Maximal rigid subcategories in $2-$Calabi-Yau triangulated categories [1]

Dedicated to Claus M. Ringel on the occasion of his 65th birthday

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Abstract

We study the functorially finite maximal rigid subcategories in $2-$CY triangulated categories and their endomorphism algebras. Cluster tilting subcategories are obviously functorially finite and maximal rigid; we prove that the converse is true if the $2-$CY triangulated categories admit a cluster tilting subcategory. As a generalization of a result of [KR], we prove that any functorially finite maximal rigid subcategory is Gorenstein with Gorenstein dimension at most 1. Similar as cluster tilting subcategory, one can mutate maximal rigid subcategories at any indecomposable object. If two maximal rigid objects are reachable via simple mutations, then their endomorphism algebras have the same representation type.

Key words. Maximal rigid subcategories; Cluster tilting subcategories; $2-$CY triangulated categories; Gorenstein algebras; Representation types.

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1 Introduction

In the categorification theory of cluster algebras [FZ], cluster categories [BMRRT, Ke1, Am], (stable) module categories over preprojective algebras [GLS1, GLS2, BIRS], and more general $2-$Calabi-Yau triangulated categories with cluster tilting objects [FuKe, Pa1] play a central role. We refer the reader to the nice surveys [GLS2, Ke2, BM, Rin] and the references there for the recent developments.

Cluster tilting objects (subcategories) in $2-$CY categories have many nice properties. For examples, the endomorphism algebras are Gorenstein algebras of dimension at most 1 [KR]; cluster tilting objects have the same number of non-isomorphic indecomposable direct summands [DK, Pa2]. Importantly, in the categorification of cluster algebras, cluster tilting objects categorify clusters of the corresponding cluster algebras, and the combinatorics structure of cluster tilting objects is the same as the combinatorics structure of the corresponding cluster algebras [CC, CK].

Cluster tilting objects (subcategories) are maximal rigid objects (subcategories), the converse is not true in general. The first examples of $2-$Calabi-Yau categories in which maximal rigid objects are not cluster tilting were given in [BIKR] (see also the example in Section 5 of [KZ] for the example of triangulated category in which maximal rigid objects are not cluster tilting). Cluster

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tubes introduced in [BKL] serves as another type of such examples. It was proved recently by Buan-Marsh-Vatne in [BMV] that cluster tubes contain maximal rigid objects, none of them are cluster tilting. Buan-Marsh-Vatne also proved that the set of maximal rigid objects in $2-$CY triangulated categories forms cluster structure satisfying the definition in [BIRS] by allowing loops, and the combinatorial structure of maximal rigid objects in a cluster tube models the combinatorics of a type B cluster algebra. In [V, Y], the authors studied the endomorphism algebras of maximal rigid objects in cluster tubes, in particular, they proved that the endomorphism algebras are Gorenstein of Gorenstein dimension at most 1.

The aim of this paper is to give a systematic study of functorially finite maximal rigid subcategories in $2-$CY triangulated categories and endomorphism algebras of maximal rigid objects. For any functorially finite maximal rigid subcategory $\mathcal{R}$ in a $2-$CY triangulated category C, one considers the extension subcategory $\mathcal{R} * \mathcal{R}[1]$ (compare [Pla]). It is in general not equal to C. Note that under the condition that $\mathcal{R}$ is cluster tilting, we have $C = \mathcal{R} * \mathcal{R}[1]$ [KR]. We observe that any rigid object belongs to $\mathcal{R} * \mathcal{R}[1]$. Using this fact, we prove that if a $2-$CY triangulated category contains a cluster tilting subcategory, then any functorially finite maximal rigid subcategory is cluster tilting. This generalizes Theorem II.1.8 in [BIRS] from algebraic $2-$CY triangulated categories to arbitrary $2-$CY triangulated categories. Then we consider $2-$CY triangulated categories with maximal rigid subcategories. It is proved that some results in [DK, Pa2] also hold in this setting. Namely, we prove that the representatives of isomorphic classes of indecomposable objects of a functorially finite maximal rigid subcategory form a basis in the split Grothendieck group of another functorially finite maximal rigid subcategory. In particular, all maximal rigid objects have the same number of non-isomorphism indecomposable direct summands. Using a recent result of Nakaoka [Na], we prove that functorially finite maximal rigid subcategories in $2-$CY triangulated categories are Gorenstein of dimension at most 1. This is a generalization of the same results in cluster tubes[V, Y]. And it also generalizes the same results on cluster tilting subcategories of [KR] to functorially finite maximal rigid subcategories. Finally, we study the endomorphism algebras of maximal rigid objects in $2-$CY triangulated categories. If two maximal rigid objects are reachable via simple mutations, then the corresponding endomorphism algebras of them have the same representation type.

The paper is organized as follows. In Section 2, we prove that the functorially finite maximal rigid subcategories are cluster tilting in $2-$CY triangulated categories with a cluster tilting subcategory. In Section 3, the notion of index of a rigid object with respect to a cluster tilting subcategory is generalized by replacing cluster tilting subcategory with functorially finite maximal rigid subcategory with respect to maximal rigid subcategory (compare [Pla]). The representatives of isomorphism classes of indecomposable objects of a functorially finite maximal rigid subcategory form a basis in the split Grothendieck group of another functorially finite maximal rigid subcategory. As a direct consequence, the numbers of indecomposable direct summands of functorially finite maximal rigid objects are the same. In the last section, we prove that any functorially finite maximal rigid subcategory is Gorenstein of dimension at most 1. Finally, for two reachable maximal rigid objects, the corresponding endomorphism algebras have same representation type.

