Smash Products of Quasi-hereditary Graded Algebras

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Abstract. The Smash product of a finite dimensional quasi-hereditary algebra graded by a finite group with the group is proved to be a quasi-hereditary algebra. Some elementary relations between the good modules of the two quasi-hereditary algebras are given.

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Quasi-hereditary algebras have been defined by Cline, Parshall, and Scott [2] in order to deal with highest weight categories as they arise in the representation theory of semisimple complex Lie-algebras. Many algebras which arise rather natural have been shown to be quasi-hereditary: the Schur algebras[9], the Auslander algebras[4], and the endomorphism algebra of direct sum of all indecomposable Δ-good modules for any $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra[13] , and so on. In this note, we will prove that the smash product $\Lambda\#G^*$ of a quasi-hereditary G-graded algebra Λ with G is a quasi-hereditary algebra.

Quasi-hereditary algebras depend heavily on the ordering of the simple modules. Let $\mathcal{K}$ be an algebraically closed field. For a finite dimensional algebra Λ over $\mathcal{K}$, we fix an ordering on the simple $\Lambda-$modules: S(1), S(2), $\cdots$, S(n). Let P(i) be the projective cover of S(i), Q(i) the injective envelope of S(i). We denote by $\Delta(i)$ the maximal factor of P(i) with composition factors of the form S(j), where $j \leq i$, and similarly, let $\bigtriangledown(i)$ be the maximal submodule of Q(i) with composition factors of form S(j),where $j \leq i$. Let $\Delta = \{\Delta(1), \cdots, \Delta(n)\}, \bigtriangledown = \{\bigtriangledown(1), \cdots, \bigtriangledown(n)\}$. We denote by $\mathcal{F}(\Sigma)$ the full subcategory of $mod\Lambda$ consisting of modules which have a filtration with factors in Σ, where $mod\Lambda$ denotes the category of f.g. left modules over Λ

and Σ is a set of modules. This modules are said to be $\Sigma-$good. The algebra Λ is called quasi-hereditary with respect to the ordering of simple modules (written as q.h. algebra for simplicity) if $End(\Delta(i))$ is a division ring, for all $i = 1, \cdots, n$, and $P(i) \in \mathcal{F}(\Delta)$ for all i. For a q.h. algebra, $\mathcal{F}(\Delta)$ has (relative) Auslander-Reiten sequences[10]. In case there are only finitely many isomorphism classes of indecomposable A-modules which belong to $\mathcal{F}(\Delta)$, we say that A is $\mathcal{F}(\Delta)$-finite.

Definition 1. A $\mathcal{K}$-algebra Λ is called a G-graded algebra if $\Lambda = \bigoplus_{g\in G} \Lambda_g$ (as $\mathcal{K}$ -subspaces) and $\Lambda_g\Lambda_h \subseteq \Lambda_{gh}$, for all $g, h \in G$.

For a G-graded algebra, we can construct another $\mathcal{K}$-algebra, which is called the smash product of Λ with G(compare [1],[3],[7]) as follows: let

$\Lambda\#G^* = \bigoplus_{g\in G} \Lambda p_g$ be the free left $\Lambda-$module on the generators $p_g,\ g \in G$. Elements $xp_g,\ yp_h$ multiply by

$$(xp_g)(yp_h) = xy_{gh^{-1}}p_h,$$

where x_g denotes the g-component of x.

With this multiplication, the free module is an associative $\mathcal{K}-$ algebra. $\Lambda\#G^*$ has no identity, but it has local identity in general. In case of G being finite , $\Lambda\#G^*$ has an identity. It was proved in [7] that $\Lambda\#G^* \cong (\Lambda_{gh^{-1}})_{G\times G}$, where$(\Lambda_{gh^{-1}})_{G\times G}$, denotes the ring consisting of functions from $G \times G$ to Λ with only finitely many non-zero values, and such that f(g,h) must be in $\Lambda_{gh^{-1}}$.

Let $\Lambda = \bigoplus_{g\in G} \Lambda_g$ be a G-graded algebra with identity 1 and with a complete set of orthogonal idempotents $e_i, i = 1, ..., n$, and with $e_i \in \Lambda_e$. We denote the category of G-graded Λ-module by Λ-Gr, denote the full subcategory of Λ-Gr consisting of finite generated Λ-module by $\Lambda-$gr. There is an isomorphic functor F from $\Lambda\#G^*$-Mod to Λ-Gr, whose restriction on $\Lambda\#G^*$-mod is again an isomorphic functor from $\Lambda\#G^*$-mod to Λ-gr, which we denote also by F [7]. For a graded module M, $M[g]$ denotes the suspension image of M [8]. A module M in Λ-mod is called gradable if there exists a graded module M_1 such that $M = M_1$ as $\Lambda-$modules. The following lemma was proved in [5] in the case Λ is Z-graded, here we point out it is true for arbitrary group graded algebras.

