\Lambda$-Gr consisting of finite generated $\Lambda$-module by $\Lambda$-Gr. There is an isomorphic functor $F$ from $\Lambda^* \# G$-mod to $\Lambda$-Gr, whose restriction on $\Lambda^* \# G$-mod is again an isomorphic functor from $\Lambda^* \# G$-mod to $\Lambda$-gr, which we denote also by $F$ [7]. For a graded module $M$, $M[g]$ denotes the suspension image of $M$ [8]. A module $M$ in $\Lambda$-mod is called gradable if there exists a graded module $M_1$ such that $M = M_1$ as $\Lambda$-modules. The following lemma was proved in [5] in the case $\Lambda$ is $Z$-graded, here we point out it is true for arbitrary group graded algebras.

**Lemma 1.** Let $\Lambda$ be a $G$-graded algebra. Then each simple $\Lambda$-module is gradable.

**Proof.** Let $S(i)$ be simple $\Lambda$-module. Then $S(i) = \frac{P(i)}{\text{rad}P(i)}$. We know that $P(i) = \Lambda e_i, e_i \in \Lambda_e$, is gradable. Then if we prove that $\text{rad} P(i)$ is gradable, this will imply that $S(i)$ is gradable. We notice that

$$\text{rad} P(i) = \Sigma_{f \in \text{Hom}(P, P(i)), P \text{ is proj}, f \text{ is not epi}} \text{Im}(f).$$

For every homomorphism $f \in \text{Hom}_\Lambda(P, P(i))$, we can write $f$ as $f = \Sigma_{g \in G} f_{g}$,
where \( f_g \) is a homomorphism of degree \( g \) from \( P \) to \( P(i) \) [8]( where \( P \) and \( P(i) \) are viewed as graded modules). Then

\[
\sum_{f \in \text{Hom}(P,P(i)), P \text{ is proj, } f \text{ is not epi}} \frac{\text{Im}(f)}{\text{Im}(f)} = \sum_{f \in \bigcup_{g \in G} \text{Hom}_{\Lambda^g}(P\cdot g, P(i)[e]).P \text{ is proj, } f \text{ is not epi}} \frac{\text{Im}(f)}{\text{Im}(f)}.
\]

Then \( radP(i) \) is gradable. This finishes the proof.

Throughout this note, we assume that the group \( G \) is a finite group. Let \( S \) be the set of simple \( \Lambda - \) modules ( we pick exactly only one from each isomorphism class ), and we fix an ordering on \( S \) i.e. \( S = \{ S(1), \cdots, S(n) \} \). We now define the simple \( \Lambda^G \)-modules. For \( (g, \lambda) \in G \times S \), let \( e_{(g,\lambda)} \) be the idempotent of \( \Lambda^G \) whose \( (g, g) \)-entry is \( e_{\lambda} \), the others are zero.

Then the set \( \{ e_{(g,\lambda)} \mid (g, \lambda) \in G \times S \} \) is a complete set of orthogonal primitive idempotents. For each primitive idempotent, we get a simple module. So we denote by \( G \times S \) the set of non-isomorphic simple \( \Lambda^G \)-modules. By the lemma above, we have \( G \times S = \bigcup_{g \in G} \{ S(g, 1), \cdots, S(g, n) \} \), where \( F(S(g,i)) = S(i)[g] \), the suspension image of a graded module \( S(i) \). We will define an ordering on this set. We first fix an ordering on set \( G \) (without any restriction). Then we define the ordering on \( G \times S \) as follows: \( (g, \lambda) \leq (h, \lambda_1) \) if and only if \( \lambda \leq \lambda_1 \) or if \( \lambda = \lambda_1 \), but \( g \leq h \).

**Theorem 2.** If \( (\Lambda, S) \) is a finite dimensional q.h. G-graded \( \mathcal{K} \)-algebra with identity, where \( \Lambda = \bigoplus_{g \in G} \Lambda_g \), then \( (\Lambda^G, G \times S) \) is a q.h. algebra.

**Proof.** Let \( P(g, \lambda) = \Lambda^G e_{(g, \lambda)} \) be the indecomposable projective \( \Lambda^G \)-module corresponding to the primitive idempotent \( e_{(g, \lambda)} \). It is the projective cover of simple \( \Lambda^G \)-module \( S(g, \lambda) \). Let \( \Delta(g, \lambda) \) denote the maximal factor module of \( P(g, \lambda) \) with composition factors of form \( S(h, \delta) \), where \( (h, \delta) \leq (g, \lambda) \). Then \( \Delta(g, \lambda) = \frac{P(g, \lambda)}{U(g, \lambda)} \), where

\[
U(g, \lambda) = \sum_{f \in \bigcup_{(g, \lambda) > (g, \lambda)} \text{Hom}(P(h, \delta), P(g, \lambda))} \text{Im}(f) = N_1 + N_2,
\]

where

\[
N_1 = \sum_{f \in \bigcup_{(g, \lambda) > (g, \lambda)} \text{Hom}(P(h, \delta), P(g, \lambda))} \text{Im}(f), \quad N_2 = \sum_{f \in \bigcup_{(g, \lambda) > (g, \lambda)} \text{Hom}(P(h, \lambda), P(g, \lambda))} \text{Im}(f).
\]