2 Relations between cluster-tilting subcategories and maximal rigid subcategories

Throughout this paper, k denotes an algebraically closed field and C denotes a $k-$linear triangulated category whose shift functor is denoted by [1]. We assume that C is Hom$-$finite and Krull-Remak-Schmidt, i.e. $\dim_k \operatorname{Hom}(X, Y) < \infty$ for any two objects X and Y in C, and every object decomposes into a finite direct sum of objects having local endomorphism rings. For basic references on representation theory of algebras and triangulated categories, we refer to [H].

For $X, Y \in C$ and $n \in \mathbf{Z}$, we put

$$\operatorname{Ext}^n(X, Y) = \operatorname{Hom}(X, Y[n]).$$

When we say that a subcategory $\mathcal{T}$ of C, we mean that $\mathcal{T}$ is a full subcategory which is closed under taking isomorphisms, direct sums and direct summands. $\mathcal{T}^\perp$ denotes the subcategory consisting of $Y \in C$ with $\operatorname{Hom}(T, Y) = 0$ for any $T \in \mathcal{T}$, and $^\perp T$ denotes the subcategory consisting of $Y \in C$ with $\operatorname{Hom}(Y, T) = 0$ for any $T \in \mathcal{T}$. For an object $T \in C$, add T denotes the subcategory consisting of direct summands of direct sums of finite copies of T.

For two subcategories $\mathcal{D} \subset \mathcal{T}$ of C, the quotient category $\mathcal{T}/\mathcal{D}$ of $\mathcal{T}$ by $\mathcal{D}$ has the same objects as $\mathcal{T}$, and its morphism spaces are defined by $\operatorname{Hom}_{\mathcal{T}/\mathcal{D}}(X, Y) = \operatorname{Hom}(X, Y)/\mathcal{D}(X, Y)$, where $\mathcal{D}(X, Y)$ denotes the set of morphisms which factor through some object in $\mathcal{D}$.

For two subcategories $\mathscr{X}, \mathscr{Y}$, we denote $\operatorname{Ext}^1(\mathscr{X}, \mathscr{Y}) = 0$ if $\operatorname{Ext}^1(X, Y) = 0$ for any $X \in \mathscr{X}, Y \in \mathscr{Y}$. $\mathscr{X} * \mathscr{Y}$ denotes the extension category of $\mathscr{X}$ by $\mathscr{Y}$, whose objects are by definition the objects E with triangle $X \to E \to Y \to X[1]$, where $X \in \mathscr{X}, Y \in \mathscr{Y}$. By the octahedral axiom, we have $(\mathscr{X} * \mathscr{Y}) * \mathcal{Z} = \mathscr{X} * (\mathscr{Y} * \mathcal{Z})$. We call $\mathscr{X}$ extension closed if $\mathscr{X} * \mathscr{X} = \mathscr{X}$.

For $X \in C$, a morphism $f : T \to X$ is called right $\mathcal{T}-$approximation of X if $\operatorname{Hom}(-, f)|_{\mathcal{T}}$ is surjective. If any object $X \in C$ has a right $\mathcal{T}-$approximation, we call $\mathcal{T}$ contravariantly finite in C. Left $\mathcal{T}-$approximation and covariantly finiteness are defined dually. We say that $\mathcal{T}$ is functorially finite if it is both covariantly finite and contravariantly finite. It is easy to see that add T is functorially finite for any object $T \in C$, since C is Hom-finite and add T contains finitely many indecomposable objects up to isomorphism.

A triangulated category C is called $2-$CY provided that there are bifunctorial isomorphisms

$$\operatorname{Ext}^1(X, Y) = D\operatorname{Ext}^1(Y, X)$$

for $X, Y \in C$, where $D = \operatorname{Hom}_k(-, k)$ is the duality of $k-$spaces.
Throughout this paper, we assume that C is $2-$CY.

An exact category is called stably $2-$CY [BIRS] if it is Frobenius, that is, it has enough projectives and injectives, which coincide, and the stable category is $2-$CY triangulated. If a triangulated category is triangulated equivalent to the stable category of a stably $2-$CY exact category, then we call it algebraic $2-$CY triangulated category [Ke3].

Examples of stably $2-$CY categories are the categories of Cohen-Macaulay modules over an isolated hypersurface singularity [BIKR]; the module categories of preprojective algebras of Dynkin quivers [GLS1]. Basic examples of $2-$CY triangulated categories are the cluster categories of

abelian hereditary categories with tilting objects [BMRRT, Ke1]; the Hom-finite generalized cluster categories of algebras with global dimension of at most 2[Am]; the stable categories of stably $2-CY$ categories [BIRS] and cluster tubes [BKL, BMV].

We recall some basic notions [BMRRT, I1, KR, GLS1, BIRS].

Definition 2.1. *Let $\mathcal{T}$ be a subcategory of C which is closed under taking direct summands and finite direct sums.*

1. *$\mathcal{T}$ is called rigid provided $Ext^1(\mathcal{T}, \mathcal{T}) = 0$.*

2. *$\mathcal{T}$ is called maximal rigid provided $\mathcal{T}$ is rigid and is maximal with respect to this property, i.e. if $Ext^1(\mathcal{T} \bigcup add\, M, \mathcal{T} \bigcup add\, M) = 0$, then $M \in \mathcal{T}$.*

3. *$\mathcal{T}$ is called cluster-tilting provided $\mathcal{T}$ is functorially finite and $\mathcal{T} = {}^\perp\mathcal{T}[1]$.*

4. *An object T is called rigid, maximal rigid, or cluster tilting if add T is rigid, maximal rigid, or cluster tilting respectively.*

Remark 2.2. *1. Any $2-CY$ triangulated category C admits rigid subcategories (0 is viewed as a trivial rigid object), and also admits maximal rigid subcategories if C is skeletally small.*

2. *There are $2-CY$ triangulated categories which contains no cluster tilting subcategories [BIKR, BMV].*

3. *Cluster tilting subcategories are functorially maximal rigid subcategories. But the converse is not true in general. It was observed by Buan-Marsh-Vatne [BMV] that the cluster tubes contain maximal rigid objects, none of them are cluster tilting objects.*

If C admits a cluster-tilting subcategory $\mathcal{T}$, we know that $C = \mathcal{T} * \mathcal{T}[1]$, i.e. for any object X in C there is a triangle $T_1 \to T_0 \to X \to T_1[1]$ with $T_i \in \mathcal{T}, i = 1, 2$ [KR, KZ]. In fact, the converse is also true.