Lemma 1. Let Λ be a G-graded algebra. Then each simple Λ-module is gradable.

Proof. Let S(i) be simple Λ-module. Then $S(i) = \frac{P(i)}{radP(i)}$. We know that $P(i) = \Lambda e_i, e_i \in \Lambda_e$, is gradable. Then if we prove that rad P(i) is gradable, this will imply that S(i) is gradable. We notice that

$$radP(i) = \Sigma_{f\in Hom(P,P(i)),P \text{ is proj, } f \text{ is not epi}} Im(f).$$

For every homomorphism $f \in Hom_\Lambda(P, P(i))$, we can write f as $f = \Sigma_{g\in G} f_g$, where f_g is a homomorphism of degree g from P to $P(i)$ [8](where P and $P(i)$ are viewed as graded modules). Then

$$\begin{aligned} & \Sigma_{f\in Hom(P,P(i)),P \ is \ proj, \ f \ is \ not \ epi} && Im(f) \\ = \ & \Sigma_{f\in\bigcup_{g\in G} Hom_{\Lambda-gr}(P[g],P(i)[e]),P \ is \ proj, \ f \ is \ not \ epi} && Im(f) \end{aligned}.$$

Then $radP(i)$ is gradable. This finishes the proof.

Throughout this note, we assume that the group G is a finite group. Let S be the set of simple $\Lambda-$ modules (we pick exactly only one from each isomorphism class), and we fix an ordering on S i.e.$S = \{S(1), \cdots, S(n)\}$. We now define the simple $\Lambda\#G^*-$ modules. For $(g, \lambda) \in G \times S$, let $e_{(g,\lambda)}$ be the idempotent of $\Lambda\#G^*$ whose (g, g)-entry is e_λ, the others are zero. Then the set $\{e_{(g,\lambda)} \mid (g, \lambda) \in G \times S\}$ is a complete set of orthogonal primitive idempotents. For each primitive idempotent, we get a simple module. So we denote by $G \times S$ the set of non-isomorphic simple $\Lambda\#G^*$- modules. By the lemma above, we have $G \times S = \bigcup_{g\in G}\{S(g, 1), \cdots, S(g, n)\}$, where F(S(g,i))= S(i)[g], the suspension image of a graded module S(i). We will define an ordering on this set. We first fix an ordering on set G(without any restriction). Then we define the ordering on $G\times S$ as follows: $(g, \lambda) \leq (h, \lambda_1)$ if and only if $\lambda \leq \lambda_1$ or if $\lambda = \lambda_1$, but $g \leq h$.

Theorem 2. If (Λ, S) is a finite dimensional q.h. G-graded $\mathcal{K}-$algebra with identity, where $\Lambda = \bigoplus_{g\in G} \Lambda_g$, then $(\Lambda\#G^*, G \times S)$ is a q.h. algebra.

Proof. Let $P(g, \lambda) = \Lambda\#G^*e(g, \lambda)$ be the indecomposable projective $\Lambda\#G^*$module corresponding to the primitive idempotent $e(g, \lambda)$. It is the projective cover of simple $\Lambda\#G^*$-module $S(g, \lambda)$. Let $\Delta(g, \lambda)$ denote the maximal factor module of $P(g, \lambda)$ with composition factors of form $S(h, \delta)$, where $(h, \delta) \leq (g, \lambda)$. Then $\Delta(g, \lambda) = \frac{P(g,\lambda)}{U(g,\lambda)}$, where

$$U(g, \lambda) = \Sigma_{f\in\bigcup_{(h,\delta)>(g,\lambda)} Hom(P(h,\delta),P(g,\lambda))} Im(f) = N_1 + N_2 ,$$

where

$$N_1 = \Sigma_{f\in\bigcup_{\delta>\lambda} Hom(P(h,\delta),P(g,\lambda))}Im(f), \ N_2 = \Sigma_{f\in\bigcup_{(h,\lambda)>(g,\lambda)} Hom(P(h,\lambda),P(g,\lambda))}Im(f).$$