We know that \( \Delta(\lambda) = \frac{P(\lambda)}{U(\lambda)} \), where \( U(\lambda) = \sum_{f \in \bigcup_{(g, \lambda) > (g, \lambda)} \text{Hom}(P(\delta), P(\lambda))} \text{Im}(f) \). It is proved in [8] that \( f = \sum_{g \in G} f_g \), where \( f_g \) is a homomorphism of degree \( g \) from \( P(\delta) \) to \( P(\lambda) \)(where \( P(\delta) \) and \( P(\lambda) \) are viewed as graded modules). So we have the equality

\[
\sum_{f \in \bigcup_{(g, \lambda) > (g, \lambda)} \text{Hom}(P(\delta), P(\lambda))} \text{Im}(f) = \sum_{f \in \bigcup_{h \in G} \text{Hom}_{\Lambda^g}(P(\delta)[h], P(\lambda)[e])} \text{Im}(f).
\]

Then \( U(\lambda) \) is a graded submodule of \( P(\lambda)[e] \). Therefore \( \Delta(\lambda) \) is a gradable module. We will prove that \( N_1 \geq N_2 \).

We denote by \( \text{Im}(\text{Hom}_{\Lambda^G}(P(g, k), P(h, k))) \) the sum of image of all homomorphisms from \( P(g, k) \) to \( P(h, k) \). Then \( \text{Im}(\text{Hom}_{\Lambda^G}(P(g, k), P(h, k))) \)
is a submodule of $P(h,k)$ generated by $e_kA_{hg^{-1}}e_k$. Then $e_kA_{hg^{-1}}e_k \subseteq \text{Im}(\text{Hom}_{\Lambda} - gr(\bigoplus_{j \geq k, g \in G} P(j)[g], P(k)[h]))$. Otherwise we have that $e_kA_{hg^{-1}}e_k \not\subseteq \text{Im}(\text{Hom}_{\Lambda}(\bigoplus_{j > k} P(j), P(k)))$. This is because

$$\text{Im}(\text{Hom}_{\Lambda} - gr(\bigoplus_{j > k} P(j)[g], P(k)[h])) = \text{Im}(\text{Hom}_{\Lambda}(\bigoplus_{j > k} P(j), P(k))).$$

So we get $[\Delta(k) : S(k)] \geq 2$, contradicting the fact that $\Lambda$ is a q.h. algebra.

Then $F(\Delta(g,k)) = \Delta(k)[g]$, and $\text{End}_{\Lambda#G^*}(\Delta(g,k)) \cong \text{End}_{\Lambda - gr}(\Delta(k)[g]) \cong \text{END}_{\Lambda}(\Delta(k))_g$, the $e$-component of $\text{END}_{\Lambda}(\Delta(k)) = \text{End}_{\Lambda}(\Delta(k))[8]$, the later being a division ring, where $\text{END}_{\Lambda}(\Delta(k)) = \bigoplus_{g \in G} \text{End}_{\Lambda}(\Delta(k))[g]$, and $\text{End}_{\Lambda}(\Delta(k))_g$ is the additive subgroup of $\text{End}_{\Lambda}(\Delta(k))$ consisting of homomorphisms of degree $g$ from $\Delta(k)$ to itself. So $\text{End}_{\Lambda#G^*}(\Delta(g,k))$ is a division ring.

We can get a $F(\Lambda#G^*)$ -filtration of $P(g,k)$ from the $F(\Delta)$ -filtration of $P(k)$. Therefore $\Lambda#G^*$ is a q.h. algebra with respect to the ordering $G \times S$. This finishes the proof.

**Example.** Let $\Lambda : 1 \rightarrow 2$, $\alpha \beta = 0$. Then $\Lambda$ is a q.h. algebra with respect to the natural ordering. Let $\Lambda_1 = Ke_1 + Ke_2$, $\Lambda_2 = K\alpha + K\beta$, $\Lambda_3 = K\beta\alpha$. Then $\Lambda$ is a $Z_3$ -graded algebra. The smash product $\Lambda#Z^*_3$ is as following,

$$\begin{array}{cccc}
1 & \alpha_1 & 4 & \beta_1 \\
\beta_3 & \uparrow & \downarrow & \alpha_2 \\
6 & \alpha_3 & 3 & \beta_2 & 5 \\
\end{array}$$

It is a q.h. algebra with respect to the natural ordering.