Remark 2.3. *If $C = \mathcal{T} * \mathcal{T}[1]$ with $\mathcal{T}$ a rigid subcategory of C, then $\mathcal{T}$ is a cluster-tilting subcategory.*

Proof. Clearly, $\mathcal{T}$ is functorially finite. Given an object X in C with $Ext^1(X, \mathcal{T}) = 0$, there is a triangle $T_1 \to T_0 \to X \to T_1[1]$. Then this triangle splits. Hence $X \in \mathcal{T}$. This proves that $\mathcal{T}$ is cluster-tilting.

$\square$

In general, for a maximal rigid subcategory $\mathcal{R}$, $\mathcal{R} * \mathcal{R}[1]$ is smaller than C, but all rigid objects belong to $\mathcal{R} * \mathcal{R}[1]$ [BIRS]. The following lemma was proved for preprojective algebras in [BMR, GLS1], it holds for any $2-CY$ triangulated category.

Lemma 2.4. *Let $\mathcal{R}$ be a contravariantly finite maximal rigid subcategory in a $2-CY$ triangulated category C. For any rigid object $X \in C$, if $Y \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} Y[1]$ is a triangle such that $R_0 \in \mathcal{R}$ and g is a right $\mathcal{R}$-approximation of X, then $Y \in \mathcal{R}$. Furthermore, there is a left $\mathcal{R}$-approximation $f_1 : X \to R_1$, which extends to a triangle $X \xrightarrow{f_1} R_1 \xrightarrow{g_1} R_2 \xrightarrow{h_1} X[1]$ with $R_2 \in \mathcal{R}$.*

Proof. Since g is a right $\mathcal{R}-$approximation of X and $\text{Ext}^1(R, R_0) = 0$ for any objects R in $\mathcal{R}$, we have that $\text{Ext}^1(\mathcal{R}, \text{add } Y) = 0$, in particular, we have $\text{Ext}^1(R_0, Y) = 0$.

By applying $\text{Hom}(-, X)$ and $\text{Hom}(Y, -)$ to the triangle $Y \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} Y[1]$ we have two exact sequences:

$$\text{Hom}(R_0, X) \xrightarrow{\text{Hom}(f, X)} \text{Hom}(Y, X) \longrightarrow \text{Ext}^1(X, X) = 0$$

and

$$\text{Hom}(Y, R_0) \xrightarrow{\text{Hom}(Y, g)} \text{Hom}(Y, X) \longrightarrow \text{Ext}^1(Y, Y) \longrightarrow \text{Ext}^1(Y, R_0) = 0.$$

Let α be an element of $\text{Hom}(Y, X)$. By the first exact sequence there is a $\beta \in \text{Hom}(R_0, X)$ such that $\alpha = \beta f$. Since g is a right $\mathcal{R}-$approximation of X, there is a $\gamma \in \text{Hom}(R_0, R_0)$ such that $\beta = g\gamma$. Then $\alpha = g\gamma f$. This shows that $\text{Hom}(Y, g)$ is surjective. Hence $\text{Ext}^1(Y, Y) = 0$ by the second exact sequence. It follows that $Y \in \mathcal{R}$.

For the second part, we apply the first part to the rigid object $X[1]$. There is a triangle $R_1 \xrightarrow{f_1} R_2 \xrightarrow{g_1} X[1] \xrightarrow{h_1} R_1[1]$. Then we have a triangle $X \xrightarrow{-h_1[-1]} R_1 \to R_2 \to X[1]$. It is easy to see $-h_1[-1]$ is a left $\mathcal{R}-$approximation of X.

$\square$

There is a dual statement for covariantly finite maximal rigid subcategory $\mathcal{R}$, we leave it to the reader.

By Lemma 2.4, we have the following result which is the second part of Proposition I.1.7 in [BIRS].

Corollary 2.5. *Let $\mathcal{R}$ be a functorially finite maximal rigid subcategory. Then every rigid object belongs to $\mathcal{R} * \mathcal{R}[1]$.*

One can see that any cluster-tilting subcategory is maximal rigid, but the converse is not true [BIKR, BMV]. The main result of this section is the following theorem which tells us that if one of functorially finite maximal rigid subcategories of C is cluster-tilting, then so are all of them. This is a generalization of Theorem II.1.8 in [BIRS], where the same conclusion was proved for algebraic $2-$CY triangulated categories.

Theorem 2.6. *Let C ba a 2-CY triangulated category admitting a cluster-tilting subcategory $\mathcal{T}$. Then every functorially finite maximal rigid subcategory is cluster-tilting.*

Proof. Assume that $\mathcal{R}$ is a functorially finite maximal rigid subcategory in C. Given an object $X \in C$ satisfying $\text{Ext}^1(\text{add } X, \mathcal{R}) = 0$, we have a triangle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} X \longrightarrow T_1[1]$$

where T_0 and T_1 belong to $\mathcal{T}$. Since $\mathcal{R}$ is functorially finite in C, there is a left $\mathcal{R}-$approximation of T_0 which extends to a triangle by Lemma 2.4,

$$R_0[-1] \longrightarrow T_0 \xrightarrow{\alpha} R \longrightarrow R_0,$$

where $R, R_0 \in \mathcal{R}$.