We know that $\Delta(\lambda) = \frac{P(\lambda)}{U(\lambda)}$, where $U(\lambda) = \Sigma_{f\in\bigcup_{\delta>\lambda} Hom(P(\delta),P(\lambda))}Im(f)$. It is proved in [8] that $f = \Sigma_{g\in G}f_g$, where f_g is a homomorphism of degree g from $P(\delta)$ to $P(\lambda)$(where $P(\delta)$ and $P(\lambda)$ are viewed as graded modules). So we have the equality

$$\Sigma_{f\in\bigcup_{\delta>\lambda} Hom(P(\delta),P(\lambda))}Im(f) = \Sigma_{f\in\bigcup_{\delta>\lambda, \ h\in G} Hom_{\Lambda-gr}(P(\delta)[h],P(\lambda)[e])}Im(f).$$

Then $U(\lambda)$ is a graded submodule of $P(\lambda)[e]$. Therefore $\Delta(\lambda)$ is a gradable module. We will prove that $N_1 \supseteq N_2$.

We denote by $Im(Hom_{\Lambda\#G^*}(P(g, k), P(h, k)))$ the sum of image of all homomorphisms from P(g,k) to P(h,k). Then $Im(Hom_{\Lambda\#G^*}((P(g, k), P(h, k)))$

is a submodule of P(h,k) generated by $e_k\Lambda_{hg^{-1}}e_k$. Then $e_k\Lambda_{hg^{-1}}e_k = 0$ or $e_k\Lambda_{hg^{-1}}e_k \subseteq Im(Hom_{\Lambda-gr}(\bigoplus_{j>k,g\in G} P(j)[g], P(k)[h]))$. Otherwise we have that $e_k\Lambda_{hg^{-1}}e_k \neq 0$, and $e_k\Lambda_{hg^{-1}}e_k \nsubseteq Im(Hom_{\Lambda}(\bigoplus_{j>k} P(j), P(k)))$. This is because

$$Im(Hom_{\Lambda-gr}(\bigoplus_{j>k,g\in G} P(j)[g], P(k)[h])) = Im(Hom_{\Lambda}(\bigoplus_{j>k} P(j), P(k)).$$

So we get $[\Delta(k) : S(k)] \geq 2$, contradicting the fact that Λ is a q. h. algebra. Then $F(\Delta(g,k)) = \Delta(k)[g]$, and $End_{\Lambda\#G^*}(\Delta(g,k)) \cong End_{\Lambda-gr}(\Delta(k)[g])$ $\cong END_{\Lambda}(\Delta(k))_e$, the e-component of $END_{\Lambda}(\Delta(k)) = End_{\Lambda}(\Delta(k))$[8], the later being a division ring, where $END_{\Lambda}(\Delta(k)) = \bigoplus_{g\in G} End_{\Lambda}(\Delta(k))_g$, and $End_{\Lambda}(\Delta(k))_g$ is the additive subgroup of $End_{\Lambda}(\Delta(k))$ consisting of homomorphisms of degree g from $\Delta(k)$ to itself. So $End_{\Lambda\#G^*}(\Delta(g,k))$ is a division ring. We can get a $\mathcal{F}({}_{\Lambda\#G^*}\Delta)$ -filtration of P(g,k) from the $\mathcal{F}({}_{\Lambda}\Delta)$ -filtration of P(k). Therefore $\Lambda\#G^*$ is a q.h. algebra with respect to the ordering $G \times S$. This finishes the proof.

Example. Let $\Lambda : 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2, \quad \alpha\beta = 0$. Then Λ is a q.h.algebra with respect to the natural ordering. Let $\Lambda_{\overline{0}} = \mathcal{K}e_1 + \mathcal{K}e_2,\ \Lambda_{\overline{1}} = \mathcal{K}\alpha + \mathcal{K}\beta,\ \Lambda_{\overline{2}} = \mathcal{K}\beta\alpha$. Then Λ is a Z_3 -graded algebra. The smash product $\Lambda\#Z_3^*$ is as following,

$$\begin{array}{ccccc} 1 & \xrightarrow{\alpha_1} & 4 & \xrightarrow{\beta_1} & 2 \\ \beta_3 \uparrow & & & & \downarrow \alpha_2 \\ 6 & \xleftarrow[\alpha_3]{} & 3 & \xleftarrow[\beta_2]{} & 5 \end{array}, \qquad \alpha_i\beta_i = 0, \quad i = 1, 2, 3.$$

It is a q.h. algebra with respect to the natural ordering.

We have mentioned above that there is a functor $F : \Lambda\#G^* - mod \longrightarrow \Lambda - gr$. In the following, we will give some relations between the two categories of $\Delta-$ good modules of the two algebras. We will call that a G-graded algebra Λ is a G-graded algebra with property P if $M[g] \ncong M$ for each indecomposable module in $\Lambda - mod$, and $g \neq e$. For example all Z-graded algebras have property P, the algebra in the example above has also the property P.