We have mentioned above that there is a functor $F : \Lambda#G^* - mod \rightarrow \Lambda - gr$. In the following, we will give some relations between the two categories of $\Delta$- good modules of the two algebras. We will call that a $G$-graded algebra $\Lambda$ is a $G$-graded algebra with property P if $M[g] \not\cong M$ for each indecomposable module in $\Lambda - mod$, and $g \neq e$. For example all $Z$-graded algebras have property P, the algebra in the example above has also the property P.

**Proposition 3.** Let $\Lambda$ be a $G$-graded algebra with property P, $M \in \Lambda - gr$, $\bar{M} \in \Lambda#G^* - mod$, with $F(\bar{M}) = M$. Then $M$ is indecomposable in $\Lambda - mod$ if and only if $\bar{M}$ is indecomposable in $\Lambda#G^* - mod$.

**Proof.** The necessity is obvious. We prove the sufficiency. $\text{End}_{\Lambda#G^*}(\bar{M}) \cong \text{End}_{\Lambda - gr}M$ is the initial ring of locally $G$-graded ring $\text{End}_{\Lambda}M$. We prove that $(\text{End}_{\Lambda}M)_g \cdot (\text{End}_{\Lambda}M)_{g^{-1}} \subseteq \text{rad}(\text{End}_{\Lambda - gr}M)$, for all $g \neq e$. Otherwise, we
have $f_i \in (\text{End}_{\Lambda} M)_g$, $h_i \in (\text{End}_{\Lambda} M)_{g^{-1}}$, such that $\sum_{i=1}^n f_i h_i = 1$. This implies that $M \cong M[g]$ by Krull-Schimdt theorem and the fact that $M$ is indecomposable, contradicting the property $P$. So we have that $\text{rad}(\text{End}_{\Lambda} M) = \text{rad}(\text{End}_{\Lambda - gr} M) + \sum_{g \neq e} (\text{End}_{\Lambda} M)_g$, and $\text{End}_{\Lambda} M$ is a local ring. This finishes the proof.

Proposition 4. Let $\Lambda$ be a q.h. $G$-graded algebra. Then $F(\mathcal{F}(\Lambda^G_\Delta)) \subseteq \mathcal{F}(\Lambda_\Delta)$. If $M \in \Lambda - gr$, $M \in \mathcal{F}(\Lambda_\Delta)$, then there is a module $\mathcal{M} \in \mathcal{F}(\Lambda^G_\Delta)$ such that $F(\mathcal{M}) = M$.

Proof. The first assertion follows from the fact that the $\Delta$-filtration of $X \in \mathcal{F}(\Lambda^G_\Delta)$ is preserved by $F$ and yields a $\Delta$-filtration of $F(X)$ in $\text{mod-}\Lambda$. So we prove the second assertion. Let $M \in \Lambda - gr$, $M \in \mathcal{F}(\Lambda_\Delta)$. Let $f : P(i) \rightarrow M$ be a homomorphism. Then $f = \sum_{\sigma} f_\sigma$, where $f_\sigma$ is a homomorphism of degree $\sigma$. Then $\text{Im}(f) \subseteq \Sigma_{\sigma, f_\sigma}$. So $\Sigma_{f \in \text{Hom}_G(P(i), M)} \text{Im}(f) = \Sigma_{f \in \bigcup_{g \in G} \text{Hom}_G(P(i)[g], M)} \text{Im}(f)$. It follows that if $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ is a $\Lambda_\Delta$-filtration, hence all $M_\ell$ are graded submodule of $M$. If we denote $F^{-1}(M)$ by $\mathcal{M}$, then $\mathcal{M} \in \mathcal{F}(\Lambda^G_\Delta)$. The proof is finished.

For a q.h. algebra, there is a (uniquely defined) basic module $T$ which is direct sum of all non-isomorphic injective objects in $\mathcal{F}(\Lambda_\Delta)$. This module is called the characteristic module of $\Lambda$ [10].

Theorem 5 Let $(\Lambda = \bigoplus_{g \in G} \Lambda_g, S)$ is a finite dimensional q.h. $G$-graded $\mathcal{K}$-algebra with property $P$. Then the characteristic modules $\Lambda^G_\Delta T = \bigoplus_{(i,g) \in S \times G} T(i,g)$, such that $F(T(i,g)) = F(T(i,e))[g]$, and $\bigoplus_{i \in S} F(T(i,e))$ is the characteristic module for $\Lambda$.

Proof. Let $\Lambda^G_\Delta T = \bigoplus_{(i,g) \in S \times G} T(i,g)$ be the characteristic module for $\Lambda^G_\Delta$. Then $\Lambda^G_\Delta T \in \mathcal{F}(\Lambda^G_\Delta) \mathcal{F}(\Lambda^G_\Delta)$. We have $F(T(i,g)) \in \mathcal{F}(\Lambda_\Delta)$, and $F(T(i,g)) \in \mathcal{F}(\Lambda_\Delta)$ by duality. So $\bigoplus_{i \in S} F(T(i,e))$ is a characteristic module for $\Lambda$, by proposition 4.

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References