Let $\alpha_1 = \alpha f$. For any object Z in $\mathcal{R}$, by applying $\mathrm{Hom}(-, Z)$ to the triangle $T_1 \xrightarrow{f} T_0 \xrightarrow{g} X \to$ $T_1[1]$, we have the exact sequence

$$\mathrm{Hom}(T_0, Z) \xrightarrow{\mathrm{Hom}(f,Z)} \mathrm{Hom}(T_1, Z) \longrightarrow \mathrm{Ext}^1(X, Z) = 0.$$

Given an element $\varphi_1 \in \mathrm{Hom}(T_1, Z)$, there is a $\varphi_0 \in \mathrm{Hom}(T_0, Z)$ such that $\varphi_1 = \varphi_0 f$. Since α is a left $\mathcal{R}$−approximation of T_0, there is a ψ such that $\varphi_0 = \psi \alpha$. Then $\varphi_1 = \psi \alpha f = \psi \alpha_1$. So α_1 is a left $\mathcal{R}$−approximation of T_1. It follows from the dual of Lemma 2.4 that the triangle which α_1 is a part is of the form:

$$R_1[-1] \longrightarrow T_1 \xrightarrow{\alpha_1} R \longrightarrow R_1,$$

where $R, R_1 \in \mathcal{R}$.

Starting with $\alpha_1 = \alpha f$, we get the following commutative diagram by the octahedral axiom:

$$
\begin{array}{ccccccc}
X[-1] & = & X[-1] & & & & \\
\downarrow & & \downarrow & & & & \\
R_1[-1] & \longrightarrow & T_1 & \xrightarrow{\alpha_1} & R & \longrightarrow & R_1 \\
\downarrow & & \downarrow f & & \| & & \downarrow \\
R_0[-1] & \longrightarrow & T_0 & \xrightarrow{\alpha} & R & \longrightarrow & R_0 \\
\downarrow & & \downarrow g & & & & \\
X & = & X & & & &
\end{array}
$$

But $\mathrm{Hom}(R_0[-1], X) = \mathrm{Ext}^1(R_0, X) = 0$, so the first column is a split triangle and then $X \in \mathcal{R}$. Thus we have proved this theorem.

$\square$

Remark 2.7. *The same conclusion is not true in arbitrary triangulated categories (which are not 2−CY). See the example in Section 2 in [BMRRT], where the derived category of the quiver $Q : 1 \to 2 \to 3$ contains a functorially finite maximal rigid subcategory which is not cluster tilting. It is well-known that the derived category of Q contains cluster tilting subcategories, see for example the example in Section 5 of [KZ], or [I2].*

3 Mutations and basis of Grothendieck groups of maximal rigid subcategories

Mutations in arbitrary triangulated categories were defined in [IY]. We recall them in the setting of 2−CY triangulated categories.

Let $\mathcal{C}$ be a 2−CY triangulated category and $\mathcal{D}$ a functorially finite rigid subcategory of $\mathcal{C}$ which is closed under taking finite direct sums and direct summands. For a subcategory $\mathcal{X}$ of $\mathcal{C}$ containing $\mathcal{D}$, put

$$\mu^{-1}(\mathcal{X}, \mathcal{D}) := (\mathcal{D} * \mathcal{X}[1]) \cap {}^{\perp}(\mathcal{D}[1]).$$

Dually, for a subcategory $\mathcal{Y}$ of $\mathcal{C}$ containing $\mathcal{D}$, put

$$\mu(\mathcal{Y}, \mathcal{D}) := (\mathcal{Y}[-1] * \mathcal{D}) \cap (\mathcal{D}[-1])^{\perp}.$$

Definition 3.1. *([IY]) The pair $(\mathscr{X}, \mathscr{Y})$ of subcategories $\mathscr{X}, \mathscr{Y}$ is called $\mathcal{D}-$mutation pair if $\mathscr{X} = \mu(\mathscr{Y}, \mathcal{D})$ and $\mathscr{Y} = \mu^{-1}(\mathscr{X}, \mathcal{D})$.*

It is not difficult to see that: for subcategories $\mathscr{X}, \mathscr{Y}$ containing $\mathcal{D}$, $(\mathscr{X}, \mathscr{Y})$ forms a $\mathcal{D}-$mutation if and only if for any $X \in \mathscr{X}$, $Y_1 \in \mathscr{Y}$ there are two triangles:

$$X \xrightarrow{f} D \xrightarrow{g} Y \to X[1],$$

$$X_1 \xrightarrow{f_1} D_1 \xrightarrow{g_1} Y_1 \to X_1[1]$$

where $D, D_1 \in \mathcal{D}$, $Y \in \mathscr{Y}$, $X_1 \in \mathscr{X}$, f and f_1 are left $\mathcal{D}-$approximations; g and g_1 are right $\mathcal{D}-$approximations.

The following result is analogous to the first part of Theorem 5.1 in [IY], where the arguments are stated for cluster tilting subcategories. We give a proof here for the convenience of the reader.