Proposition 3. Let Λ be a G-graded algebra with property P, $M \in \Lambda - gr$, $\overline{M} \in \Lambda\#G^* - mod$, with $F(\overline{M}) = M$. Then M is indecomposable in $\Lambda - mod$ if and only if $\overline{M}$ is indecomposable in $\Lambda\#G^* - mod$.

Proof. The necessity is obvious. We prove the sufficiency. $End_{\Lambda\#G^*}\overline{M} \cong End_{\Lambda-gr}M$ is the initial ring of locally G-graded ring $End_{\Lambda}M$. We prove that $(End_{\Lambda}M)_g \cdot (End_{\Lambda}M)_{g^{-1}} \subseteq rad(End_{\Lambda-gr}M)$, for all $g \neq e$. Otherwise, we have $f_i \in (End_\Lambda M)_g, h_i \in (End_\Lambda M)_{g^{-1}}$, such that $\Sigma_{i=1}^n f_i h_i = 1$. This implies that $M \cong M[g]$ by Krull-Schimdt theorem and the fact that M is indecomposable, contradicting the property P. So we have that $rad(End_\Lambda M) = rad(End_{\Lambda-gr}) + \Sigma_{g\neq e}(End_\Lambda M)_g$, and $End_\Lambda M$ is a local ring. This finishes the proof.

Proposition 4. Let Λ be a q.h. G-graded algebra. Then $F(\mathcal{F}({}_{\Lambda\#G^*}\Delta)) \subseteq \mathcal{F}({}_\Lambda\Delta)$. If $M \in \Lambda - gr, M \in \mathcal{F}({}_\Lambda\Delta)$, then there is a module $\overline{M} \in \mathcal{F}({}_{\Lambda\#G^*}\Delta)$ such that $F(\overline{M}) = M$.

Proof. The first assertion follows from the fact that the Δ- filtration of $X \in \mathcal{F}({}_{\Lambda\#G^*}\Delta$ is preserved by F and yields a Δ-filtration of F(X) in mod-Λ. So we prove the second assertion. Let $M \in \Lambda - gr$, $M \in \mathcal{F}({}_\Lambda\Delta)$. Let $f : P(i) \longrightarrow M$ be a homomorphism. Then $f = \Sigma_\sigma f_\sigma$, where f_σ is a homomorphism of degree σ. Then $Im(f) \subseteq \Sigma_\sigma Im(f_\sigma)$. So $\Sigma_{f\in Hom(P(i),M)} Im(f) = \Sigma_{f\in\bigcup_{g\in G} Hom_{gr}(P(i)[g],M)} Im(f)$. It follows that if $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ is a ${}_\Lambda\Delta$-filtration, hence all M_i are graded submodule of M. If we denote $F^{-1}(M)$ by $\overline{M}$, then $\overline{M} \in \mathcal{F}({}_{\Lambda\#G^*}\Delta)$. The proof is finished.

For a q.h. algebra, there is a (uniquely defined) basic module T which is direct sum of all non-isomorphic injective objects in $\mathcal{F}({}_\Lambda\Delta)$. This module is called the characteristic module of Λ[10].

Theorem 5 Let $(\Lambda = \bigoplus_{g\in G} \Lambda_g, S)$ is a finite dimensional q.h. G-graded $\mathcal{K}-$algebra with property P. Then the chacteristic modules ${}_{\Lambda\#G^*}T = \bigoplus_{(i,g)\in S\times G} T(i,g)$, such that $F(T(i,g)) = F(T(i,e))[g]$, and $\bigoplus_{i\in S} F(T(i,e))$ is the characteristic module for Λ .

Proof. Let ${}_{\Lambda\#G^*}T = \bigoplus_{(i,g)\in S\times G} T(i,g)$ be the characteristic module for $\Lambda\#G^*$. Then ${}_{\Lambda\#G^*}T \in \mathcal{F}({}_{\Lambda\#G^*}\Delta) \bigcap \mathcal{F}({}_{\Lambda\#G^*}\bigtriangledown)$. We have $F(T(i,g)) \in \mathcal{F}({}_\Lambda\Delta)$, and $F(T(i,g)) \in \mathcal{F}({}_\Lambda\bigtriangledown)$ by duality. So $\bigoplus_{i\in S} F(T(i,e))$ is a characteristic module for Λ, by proposition 4.

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