Proposition 3.2. *Let $\mathcal{R}$ be a functorially finite maximal rigid subcategory containing $\mathcal{D}$. Then its mutation $\mathcal{R}' = \mu^{-1}(\mathcal{R}, \mathcal{D})$ is a functorially finite maximal rigid subcategory, and $(\mathcal{R}, \mathcal{R}')$ is a $\mathcal{D}-$mutation pair.*

Proof. Let $\mathcal{Z} = {}^{\perp}\mathcal{D}[1] = \mathcal{D}[-1]^{\perp}$, and $\mathscr{U} := \mathcal{Z}/\mathcal{D}$ the quotient triangulated category, whose shift functor is denoted by $< 1 >$ (for details of the triangulated structure of the quotient triangulated category, see Section 4 of [IY]). The images of the morphism f and the subcategory $\mathcal{R}$ in the quotient $\mathscr{U}$ are denoted by $\underline{f}$ and $\underline{\mathcal{R}}$ respectively.
We first sketch the proof of the fact $\mathcal{R}$ is a functorially finite maximal rigid subcategory in C if and only if so is $\underline{\mathcal{R}}$ in $\mathscr{U}$.
It is easy to see that $\mathcal{R}$ is functorially finite if and only if so is $\underline{\mathcal{R}}$.
We will prove that $\underline{\mathcal{R}}$ is maximal rigid in $\mathscr{U}$ provided $\mathcal{R}$ is maximal rigid in C. Let $M \in \mathscr{U}$ satisfy that $\mathrm{Hom}(M, X < 1 >) = 0, \mathrm{Hom}(X, M < 1 >) = 0, \mathrm{Hom}(M, M < 1 >) = 0$, for any $X \in \underline{\mathcal{R}}$. For $X \in \mathcal{R}$, we have a triangle in C:

$$X \xrightarrow{f} D \xrightarrow{g} X < 1 > \xrightarrow{h} X[1].$$

which is from the definition of the functor $< 1 >$. Now suppose that $\alpha \in \mathrm{Hom}(M, X[1])$. Since $\mathrm{Hom}(M, D[1]) = 0$, α factors through h by $\beta : M \to X < 1 >$. Since $\mathrm{Hom}(M, X < 1 >) = 0$ in $\mathscr{U}$, we have that β factors through g. Then α factors through $h \circ g$, and then $\alpha = 0$. This proves that $\mathrm{Hom}(M, X[1]) = 0$. One can prove that $\mathrm{Hom}(X[1], M) = 0, \mathrm{Hom}(M, M[1]) = 0$ in a similar way. Then $\underline{\mathcal{R}}$ is maximal rigid in $\mathscr{U}$. The converse implication that $\mathcal{R}$ is maximal rigid in C provided $\underline{\mathcal{R}}$ is maximal rigid in $\mathscr{U}$ can be proved in a similar way. We omit the details here.
It follows from the fact $(\mathcal{R}, \mathcal{R}')$ is a $\mathcal{D}-$mutation in C that $(\underline{\mathcal{R}}, \underline{\mathcal{R}'})$ is a $0-$mutation in $\mathscr{U}$. Then $\underline{\mathcal{R}'} = \underline{\mathcal{R}} < 1 >$ in $\mathscr{U}$, and then $\underline{\mathcal{R}'}$ is maximal rigid in $\mathscr{U}$. It follows that $\mathcal{R}'$ is maximal rigid in C. $\square$

We call a subcategory $\mathcal{R}_1$ an almost complete maximal rigid subcategory if there is an indecomposable object R which is not isomorphic to any object in $\mathcal{R}_1$ such that $\mathcal{R} = \mathrm{add}\,(\mathcal{R}_1 \bigcup \{R\})$ is a functorially finite maximal rigid subcategory in C. Such R is called a complement of an almost complete maximal rigid subcategory $\mathcal{R}_1$. It is easy to see that any almost complete maximal rigid subcategory is functorially finite. Combining the proposition above with Corollary 2.5, we have the following corollary, which was indicated in [BIRS].

Corollary 3.3. *Let $\mathcal{R}_1$ be an almost complete maximal rigid subcategory of C. Then there are exactly two complements of $\mathcal{R}_1$, say R and R^*. Denote by $\mathcal{R} = add\,(\mathcal{R}_1 \bigcup\{R\})$, $\mathcal{R}' = add\,(\mathcal{R}_1 \bigcup\{R^*\})$. Then $(\mathcal{R}, \mathcal{R}'), (\mathcal{R}', \mathcal{R})$ are $\mathcal{R}_1-$mutations.*

Proof. This follows from [IY, 5.3]. Note that the arguments there are stated only for cluster tilting subcategories, but work also for functorially finite maximal rigid subcategories with the help of Corollary 2.5.

$\square$

Definition 3.4. *Let $\mathcal{R}_1$ be an almost complete maximal rigid subcategory of C with the complements R and R^*. Denote $\mathcal{R} = add\,(\mathcal{R}_1 \bigcup\{R\})$, $\mathcal{R}' = add\,(\mathcal{R}_1 \bigcup\{R^*\})$. If $dim_k Ext^1(R, R') = 1$ (equivalently $dim_k Ext^1(R', R) = 1$), then the mutation $(\mathcal{R}, \mathcal{R}')$ is called a simple mutation [Ke2,Pla].*

There are mutations of some maximal rigid objects in cluster tubes which are not simple [Y]. It was proved in [Pla] that for a simple mutation of maximal rigid subcategories $\mathcal{R}, \mathcal{R}'$, $\mathcal{R} * \mathcal{R}[1] = \mathcal{R}' * \mathcal{R}'[1]$.

Let $\mathcal{R}$ be a functorially maximal rigid subcategory of $2-CY$ triangulated category C. Let $K_0^{split}(\mathcal{R})$ be the (split) Grothendieck group, which by definition, the free abelian group with a basis $[R]$, where R runs through the representatives of isomorphism classes of indecomposable objects in $\mathcal{R}$. Let X be a rigid object of C. By the corollary 2.5 above, there is a triangle $R_1 \rightarrow R_0 \rightarrow X \rightarrow R_1[1]$. So we can define the index $ind_{\mathcal{R}}(X) = [R_0] - [R_1] \in K_0^{split}(\mathcal{R})$ as in [Pa1, DK, Pla].

Proposition 3.5. *Let $\mathcal{R}$ be a functorially finite maximal rigid subcategory.*
(1) If X and Y are rigid objects in C such that $ind_{\mathcal{R}}(X) = ind_{\mathcal{R}}(Y)$, then X and Y are isomorphic.
(2) Let X be a rigid object of C and let $X_i, i \in I$, be a finite family of pairwise nonisomorphic indecomposable direct summands of X. Then the elements $ind_{\mathcal{R}}(X_i), i \in I$, are linearly independent in $K_0^{split}(\mathcal{R})$.

Proof. All conclusions follow from Sections 2.1, 2.2, 2.3, 2.5 in [DK]. Note that the arguments there are stated only for cluster-tilting subcategories, but work also for functorially finite maximal rigid subcategories by the Corollary 2.5 above.

$\square$

Theorem 3.6. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two functorially finite maximal rigid subcategories in C. Then the elements $ind_{\mathcal{R}}(R')$, where R' runs through a system of representatives of the isomorphism classes of indecomposables of $\mathcal{R}'$, form a basis of the free abelian group $K_0^{split}(\mathcal{R})$.*

Proof. The proof of Theorem 2.4 in [DK, 2.6] works also in this setting.

$\square$

Corollary 3.7.   *1. The category C has a maximal rigid object if and only if all functorially maximal rigid subcategories have a finite number of pairwise non-isomorphic indecomposable objects.*

   2. All maximal rigid objects have the same number of indecomposable direct summands (up to isomorphism).

4 Gorenstein property of maximal rigid subcategories

Let $\mathcal{R}$ be a functorially finite maximal rigid subcategory of C and $\mathcal{A}$ the quotient category of $\mathcal{D} = \mathcal{R}[-1] * \mathcal{R}$ by $\mathcal{R}$. Let $\mathrm{mod}\mathcal{R}$ denote the category of finitely presented $\mathcal{R}$−modules where a $\mathcal{R}$−module means a contravariantly functor from $\mathcal{R}$ to the category of k−vector spaces. We know that $\mathcal{A}$ is an abelian category whose abelian structure is induced by the triangulated structure of C and there is an equivalence $F : \mathcal{A} \to \mathrm{mod}\mathcal{R}$ [IY]. As in section 2, we put

$$\mathcal{R}^{\perp} := \{X \in C \mid \mathrm{Hom}(\mathcal{R}, X) = 0\} \text{ and } {}^{\perp}\mathcal{R} := \{X \in C \mid \mathrm{Hom}(X, \mathcal{R}) = 0\}.$$

By 2−CY property of C, ${}^{\perp}\mathcal{R}[1] = \mathcal{R}[-1]^{\perp}$, which is denoted by $\mathcal{S}$. Clearly, both $(\mathcal{R}, \mathcal{S})$ and $(\mathcal{S}, \mathcal{R})$ are cotorsion pairs in the sense in [Na] (equivalently, $(\mathcal{R}, \mathcal{S}[1])$ and $(\mathcal{S}[-1], \mathcal{R})$ are torsion pairs in the sense in [IY]).

For the convenience of the reader we recall briefly the abelian structure of $\mathcal{A}$ from [Na]. Let $\underline{f} \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ with $X, Y \in \mathcal{D}$ and $f \in \mathrm{Hom}_{C}(X, Y)$ where f is a part of the triangle $Z[-1] \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$. Let $f_1 : X \to R_0$ be a left $\mathcal{R}$−approximation of X which extends to a triangle $R_1[-1] \to X \xrightarrow{f} R_0 \to R_1$. Then $R_1 \in \mathcal{R}$ by Lemma 2.4. We have the following commutative diagram which is constructed from the octahedral axiom:

$$
\begin{array}{ccccccc}
 & & Z[-1] & = & Z[-1] & & \\
 & & \downarrow h & & \downarrow & & \\
R_1[-1] & \longrightarrow & X & \longrightarrow & R_0 & \longrightarrow & R_1 \\
\| & & \downarrow f & & \downarrow & & \| \quad (*). \\
R_1[-1] & \longrightarrow & Y & \xrightarrow{m_f} & M_f & \longrightarrow & R_1 \\
 & & \downarrow g & & \downarrow & & \\
 & & Z & = & Z & &
\end{array}
$$

The map m_f is the cokernel of $\underline{f}$ [Na] (note that $M_f \in (\mathcal{R}[-1] * \mathcal{R}) * \mathcal{R} = \mathcal{R}[-1] * (\mathcal{R} * \mathcal{R}) = \overline{\mathcal{R}[-1] * \mathcal{R}})$.

The kernel of $\underline{f}$ is obtained similarly. Since $(\mathcal{R}, \mathcal{S}[1])$ is a torsion pair, we have a triangle $S_0' \to R_0' \to Y \to S_0'[1]$, where $R_0' \in \mathcal{R}$ and $S_0' \in \mathcal{S}$. Using the octahedral axiom, we have the first diagram of the following two commutative diagrams. Since $(\mathcal{R}[-1], \mathcal{S})$ is a torsion pair, we have a triangle $R[-1] \xrightarrow{\psi} L_f \to S \to R$ with $R \in \mathcal{R}$ and $S \in \mathcal{S}$. Since $(\mathcal{R}, \mathcal{S}[1])$ is also a torsion pair, we have another triangle $S' \xrightarrow{\vec{R}'} S \to S'[1]$ with $R' \in \mathcal{R}$ and $S' \in \mathcal{S}$. Using the octahedral axiom, we have the second diagram of the following two commutative diagrams:

$$
\begin{array}{ccccccc}
 & & Z[-1] & = & Z[-1] & & \\
 & & \downarrow & & \downarrow h & & \\
S_0' & \longrightarrow & L_f & \xrightarrow{l_f} & X & \longrightarrow & S_0'[1] \\
\| & & \downarrow & & \downarrow f & & \| \quad (**) \\
S_0' & \longrightarrow & R_0' & \longrightarrow & Y & \longrightarrow & S_0'[1] \\
 & & \downarrow & & \downarrow g & & \\
 & & Z & = & Z & &
\end{array}
$$

and

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$$
\begin{array}{ccc}
S' & = & S' \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccc}
R[-1] & \xrightarrow{\varphi} & K_L & \longrightarrow & R' & \longrightarrow & R \\
\| & & \downarrow k_L & & \downarrow & & \| \;\; (***). \\
R[-1] & \xrightarrow{\psi} & L_f & \longrightarrow & S & \longrightarrow & R \\
& & \downarrow & & \downarrow & & \\
& & S'[1] & = & S'[1] & &
\end{array}
$$

The composition $l_f k_L$ is the kernel of $\underline{f}$.

Remark 4.1. *[Na, Remark 4.5]For any $X \in \mathcal{R}[-1] * \mathcal{R}$ and any $\underline{y} \in Hom_{C/\mathcal{R}}(X, L_f)$, there exists a unique morphism $\underline{x} \in Hom_{\mathcal{A}}(X, K_L)$ such that $\underline{y} = \underline{k_L}\underline{x}$.*

Thus K_L is determined uniquely up to a canonical isomorphism in $\mathcal{A}$.

The following lemma is a suitable version of Proposition 6.1 in [Na] in our setting. We include a proof for the convenience of the reader.

Lemma 4.2. *Let $f : X \to Y$ be a morphism in C which is a part of triangle $Z[-1] \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$. Then $\underline{f}$ is an epimorphism if and only if $M_f \in \mathcal{R}$; $\underline{f}$ is a monomorphism if and only if $L_F \in \mathcal{S}$. In particular, if Z is in $\mathcal{R}$, then f is an epimorphism; if $Z[-1]$ is in $\mathcal{S}$, then f is a monomorphism.*

Proof. (1) $\underline{f} : X \to Y$ is an epimorphism in $\mathcal{A}$ if and only if $M_f \cong 0$ in $\mathcal{A}$, i.e. $M_f \in \mathcal{R}$.
(2) $\underline{f} : X \to Y$ is a monomorphism in $\mathcal{A}$ if and only if $K_L \cong 0$ in $\mathcal{A}$, i.e. $K_L \in \mathcal{R}$. We claim that $K_L \in \mathcal{R}$ if and only if $L_F \in \mathcal{S}$. If $K_L \in \mathcal{R}$, then $\varphi = 0$ by $Hom(\mathcal{R}[-1], \mathcal{R}) = 0$. So $\psi = k_L \varphi = 0$, and hence $L_F \in \mathcal{S}$. If $L_F \in \mathcal{S}$, then $\psi = 0$. So $k_L \varphi = 0$ that implies that k_L factors through R'. Then $\underline{k_L} = \underline{0}$, and then $K_L \in \mathcal{R}$ by Remark 4.1.
(3)If Z is in $\mathcal{R}$, then the third column in (*) is a splitting triangle. So $M_f \in \mathcal{R}$. Dually, if $Z[-1]$ is in $\mathcal{S}$, then the second column in (**) is a splitting triangle. So $L_f \in \mathcal{S}$.
$\square$

Now we determine the projective objects and injective objects in $\mathcal{A}$.

Proposition 4.3. *An object M of $\mathcal{A}$ is a projective object if and only if $M \in \mathcal{R}[-1]$. An object N of $\mathcal{A}$ is an injective object if and only if $N \in \mathcal{R}[1]$.*

Proof. (1)Given $R \in \mathcal{R}$. For any epimorphism $\underline{f} : X \to Y$ in $\mathcal{A}$, and any morphism $\underline{\alpha} : R[-1] \to Y$, $m_f \alpha = 0$ by $M_f \in \mathcal{R}$ and $Hom(\mathcal{R}[-1], \mathcal{R}) = \overline{0}$. So $g\alpha = 0$. Then α factors through f, hence $\underline{\alpha}$ factors through $\underline{f}$. This proves that $R[-1]$ is projective in $\mathcal{A}$.
Conversely assume M is a projective object in $\mathcal{A}$. Since $M \in \mathcal{R}[-1] * \mathcal{R}$, there is a triangle $\mathcal{R}_0[-1] \xrightarrow{\sigma} M \to R_1 \to R_0$ with $R_0, R_1 \in \mathcal{R}$. Then $\underline{\sigma}$ is an epimorphism in $\mathcal{A}$ by Lemma 4.2. So the epimorphism $\underline{\sigma} : \mathcal{R}_0[-1] \to M$ splits. Hence $M \in \mathcal{R}[-1]$.

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(2)Note that $R[1] \in \mathcal{R}[-1] * \mathcal{R}$, for all $R \in \mathcal{R}$, by Corollary 2.5.

Given $R \in \mathcal{R}$. For any monomorphism $f : X \to Y$ in $\mathcal{A}$, and any morphism $\beta : X \to R[1]$, $\beta l_f = 0$ by $L_f \in \mathcal{S}$. So $\beta h = 0$. Then β factors through f, hence $\underline{\beta}$ factors through $\underline{f}$. This proves that $R[1]$ is injective in $\mathcal{A}$.

Conversely assume M is an injective object in $\mathcal{A}$. Since $M \in C = \mathcal{S} * \mathcal{R}[1]$, there is a triangle $S \to M \xrightarrow{\tau} R[1] \to S[1]$ with $R \in \mathcal{R}$ and $S \in \mathcal{S}$. Then $\underline{\tau}$ is a monomorphism in $\mathcal{A}$ by Lemma 4.2. So $\underline{\tau}$ splits, hence $M \in \mathcal{R}[1]$.

$\square$

The main result in this section is the following theorem which is a generalization of Proposition 2.1 in [KR], Theorem 4.3 in [KZ]. This has been proved in [V, Y] for C being cluster tubes.

Theorem 4.4. *Let C be a 2-Calabi-Yau triangulated category with a functorially finite maximal rigid subcategory $\mathcal{R}$ and let $\mathcal{A}$ be the abelian quotient category of $\mathcal{R}[-1] * \mathcal{R}$ by $\mathcal{R}$. Then*
(1) The abelian category $\mathcal{A}$ has enough projective objects.
(2) The abelian category $\mathcal{A}$ has enough injective objects.
(3) The abelian category $\mathcal{A}$ is Gorenstein of Gorenstein dimension at most one.

Proof. (1) Given $X \in \mathcal{R}[-1] * \mathcal{R}$. There is a triangle $R_1[-1] \to R_0[-1] \xrightarrow{f} X \to R_1$ with $R_0, R_1 \in \mathcal{R}$. Then $\underline{f} : R_0[-1] \to X$ is an epimorphism with $R_0[-1]$ a projective object.

(2) Given $X \in \mathcal{R}[-1] * \mathcal{R}$. Since $X \in C = \mathcal{S} * \mathcal{R}[1]$. There is a triangle $S \to X \xrightarrow{g} R[1] \to S[1]$ with $R \in \mathcal{R}$ and $S \in \mathcal{S}$. Then $\underline{g} : X \to R[1]$ is a monomorphism with $R[1]$ an injective object.

(3) For an injective object $R[1]$ in $\mathcal{A}$, since $R[1] \in \mathcal{R}[-1] * \mathcal{R}$, then there is a triangle $R_1[-1] \xrightarrow{h} R_0[-1] \xrightarrow{f} R[1] \to R_1$ with $R_0, R_1 \in \mathcal{R}$. Then $\underline{f}$ is an epimorphism by Lemma 4.2 and $\underline{h}$ is the kernel of $\underline{f}$ by the structure of kernel. So we have an exact sequence $0 \to R_1[-1] \xrightarrow{h} R_0[-1] \xrightarrow{f} R[1] \to 0$ which is a projective resolution of the injective object $R[1]$ in $\mathcal{A}$. Therefore proj.dim.$R[1] \le 1$.

For a projective object $R[-1]$ in $\mathcal{A}$, since $R[-1] \in \mathcal{R} * \mathcal{R}[1]$ by Corollary 2.5, there is a triangle $R_0 \to R[-1] \xrightarrow{f} R_1[1] \xrightarrow{g} R_0[1]$ with $R_0, R_1 \in \mathcal{R}$. Then $\underline{f}$ is a monomorphism in by Lemma 4.2 and $\underline{g}$ is the cokernel of $\underline{f}$ by the structure of cokernel. So we have an exact sequence $0 \to R[-1] \xrightarrow{f} R_1[1] \xrightarrow{g} R_0[1] \to 0$ which is a injective resolution of the projective object $R[-1]$ in $\mathcal{A}$. Therefore inj.dim.$R[-1] \le 1$.

Therefore $\mathcal{A}$ is Gorenstein of Gorenstein dimension at most one.

$\square$

As in [KZ], we have the following corollary.

Corollary 4.5. *Let C be a 2-Calabi-Yau triangulated category and $\mathcal{R}$ a functorially finite maximal rigid subcategory. Then $\mathcal{A}$ is a Frobenius category if and only if $\mathcal{R} = \mathcal{R}[2]$.*

Proof. $\mathcal{A}$ is Frobenius if and only if the sets of projective objects and of injective objects of $\mathcal{A}$ coincide, i.e. $\mathcal{R}[-1] = \mathcal{R}[1]$ if and only if $\mathcal{R} = \mathcal{R}[2]$.

$\square$

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In the last part of this section, we assume that the $2-CY$ triangulated category C admits a maximal rigid object. It follows from Corollary 3.7 that all maximal rigid subcategories are of form add R, where R is a maximal rigid object. The numbers of indecomposable direct summands of all basic maximal rigid objects are the same. Two maximal rigid objects are called reachable via simple mutations if one of them can be obtained from another by finite steps of simple mutations.

The $2-CY$ tilted algebras [BIRS] which by definition the endomorphism algebras of cluster tilting objects in a $2-CY$ triangulated category are a special case of endomorphism algebras of maximal rigid objects in a $2-CY$ triangulated category. The converse is not true in general since the endomorphism algebras of maximal rigid objects in a $2-CY$ triangulated category may contain loops [BIKR][BMV].

Now we collect the representation theoretic properties of endomorphism algebras of maximal rigid objects in a $2-CY$ triangulated category.

Proposition 4.6. *1. All endomorphism algebras of maximal rigid objects in a $2-CY$ triangulated category are Gorenstein algebras of dimension at most 1.*

2. *Let R and R' form a simple mutation pair. Then $EndR$ and $EndR'$ are nearly Morita equivalent, i.e.* mod $EndR/$add $S_i \approx$ mod $EndR'/$add S'_i *(where the symbol $\approx$ means the categories on its both sides are equivalent).*

3. *If R and R' are reachable via simple mutations, then $EndR$ and $EndR'$ have the same representation type.*

Proof. 1. This is direct consequence of Theorem 4.4.

2. This was proved in [Y].

3. Denote $A = EndR$ and $A' = EndR'$. From the assumption, we have that $R[-1] * R = R'[-1] * R'$ by [Pla], which is denoted by $\mathcal{D}$. By Theorem 4.4, A-mod$\approx \mathcal{D}/$add R and A'-mod$\approx \mathcal{D}/$add R'. Therefore A-mod/add $R' \approx \mathcal{D}/$add $(R \cup R') \approx A'$-mod/add R. Hence, ind$A$ is a finite set if and only if ind$\mathcal{D}$ is a finite set. Thus A is of finite type if and only if A' is so. Moreover, by the proof in [Kr], A-mod is wild if and only if A'-mod is wild. Therefore, by tame-wild dichotomy, A and A' have the same representation type. $\square$

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